Short recap on lecture 2:

- Joint proba. density function (PDF) of eigenvalues in GOE / GUE

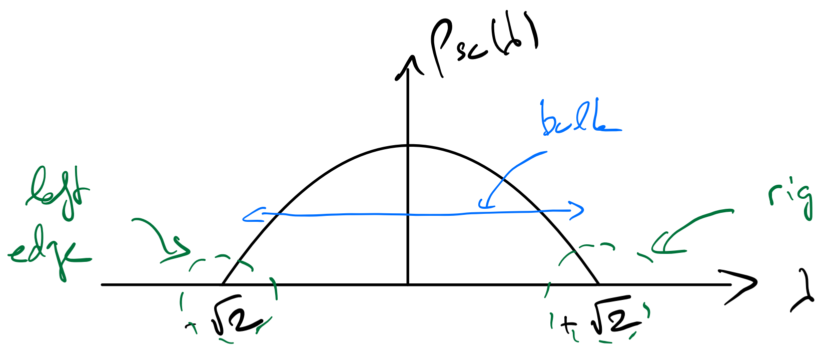
$$\boxed{P(\lambda_1,\dots,\lambda_N) = B_N\, e^{-\beta\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2} \prod_{i<j} |\lambda_i-\lambda_j|^{\beta}}$$

with $\beta = 1$ for GOE & $\beta = 2$ for GUE

- $$\rho_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)$$

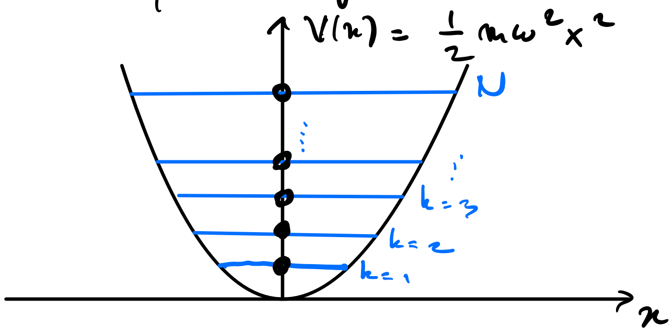
Wigner semi-circle:

$$\rho_N(\lambda) \xrightarrow[N\to\infty]{} \rho_{sc}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2} \quad -\sqrt{2}\le\lambda\le\sqrt{2}$$



Lecture 3 : Fermions and connections to RMT

N spinless fermions without interactions in a 1d harmonic trap



$$\hat{H} = \sum_{i=1}^{N} \hat{h}_i \, , \quad \hat{h}_i = \frac{\hat{p}_i^2}{2m} + V(\hat{x}_i)$$

Single-particle eigenfunct°:

$$\varphi_k(x) \propto e^{-\frac{\alpha^2 x^2}{2}} H_{k-1}(\alpha x)$$

$$\alpha = \sqrt{\frac{m\omega}{\hbar}} \qquad k = 1, 2, \dots$$

$$\epsilon_k = \hbar\omega \left(k - \frac{1}{2} \right)$$

H_k's are Hermite poly.:

$$H_0(x) = 1, \; H_1(x) = 2x, \; H_2(x) = 4x^2 - 2$$

$$\dots$$

I) $T = 0$: ground-state

N-particle wave-function

$$\Psi_0(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{N!}} \det_{1 \le k, \ell \le N} \varphi_k(x_\ell)$$

$$\propto e^{-\frac{\alpha^2}{2}(x_1^2 + \dots + x_N^2)} \det_{1 \le k, \ell \le N} H_{k-1}(x_\ell)$$

For ex, $N = 3$:

$$\det H_{k-1}(x_\ell) = \begin{vmatrix} 1 & 2x_1 & 4x_1^2 - 2 \\ 1 & 2x_2 & 4x_2^2 - 2 \\ 1 & 2x_3 & 4x_3^2 - 2 \end{vmatrix} = \begin{vmatrix} 1 & 2x_1 & 4x_1^2 \\ 1 & 2x_2 & 4x_2^2 \\ 1 & 2x_3 & 4x_3^2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} \leftarrow \text{Vandermonde det}$$

$$\propto \quad (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

$$\Rightarrow \Psi_0(x_1, .., x_N) \propto e^{-\frac{\alpha^2}{2}(x_1^2 + .. + x_N^2)} \prod_{i<j} (x_j - x_i)$$

Quantum joint proba. density funct° of x_i's:

$$\boxed{|\Psi_0(x_1, .., x_N)|^2 = \frac{1}{Z_N} e^{-\alpha^2 \sum_{i=1}^{N} x_i^2} \prod_{i<j} (x_i - x_j)^2}$$

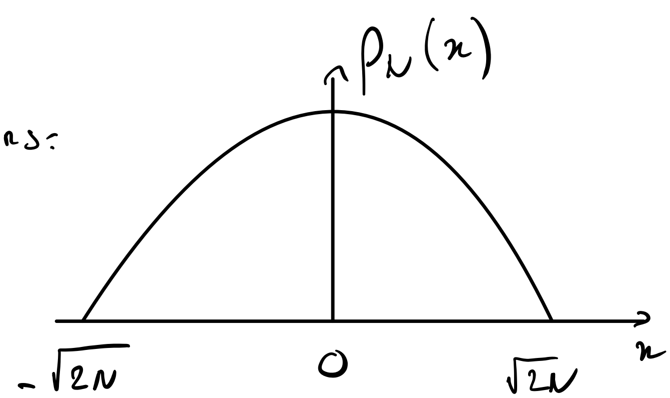
$\Rightarrow$ one-to-one mapping with the eigenvalues λ_i's of a matrix belonging to GUE:

$$\alpha x_i \longleftrightarrow \sqrt{N} \lambda_i$$

$\underline{\alpha = 1}$

- Density of Fermions:

$$\rho_N(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)$$



- Determinantal point process

$\longrightarrow$ Kernel :

$$|\Psi_0(x_1, \dots, x_N)|^2 = \frac{1}{N!} \det_{k,l} \varphi_k^*(x_l) \det_{k,l} \varphi_k(x_l)$$

Define $\quad A_{k,l} = \varphi_l^*(x_k) \, ; \quad B_{kl} = \varphi_k(x_l)$

$$\Rightarrow |\Psi_0|^2 = \frac{1}{N!} \det {}^t A \quad \det B$$

$$= \frac{1}{N!} \det A \det B = \frac{1}{N!} \det AB$$

$$(AB)_{k,l} = \sum_{m=1}^{N} A_{k,m} B_{m,l} = \underbrace{\sum_{m=1}^{N} \varphi_m^*(x_k) \varphi_m(x_l)}_{= K_N(x_k, x_l)}$$

kernel

$$\Rightarrow |\Psi_0(x_1, \dots, x_N)|^2 = \frac{1}{N!} \det_{1 \le k,l \le N} K_N(x_k, x_l)$$

The Kernel is reproducible :

$$\int_{-\infty}^{\infty} K_N(x,z) K_N(z,y) \, dz = K_N(x,y)$$

$$= \sum_{k=1}^{\tilde{N}} \sum_{k'=1}^{\tilde{N}} \int_{-\infty}^{\infty} dz \, \varphi_k^*(x) \underbrace{\varphi_k(z) \varphi_{k'}^*(z)}_{\delta_{k,k'}} \varphi_{k'}(y)$$

$$= \sum_{k=1}^{\tilde{N}} \varphi_k^*(x) \varphi_k(y) = K_N(x,y)$$

$\Rightarrow$ Correlation functions are determinantal

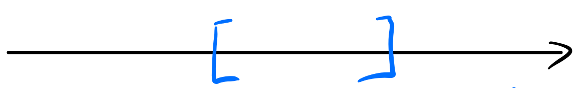
$$R_n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} dx_{n+1} \cdots dx_N \, |\Psi_0(x_1, \dots, x_n, x_{n+1}, \dots x_N)|^2$$

$$= \det_{1 \le k,\ell \le n} K_N(x_k, x_\ell)$$

Rk: $K_N(x,y) = \langle \Psi_0 | \Psi^\dagger(x) \Psi(y) | \Psi_0 \rangle$

determinantal structure $\Longleftrightarrow$ Wick's theorem.

<u>Hole probability</u>: Proba. $(N_J = 0)$



$J \equiv$ interval

$N_J \equiv$ nber of fermions in J

Generating function: $\langle e^{-pN_J} \rangle = \sum_{n=0}^{\infty} e^{-pn} \, \text{Proba.}(N_J = n)$

$p \in \mathbb{R}^+$

for DPP

$$= \text{Det}\left[\mathbb{1} - (1-e^{-p}) P_J K_N P_J\right]$$

DPP: det. point process

$$\mathrm{Det}(\mathbb{1} - \tilde{K}) = \exp\left(-\sum_{m=1}^{\infty} \frac{1}{m} \mathrm{Tr}(\tilde{K})^m\right)$$

Fredholm det. for $\tilde{K}(x,y)$

where $P_J(x) = \begin{cases} 1 & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases}$

<u>Example</u>: $\mathrm{Tr}(P_J K_N P_J) = \int dx\, P_J(x) K_N(x,x) P_J(x)$

Application to $x_{max} = \max_{1 \le i \le N} x_i$:

1^{st} observation : $\mathrm{Proba}(N_J = 0) = \lim_{p \to \infty} \langle e^{-pN_J} \rangle$

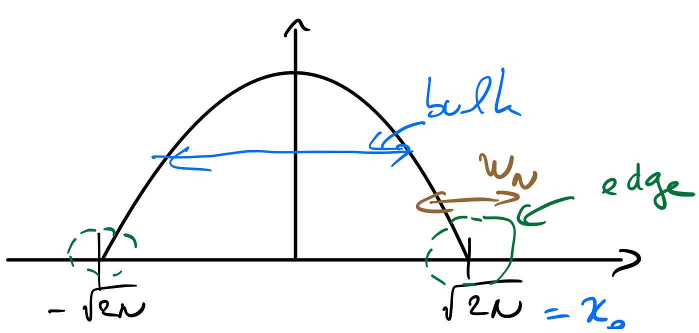
2^{nd} observat°: $\mathrm{Proba}(x_{max} \le M)$

$= \mathrm{Proba.}$ (no particle in $[M, +\infty)$)

$= \mathrm{Det}(\mathbb{1} - P_{[M,+\infty)} K_N P_{[M,+\infty)})$

→ Large N limit of the kernel: bulk vs edge.

$V(x) = \frac{1}{2} x^2$



Rk: $\hat{h} = \frac{\hat{p}^2}{2m} + V(x)$, for $N \gg 1$ the density is given by Local Density Approx:

$$\rho(x) \sim \sqrt{\mu - V(x)}$$

Fermi energy

Kernel in the bulk:

$$K_N(x,y) = \frac{1}{\ell_N} K_{\text{sine}}\left(\frac{x-y}{\ell_N}\right)$$

$$\ell_N = \frac{2}{\pi N \rho(x)}$$

$$K_{\text{sine}}(z) = \frac{\sin 2z}{\pi z} \qquad \text{sine-kernel}$$

Kernel at the edge

$$K_N(x,y) \approx \frac{1}{w_N} K_{Ai}\left(\frac{x-x_e}{w_N}, \frac{y-x_e}{w_N}\right)$$

For $V(x) = \frac{1}{2}x^2$, $x_e = \sqrt{2N}$, $w_N = N^{-\frac{1}{6}}$

$$K_{Ai}(x,y) = \frac{Ai(x)\,Ai'(y) - Ai'(x)\,Ai(y)}{x-y}$$

$$= \int_0^\infty d\lambda \; Ai(x+\lambda)\, Ai(y+\lambda)$$

↳ Airy-kernel

Application: $\text{Proba.}\,(x_{max} \leq M) \underset{N\to\infty}{\simeq} \mathcal{F}_2\left(\frac{M - x_e}{w_N}\right)$

with $\mathcal{F}_2(s) = \text{Det}\left(\mathbb{1} - P_s K_{Ai} P_s\right)$

Tracy-Widom distribution

↳ connection with Painlevé II

eq. Tracy-Widom '94.