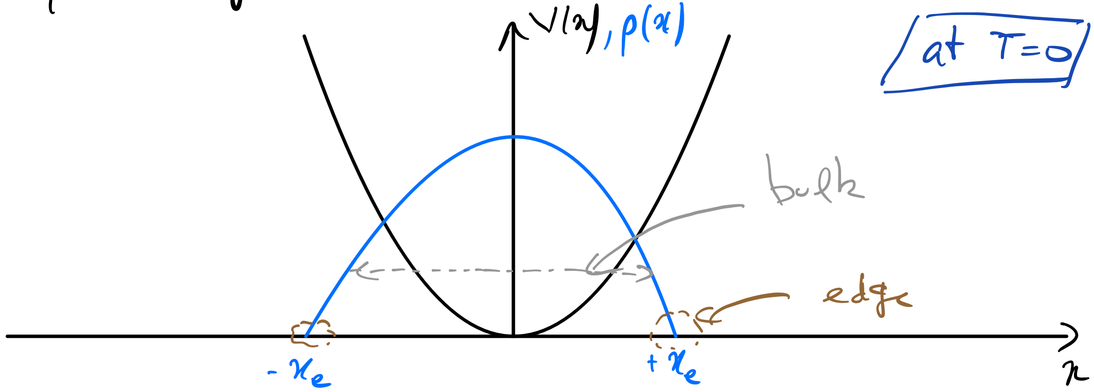
A short recap on lecture 3:

N spinless fermions without interactions in a 1d harmonic trap $\boxed{\text{at } T=0}$



Non-interacting fermions form a determinantal point process (DPP) at $T=0$:

→ n-point correlation function is given by an $n\times n$ determinant

$$R_n(x_1,\dots,x_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{\infty} dx_{n+1}\dots dx_N \, |\Psi_0(x_1,\dots,x_n,x_{n+1},\dots x_N)|^2$$

$$= \det_{1\le k,\ell\le n} K_N(x_k, x_\ell)$$

where $K_N(x,y)$ is the so-called KERNEL:

$$K_N(x_k,x_\ell) = \sum_{m=1}^{N} \varphi_m^*(x_k)\,\varphi_m(x_\ell)$$

single particle eigenfunctions

Kernel in the bulk :

$$K_N(x,y) = \frac{1}{\ell_N} K_{\text{sine}}\left(\frac{x-y}{\ell_N}\right)$$

$$\ell_N = \frac{2}{\pi N \rho(x)}$$

$$K_{\text{sine}}(z) = \frac{\sin 2z}{\pi z} \qquad \text{sine-kernel}$$

Kernel at the edge

$$K_N(x,y) \approx \frac{1}{w_N} K_{Ai}\left(\frac{x-x_e}{w_N}, \frac{y-x_e}{w_N}\right)$$

For $V(x) = \frac{1}{2}x^2$, $x_e = \sqrt{2N}$, $w_N = N^{-\frac{1}{6}}$

$$K_{Ai}(x,y) = \frac{Ai(x)\,Ai'(y) - Ai'(x)\,Ai(y)}{x-y}$$

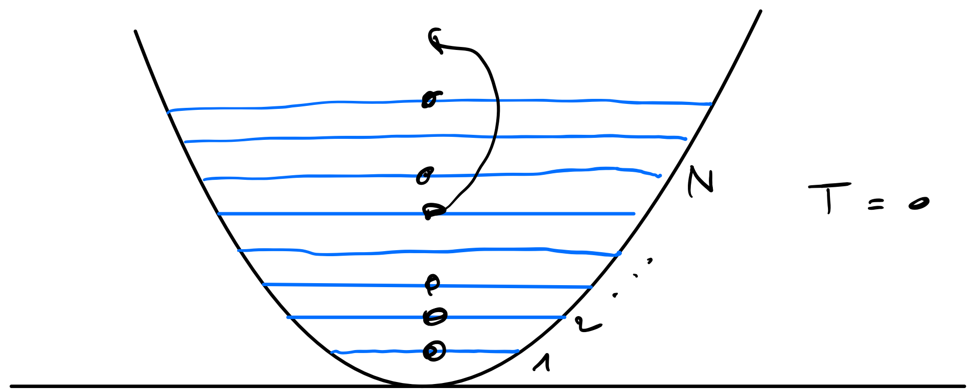
$$= \int_0^{\infty} d\lambda \; Ai(x+\lambda)\,Ai(y+\lambda)$$

$\boxed{\text{Q: What happens at finite temperature } T > 0 \text{ ?}}$

$$P_{joint}(x_1, \dots, x_N) = \langle x_1, \dots, x_N | \hat{\rho} | x_1, \dots, x_N \rangle$$

$$\hat{\rho} = \frac{1}{Z_N} e^{-\beta \hat{H}}$$

N-body Hamiltonian



$T = 0$

$$P_{joint}(x_1, \dots, x_N) = \frac{1}{N! Z_N} \sum_{k_1 < k_2 < \dots < k_N} \left| \det_{1 \le i,j \le N} \phi_{k_i}(x_j) \right|^2 e^{-\beta(\epsilon_{k_1} + \dots + \epsilon_{k_N})}$$

$\longrightarrow$ is not determinantal anymore

$\longrightarrow$ the canonical ensemble is very complicated

Trick: work in the grand-canonical ensemble

and fix the chemical potential μ.

$\Rightarrow$ the positions of the fermions form a determinantal point process!

with kernel: $K_\mu(x,y) = \sum_{k=1}^{\infty} \varphi_k^*(x)\varphi_k(y) \boxed{\frac{1}{e^{\beta(\epsilon_k-\mu)}+1}}$

$\beta = \frac{1}{k_B T}$ Fermi factor

with the relation $N = \sum_{k=1}^{\infty} \frac{1}{e^{\beta(\epsilon_k-\mu)}+1}$

Relevant length scales

At $T=0$: bulk $\ell_{bulk} \sim N^{-\frac{1}{2}}$

edge $\ell_{edge} \equiv w_N \sim N^{-\frac{1}{6}}$

At $T>0$: de Broglie wave length

$$\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}} \sim \frac{1}{\sqrt{T}}$$

$\Rightarrow$ controls the crossover from quantum/classical

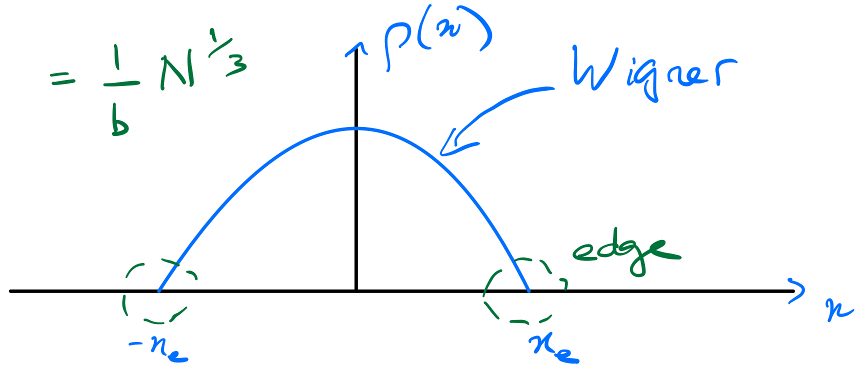
. In the bulk this crossover happens for $\lambda_T \sim \ell_{bulk}$

$\rightarrow \quad T \sim O(N)$

- At the edge, the crossover occurs for $\lambda_T \sim \ell_{edge}$

$$\Rightarrow k_B T \sim \hbar\omega N^{1/3} \ll N$$

If $T = O(N^{1/3})$ $= \frac{1}{b} N^{1/3}$

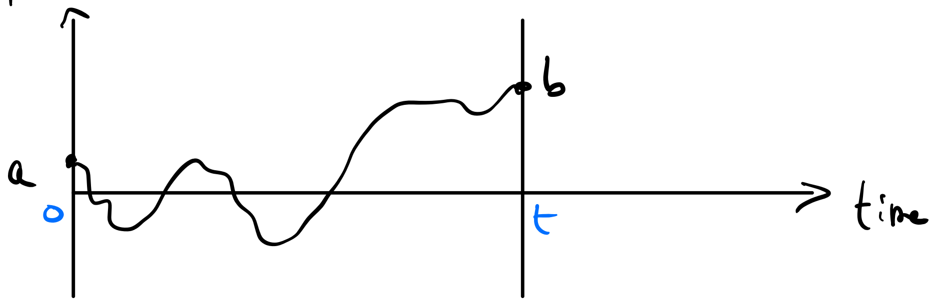


$$K_N(x,y) \xrightarrow[N\to\infty]{} \frac{1}{w_N} K_b\left(\frac{x-x_e}{w_N}, \frac{y-y_e}{w_N}\right)$$

$$\boxed{K_b(r,r') = \int_{-\infty}^{\infty} du \, \frac{Ai(r+u)\,Ai(r'+u)}{1+e^{-bu}}}$$

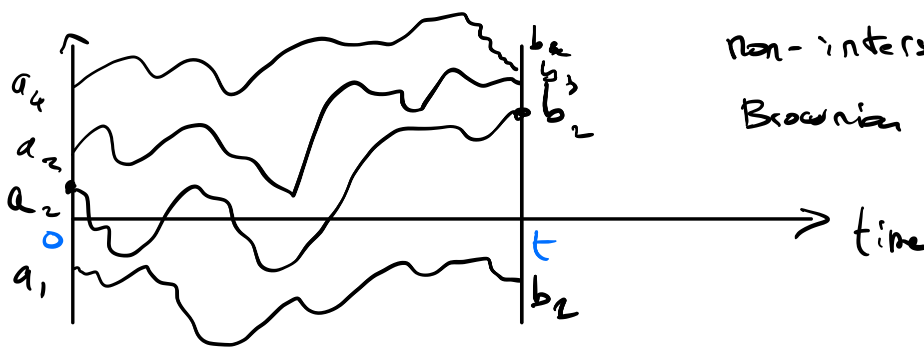
↳ deformed Airy kernel

III] Non-intersecting paths and Dyson's Brownian motion.



$$\dot{x}(t) = \xi(t) \qquad \text{(Brownian motion)}$$

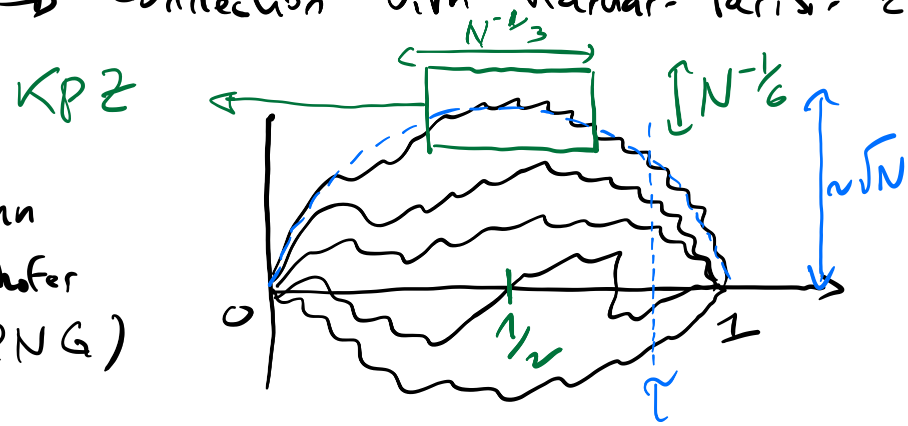
$$\langle \xi(t)\xi(t') \rangle = 2D\,\delta(t-t') \ , \ \langle \xi(t) \rangle = 0$$



non-intersecting Brownian motions

→ Connection with Kardar-Parisi-Zhang eq:

Spohn Praehofer (PNG)



N vicious walkers

$N \gg 1$

watermelons