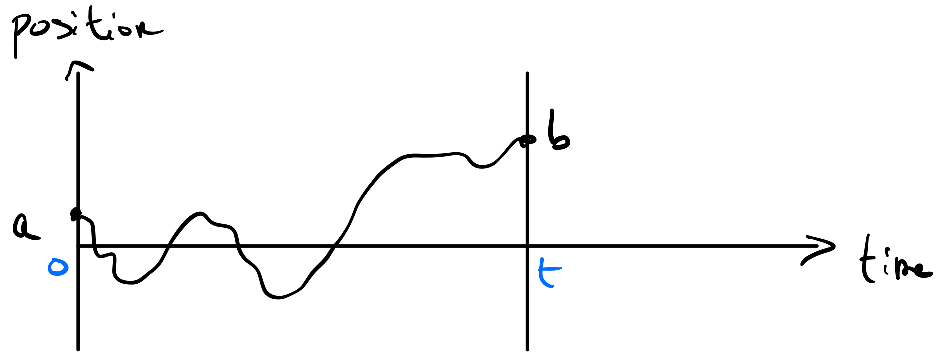
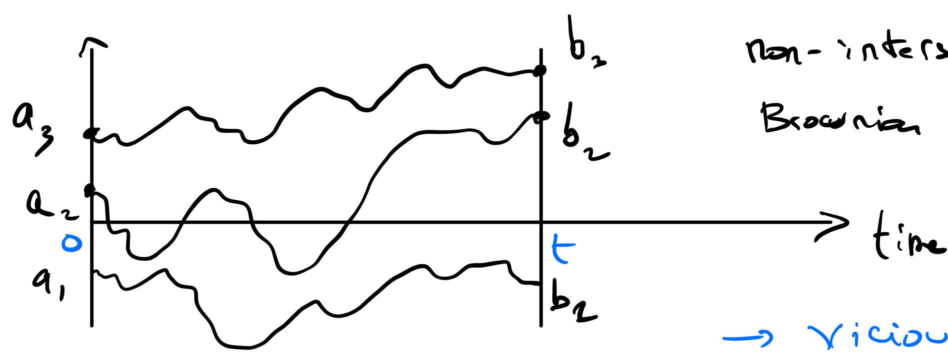
III] Non-intersecting paths and Dyson's Brownian motion.



$$\dot{x}(t) = \xi(t) \qquad \text{(Brownian motion)}$$

$$\langle \xi(t)\,\xi(t') \rangle = 2D\,\delta(t-t') \,, \quad \langle \xi(t) \rangle = 0$$



non-intersecting Brownian motions

→ vicious walkers

$N = 3$

$\underline{Q}$: $\quad \mathcal{S}_t(\underline{a} \to \underline{b}) \equiv$ transition proba. density from $\underline{a}$ to $\underline{b}$

$$\underline{a} = (a_1, \dots, a_N)$$

$$\underline{b} = (b_1, \dots, b_N)$$

For $N=1$: $\mathcal{S}_t(a\to b) = p_t(a\to b)$

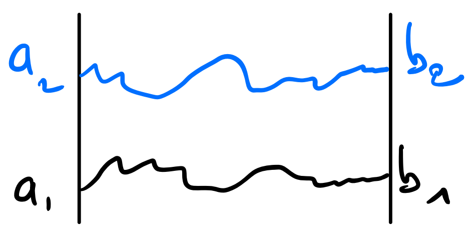
$D=\frac{1}{2}$ $\qquad = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(b-a)^2}{2t}\right)$

$N>1$?

1) Karlin - Mc Gregor formula

Consider $N=2$: divide the sets of trajectories from $\underline{a}$ to $\underline{b}$ (in $[0,t]$) into 2 subsets A & B ($A\cap B=\emptyset$):

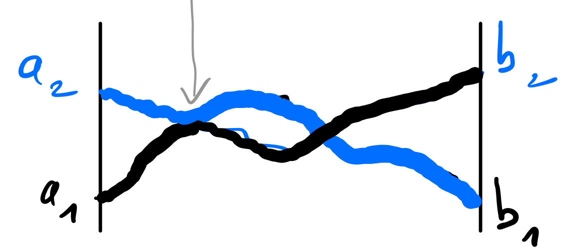
A



no crossing

permute the 2 particles

B



at least 2 cossings

contribute to

x_1: $a_2 \longrightarrow b_1$

x_2: $a_1 \longrightarrow b_2$

$$\Rightarrow \quad S_t\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = P_t(a_1 \to b_1)\, P_t(a_2 \to b_2) - P_t(a_1 \to b_2)\, P_t(a_2 \to b_1)$$

$$= \det \begin{pmatrix} P_t(a_1 \to b_1) & P_t(a_1 \to b_2) \\ P_t(a_2 \to b_1) & P_t(a_2 \to b_2) \end{pmatrix}$$

For N particles:

$$\boxed{S_t(\underline{a} \to \underline{b}) = \det_{1 \le i, j \le N} P_t(a_i \to b_j)}$$

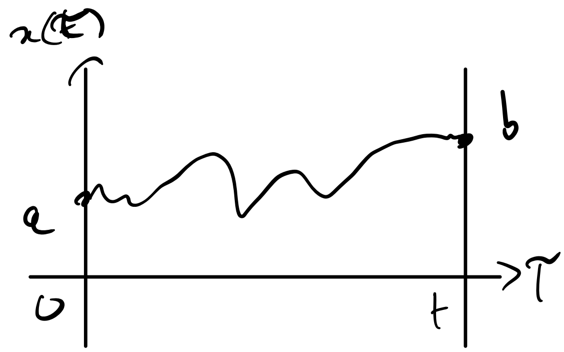
Karlin-Mc Gregor formula (1959)

single particle propagator

$\to$ a generalisat° of the reflexion principle

2) Karlin-McGregor via fermions

2.1) Path integral for 1 Brownian motion



$$\partial_\tau x(\tau) = \xi(\tau)$$

$$x(0) = a$$

$$x(t) = b$$

Gaussian white noise

$P(\{x(\tau)\}_{0\le\tau\le t}) \equiv$ proba. weight of a trajectory

$$\propto e^{-\frac{1}{2}\int_0^t d\tau\, [\underbrace{\dot{x}(\tau)}_{= \xi(\tau)}]^2}\, \delta(x(0)-a)\,\delta(x(t)-b)$$

$$P_t(a \to b) = \underbrace{\int \mathcal{D}x(\tau)\, e^{-\frac{1}{2}\int_0^t d\tau\, [\dot{x}(\tau)]^2}\, \delta(x(0)-a)\,\delta(x(t)-b)}_{\text{path integral in imaginary time for a free quantum particle w. a Hamiltonian: } \hat{h} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}}$$

$$\Rightarrow \quad p_t(a \to b) = \langle b | e^{-t\hat{h}} | a \rangle$$

$$p_{t=0}(a \to b) = \delta(a-b) .$$

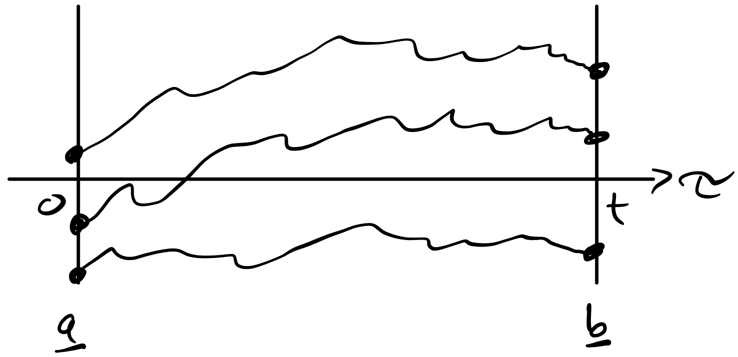
Eigenstates of $\hat{h}$: $\quad \varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \, k \in \mathbb{R}$

$$\Rightarrow p_t(a \to b) = \int_{-\infty}^{\infty} dk \;\; \varphi_k(b) \, \varphi_k^*(a) \, e^{-\frac{k^2}{2}t}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ik(b-a) - \frac{k^2}{2}t}$$

$$p_t(a \to b) = \frac{1}{\sqrt{2\pi t}} \, e^{-\frac{(b-a)^2}{2t}}$$ as it should!

2.2) Path integral for N intersecting Brownian mot.



$\Rightarrow$ quantum mechanics interpretation:

trajectories of non-interacting fermions

in imaginary time.

$$\Rightarrow \mathcal{S}_t(\underline{a} \to \underline{b}) = \langle \underline{b} | e^{-t\hat{H}_N} | \underline{a} \rangle$$

where $$\hat{H}_N = -\frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$$

+ Fermions

$$\Rightarrow \mathcal{S}_t(\underline{a} \to \underline{b}) = \sum_E \Psi_E(\underline{b}) \Psi_E(\underline{a}) e^{-Et}$$

where $$\hat{H}_N \Psi_E(\underline{x}) = E\, \Psi_E(\underline{x})$$

N-particle fermionic state

$\Psi_E(\underline{x}) \equiv$ Slater determinant

$$\Psi_E(\underline{x}) = \frac{1}{\sqrt{N!}} \det_{1 \le i,j \le N} \varphi_{k_i}(x_j)$$

where $\varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$

$$E = E(k_1, .., k_N) = \sum_{i=1}^{N} \frac{k_i^2}{2}$$

$$\Rightarrow S_t(\underline{a} \longrightarrow \underline{b}) = \frac{1}{N!} \int_{-\infty}^{\infty} dk_1 \dots \int_{-\infty}^{\infty} dk_N \ \det_{i,j} \varphi_{k_i}(b_j) \det_{i,j} \varphi^*_{k_i}(a_j)$$
$$\times \underbrace{e^{-\left(\frac{k_1^2}{2} + \frac{k_2^2}{2} .. + \frac{k_N^2}{2}\right)t}}_{e^{-Et}}$$

Cauchy-Binet formula: $\{f_i(x), g_i(x)\}_{1 \le i \le N}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \omega$

$$\int dx_1 .. dx_N \det_{1 \le i,j \le N} f_i(x_j) \det_{1 \le i,j \le N} g_i(x_j) \prod_{i=1}^{N} \omega(x_i)$$

$$= N! \det_{1 \le i,j \le N} \int dx \ f_i(x) g_j(x) \omega(x)$$

Apply it to compute $S_t(a \to b)$:

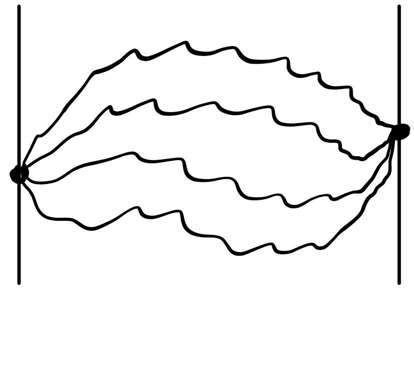
$$S_t(a\to b) = \frac{1}{N!} \times N! \det_{1\le k,l\le N} \underbrace{\int_{-\infty}^{\infty} dk \, \frac{1}{\sqrt{2\pi}} e^{-ika_k} \frac{1}{\sqrt{2\pi}} e^{ikb_l} e^{-\frac{k^2 t}{2}}}_{p_t(a_k \to b_l)}$$

$$= \det_{1\le k,l\le N} p_t(a_k \to b_l)$$

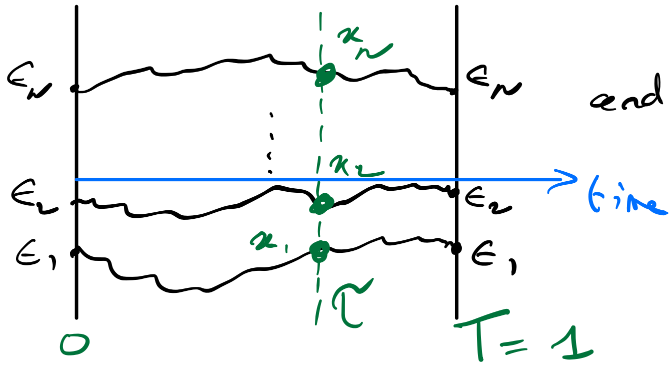
$\longrightarrow$ Karlin-Mc Gregor formula.

<u>See</u>: G.S., S.N. Majumdar, A. Comtet, J-Randon-Furling
PRL <u>101</u>, 150601 (2008)

2) "Watermelons" and RMT $\to$ Dyson's Brownian mot.



$\to$ requires regularizat°



$$P_{joint}(x_1, \dots, x_N; \tau) = \lim_{\underline{\epsilon} \to \underline{0}} \frac{\mathcal{P}_\tau(\underline{\epsilon} \to \underline{x})\,\mathcal{P}_{1-\tau}(\underline{x} \to \underline{\epsilon})}{\mathcal{P}_1(\underline{\epsilon} \to \underline{\epsilon})}$$

$$= \frac{1}{Z_N(\tau)} \prod_{i<j} (x_i - x_j)^2 \, e^{-\frac{1}{2\sigma^2(\tau)} \sum_{i=1}^{N} x_i^2}$$

$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

$$\Rightarrow \frac{x_i}{\sigma(\tau)} \overset{law}{=} \text{eigenvalues of GUE random matrices.}$$

→ Dyson's Brownian motion

$H(t) \equiv N \times N$ Hermitian matrix

$$H_{mn}(t) = \begin{cases} \frac{1}{\sqrt{2}} \left(B_{m,n}(t) + i\, \widetilde{B}_{m,n}(t) \right), & m < n \\ B_{m,m}(t) & \\ \frac{1}{\sqrt{2}} \left(B_{n,m}(t) - i\, \widetilde{B}_{n,m}(t) \right), & m > n \end{cases}$$

$B_{m,n}(t)$ and $\widetilde{B}_{m,n}(t)$ are Brownian bridges (starting & ending at 0).

Dynamics of the eigenvalues $\lambda_1(t) < \ldots < \lambda_N(t)$ is called the Dyson's Brownian motion

is exactly the same as the one of vicious walkers in watermelon config.