b) collinear region: $k^\mu \sim (\lambda^2, 1, \lambda)\, Q$

Expansions of propagators:

$$(k+p_1)^2 = k^2 + 2k\cdot p_1 + p_1^2$$

$$= \underset{\lambda^2}{k^2} + \underset{\lambda^2}{n\cdot k}\, \underset{1}{\bar{n}\cdot p_1} + \underset{1}{\bar{n}\cdot k}\, \underset{\lambda^2}{n\cdot p_1} + 2\underset{\lambda}{k_\perp}\cdot \underset{\lambda}{p_{1\perp}} + \underset{\lambda^2}{p_1^2} \sim \lambda^2$$

↪ nothing to expand

$$(k+p_2)^2 = \underset{\lambda^2}{k^2} + \underset{\lambda^2}{n\cdot k}\, \underset{\lambda^2}{\bar{n}\cdot p_2} + \underset{1}{\bar{n}\cdot k}\, \underset{1}{n\cdot p_2} + 2\underset{\lambda}{k_\perp}\cdot \underset{\lambda}{p_{2\perp}} + \underset{\lambda^2}{p_2^2} \sim \lambda^0$$

$$= \bar{n}\cdot k\; n\cdot p_2 + \underset{\uparrow\,\text{drop}}{\mathcal{O}(\lambda^2)} \simeq 2k\cdot p_{2+}$$

This gives:

$$I_c = i\,\pi^{-D/2}\mu^{2\epsilon}\int d^Dk\, \frac{2p_{1-}\cdot p_{2+}}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[2k\cdot p_{2+}+i0\right]}$$

$$= \Gamma(1+\epsilon)\left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right]$$

$(P_1^2 \equiv -p_1^2)$

↪ appearance of double and single poles in ϵ (IR divergences, since integral is UV-finite)

↪ result depends on collinear scale P_1^2 only

c) anti-collinear region: $k^\mu \sim (1, \lambda^2, \lambda)\, Q$

We find an analogous contribution:

$$I_{\bar{c}} = \Gamma(1+\epsilon)\left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_2^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right]$$

$$(P_2^2 \equiv -p_2^2)$$

$\Rightarrow$ sum of the three contributions:

$$I_h + I_c + I_{\bar{c}}$$

$$= \Gamma(1+\epsilon)\Bigg[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\frac{\mu^2}{Q^2} - \ln\frac{\mu^2}{P_1^2} - \ln\frac{\mu^2}{P_2^2}\right)$$

$$+ \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\Bigg]$$

Surprisingly, this does not reproduce the exact result on p. 4 (lecture 3), and uncancelled IR divergences remain. It follows that we have failed to identify (at least) one relevant region. Combining the three logs in the coefficient of the $1/\epsilon$ pole, we get:

$$[\text{ above }] = -\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{P_1^2 P_2^2} + \ldots$$

↪ suggest that missing region corresponds to scale:

$$\frac{P_1^2 P_2^2}{Q^2} \sim \lambda^4 Q^2 \ll \text{collinear scale } P_i^2 \sim \lambda^2 Q^2$$

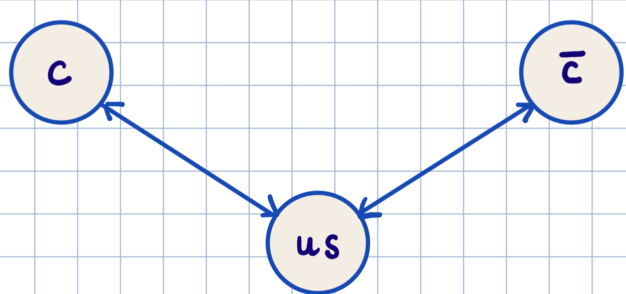
d) ultra-soft contribution:

There is a strong physics reason suggesting that we need another mode (corresponding to a momentum region) in the low-energy effective theory. An EFT built out of colliner and anti-collinear particles would contain two disjunct sectors, because no vertices connecting both types of particles are allowed:

$$\underset{(\lambda^2,1,\lambda)}{p_c^\mu} + \underset{(1,\lambda^2,\lambda)}{p_{\bar{c}}^\mu} \sim (1,1,\lambda) \quad \text{hard!}$$

Physically, it would be strange if the two jets could not interact in the low energy theory, since they need to neutralize their color. The "largest" on-shell mode that can connect to both collinear and anti-collinear particles without taking them far off-shell is the <u>ultra-soft mode:</u>

$$p_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2) \Rightarrow p_{us}^2 \sim \lambda^4 Q^2$$



"messenger"

Let us evaluate the ultra-soft contribution to the Sudakov form factor: $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)\, Q$

$$(k+p_1)^2 = \underset{\lambda^4}{k^2} + \underset{\lambda^2}{n\cdot k}\,\underset{1}{\bar{n}\cdot p_1} + \underset{\lambda^2}{\bar{n}\cdot k}\,\underset{\lambda^2}{n\cdot p_1} + 2\,\underset{\lambda^2}{k_\perp}\cdot \underset{\lambda}{p_{1\perp}} + \underset{\lambda^2}{p_1^2} \sim \lambda^2$$

$$= n\cdot k\;\bar{n}\cdot p_1 + p_1^2 + \mathcal{O}(\lambda^3) \simeq 2k\cdot p_{1-} + p_1^2$$

drop

$$(k+p_2)^2 = \bar{n}\cdot k\;n\cdot p_2 + p_2^2 + \mathcal{O}(\lambda^3) \simeq 2k\cdot p_{2+} + p_2^2$$

This gives:

$$I_{us} = i\,\pi^{-D/2}\mu^{2\epsilon}\int d^Dk\,\frac{2p_{1-}\cdot p_{2+}}{(k^2+i0)(2k\cdot p_{1-}+p_1^2+i0)(2k\cdot p_{2+}+p_2^2+i0)}$$

$$= \Gamma(1+\epsilon)\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2Q^2}{p_1^2p_2^2} + \frac{1}{2}\ln^2\frac{\mu^2Q^2}{p_1^2p_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right]$$

Adding this result to the expression on page 2, we find:

$$I_h + I_c + I_{\bar{c}} + I_{us}$$

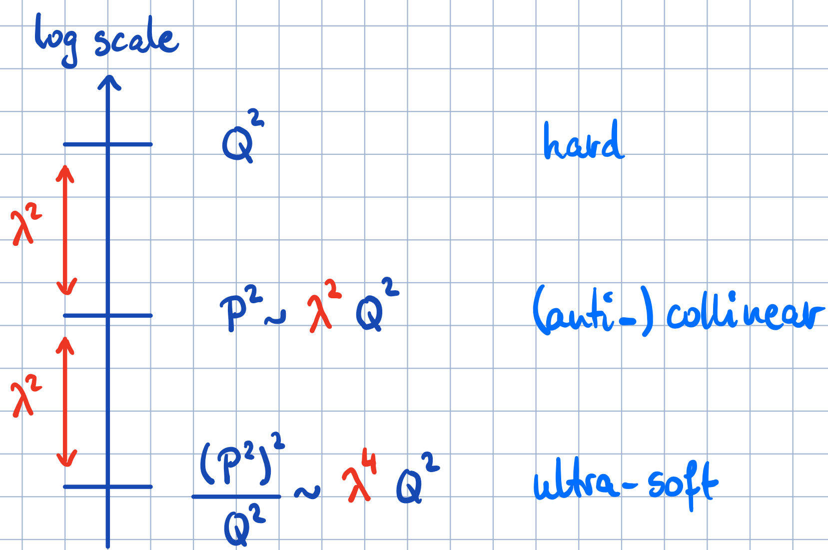
$$= \underset{\text{hard}}{\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}} - \underset{\text{collinear}}{\frac{1}{2}\ln^2\frac{\mu^2}{p_1^2}} - \underset{\text{anti-collinear}}{\frac{1}{2}\ln^2\frac{\mu^2}{p_2^2}} + \underset{\text{ultra-soft}}{\frac{1}{2}\ln^2\frac{\mu^2Q^2}{p_1^2p_2^2}} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon)$$

$$= \frac{1}{2}\ln\frac{Q^2}{p_1^2}\ln\frac{Q^2}{p_2^2} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon)$$

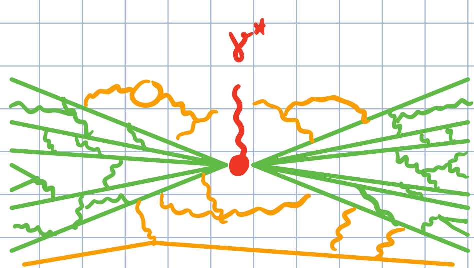
This agrees with the original expression on p. 4 of lecture 3!

Comments:

The decomposition of Sudakov double logarithms into a sum of logarithms depending on a single physical scale requires the presence of <u>three correlated scales</u>:



The ultra-soft scale is physical and characterizes soft exchanges between the two jets:



ultra-soft exchanges are needed for color neutralization

The process can be calculated in perturbative QCD only if the ultra-soft scale is much larger that Λ_{QCD}.

↪ else, need nonperturbative soft functions

Effective Lagrangian of Soft-Collinear Effective Theory

Our goal is to construct an effective Lagrangian built out of collinear, anti-collinear and ultra-soft quark and gluon fields (and ghost fields, but we will not write them out explicitly). Momentum conservation allows the following interactions involving different modes:

$$\underbrace{\phi_c \dots \phi_c}_{n_c \geq 2} \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \qquad \checkmark$$

$$\underbrace{\phi_{\bar{c}} \dots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 2} \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \qquad \checkmark$$

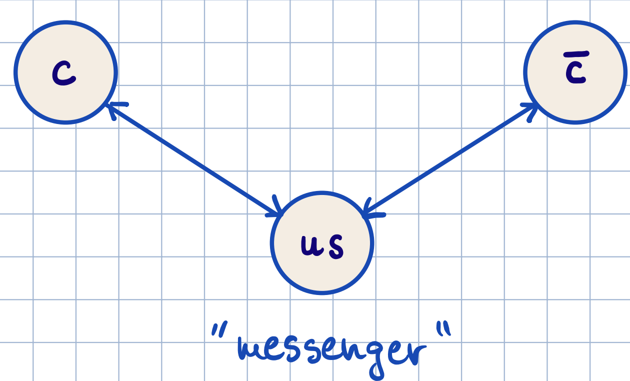
but not:

$$\underbrace{\phi_c \dots \phi_c}_{n_c \geq 1} \; \underbrace{\phi_{\bar{c}} \dots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 1} \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 0}$$ forbidden (hard interaction)

$$\phi_c \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1} \, , \quad \phi_{\bar{c}} \; \underbrace{\phi_{us} \dots \phi_{us}}_{n_{us} \geq 1}$$ momentum not conserved

It follows that:

$$\mathcal{L}_{SCET}^{(c,\bar{c},us)} = \mathcal{L}_c + \mathcal{L}_{us} + \mathcal{L}_{\bar{c}} + \mathcal{L}_{c+us} + \mathcal{L}_{\bar{c}+us}$$



It suffices to study the Lagrangian of collinear and ultra-soft fields.

Collinear quark field:

The spinor of a highly energetic (along z-axis), Light fermion satisfies:

$$\not{p}\, u_s(p) = m\, u_s(p) \quad ; \quad m \ll E = p^0 \simeq p^3$$

$$\Rightarrow \quad \not{n}\, u_s(p) \simeq 0 \qquad (\text{up to } m/E \text{ corrections})$$

In analogy with HQET, we identify the large and small components of such a spinor using projection operators:

$$P_n = \frac{\not{n}\not{\bar{n}}}{4} \quad , \quad P_{\bar{n}} = \frac{\not{\bar{n}}\not{n}}{4} \qquad (\bar{P}_n = \gamma^0 P_n^\dagger \gamma^0 = P_{\bar{n}})$$

with:

$$P_n + P_{\bar{n}} = \frac{\{\not{n}, \not{\bar{n}}\}}{4} = \frac{2\, n\cdot\bar{n}}{4} = 1$$

$$P_n^2 = P_n \quad , \quad P_{\bar{n}}^2 = P_{\bar{n}} \quad , \quad P_n P_{\bar{n}} = 0 = P_{\bar{n}} P_n$$

We define:

modes restricted to collinear region

$$\xi_n = P_n \psi_c \quad , \quad \eta_n = P_{\bar{n}} \psi_c$$

$$\Rightarrow \quad \not{n}\, \xi_n = 0 \qquad (\xi_n \text{ will describe a collinear quark in SCET})$$

$$\not{\bar{n}}\, \eta_n = 0$$

To derive the power counting in λ, we consider (massless fermion):

$$n\cdot p\,\frac{\bar{n}\!\!\!/}{2} + \bar{n}\cdot p\,\frac{n\!\!\!/}{2} + p\!\!\!/_\perp$$

$$\langle 0|\, T\{\, \psi_c(x)\, \bar{\psi}_c(0)\}\,|0\rangle = \underset{\lambda^4}{\int \frac{d^4p}{(2\pi)^4}}\; \underset{1}{e^{-ip\cdot x}}\; \underset{\frac{1}{\lambda^2}}{\frac{i\,p\!\!\!/}{p^2+i\epsilon}} \quad (\lambda^2, 1, \lambda)$$

Hence:

$$\langle 0|\, T\{\, \xi_n(x)\, \bar{\xi}_n(0)\}\,|0\rangle = \underset{\lambda^4}{\int \frac{d^4p}{(2\pi)^4}}\; \underset{1}{e^{-ip\cdot x}}\; \underset{\lambda^{-2}}{\frac{i\,\bar{n}\cdot p}{p^2+i\epsilon}}\; \frac{n\!\!\!/}{2} \sim \lambda^2$$

$$\boxed{P_n\, p\!\!\!/\, P_{\bar{n}} = \bar{n}\cdot p\,\frac{n\!\!\!/}{2}}$$

$$\langle 0|\, T\{\, \eta_n(x)\, \bar{\eta}_n(0)\}\,|0\rangle = \underset{\lambda^4}{\int \frac{d^4p}{(2\pi)^4}}\; \underset{1}{e^{-ip\cdot x}}\; \underset{\lambda^{0}}{\frac{i\,n\cdot p}{p^2+i\epsilon}}\; \frac{\bar{n}\!\!\!/}{2} \sim \lambda^4$$

$$\langle 0|\, T\{\, \xi_n(x)\, \bar{\eta}_n(0)\}\,|0\rangle = \underset{\lambda^4}{\int \frac{d^4p}{(2\pi)^4}}\; \underset{1}{e^{-ip\cdot x}}\; \underset{\lambda^{-1}}{\frac{i\,p_\perp^\mu}{p^2+i\epsilon}}\; P_n\,\gamma_\mu\,\bar{P}_{\bar{n}} \sim \lambda^3$$

It follows that:

large components   small components

$$\boxed{\xi_n \sim \lambda\,, \qquad \eta_n \sim \lambda^2}$$

Note that these rules do <u>not</u> agree with naive dimensional analysis!

In analogy with HQET, we will integrate out the small components η_n and use the field ξ_n to describe a collinear quark in SCET.