A Mathematical Introduction to 3d $\mathcal{N} = 4$ Gauge Theory III

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Recap I

We consider a 3d $\mathcal{N} = 4$ supersymmetric gauge theory:

1. A compact Lie group G
2. A unitary representation $\rho : G \to \mathsf{U}(n)$ acting on $R = \mathbb{C}^n$.

Global symmetries:

- R-symmetry $SU(2)_H \times SU(2)_C$
- Flavour symmetry $G_H \times G_C$

$$G_H = N_{\mathsf{U}(n)}(\rho(G))/\rho(G)$$

$$G_C \supset T_C = \mathsf{Hom}(\pi_1(G), U(1))$$

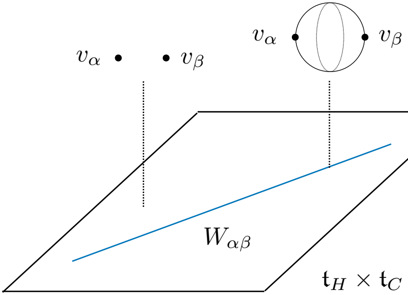
Recap 2

Introduce mass and FI parameters:

- Mass parameters $m \in \mathfrak{t}_H$
- FI parameters $\zeta \in \mathfrak{t}_C$

Study supersymmetric vacua at points on the parameter space $\mathfrak{t}_H \times \mathfrak{t}_C$.

- Assume isolated massive vacua v_α at generic points.
- Moduli spaces open up on loci $W_{\alpha\beta}$ where domain walls become tensionless.



Today

What is the general structure of supersymmetric vacua?

Two extreme cases:

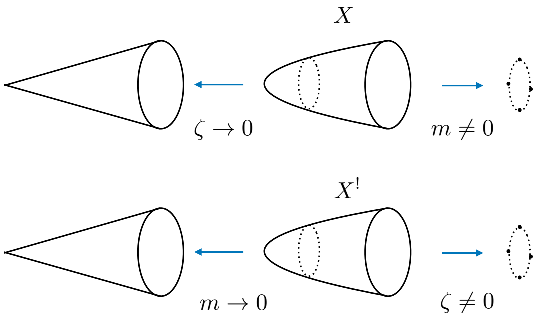
- m, ζ both generic: isolated massive vacua.
- $m = 0$, $\zeta = 0$: intricate moduli space with intersecting branches or strata of various types - see Hanany's lectures.

Intermediate cases:

1. $m = 0$, ζ generic $\rightarrow$ Higgs branch X
2. $\zeta = 0$, m generic $\rightarrow$ Coulomb branch $X^!$

Broad Picture

From the moduli spaces X, $X^!$ we can study the two extreme limits.



- Turning on m, ζ leads to isolated massive vacua again.
- Turning off ζ, m leads to conical spaces forming strata of a larger intricate moduli space at $m = 0$, $\zeta = 0$.

Some Terminology

Higgs and Coulomb refer to the infrared behaviour of the gauge theory.

Higgs branch:

- Gauge symmetry G broken.
- Hypermultiplet fields X, Y have non-zero expectations values.

Coulomb branch

- Gauge symmetry broken to maximal torus $T \subset G$.
- Hypermultiplet fields X, Y vanish.

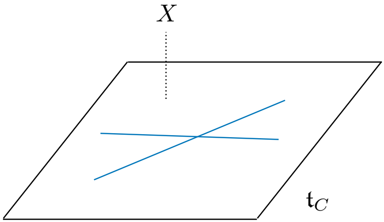
Higgs Branch X

Higgs Branch

To access the Higgs branch X,

- $m = 0$
- ζ generic

Generic means in the complement of the hyperplanes $W_{\alpha\beta}$ in $\mathfrak{t}_C$.



Classical Vacua

Setting $m = 0$, the equations for classical vacua become:

$$\begin{aligned} \mu_{\mathbb{R}} - \zeta &= 0 & \mu_{\mathbb{C}} &= 0 \\ (\sigma \cdot J) \cdot X &= 0 & (\varphi \cdot J) \cdot X &= 0 \\ (\sigma \cdot J) \cdot Y &= 0 & (\varphi \cdot J) \cdot Y &= 0 \\ [\varphi, \varphi^\dagger] &= 0 & [\sigma, \varphi] &= 0 \end{aligned}$$

Under our assumptions, for generic ζ:

- Only solutions have σ, $\varphi = 0$.
- G completely broken.

We can therefore focus on the top line.

Quotient Construction

The Higgs branch X parametrises solutions to

$$\mu_{\mathbb{R}} - \zeta = 0 \qquad \mu_{\mathbb{C}} = 0$$

modulo G gauge transformations.

- This is a hyper-Kähler or holomorphic symplectic quotient

$$\begin{aligned} X &= \mu_{\mathbb{R}}^{-1}(\zeta) \cap \mu_{\mathbb{C}}^{-1}(0)/G \\ &= T^*R \,///_{\zeta}\, G \end{aligned}$$

- A non-renormalisation theorem ensures there are no quantum corrections to this description!

Some Geometry

The hyper-Kähler quotient is defined for any quaternionic representation and produces a hyper-Kähler manifold X.

For a representation of cotangent type T^*R with complex coordinates (X, Y) it is natural to regard X as a complex manifold together with:

- A Kähler form ω. This is a closed $(1,1)$-form on X inherited from

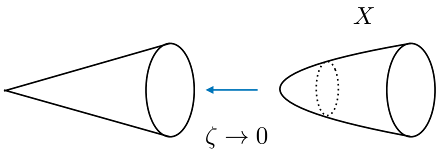
$$\omega = dX \wedge d\bar{X} + dY \wedge d\bar{Y} \, .$$

- A holomorphic symplectic form Ω. This is a closed $(2,0)$-form on X inherited from

$$\Omega = dX \wedge dY \, .$$

ζ Dependence

Under our assumptions, for generic ζ, X is a smooth holomorphic symplectic manifold.



- When $\zeta \to 0$, X develops a conical singularity where the gauge symmetry G is unbroken.
- This forms part of a larger moduli space of vacua at the superconformal point $m = 0$, $\zeta = 0$.

Example: Supersymmetric QED

Consider supersymmetric QED with $G = \mathrm{U}(1)$ and w hypermultiplets.

$$\sum_{j=1}^{w}(|X_j|^2 - |Y_j|^2) = \zeta \qquad \sum_{j=1}^{w} X_j Y_j = 0$$

- Suppose $\zeta > 0$ and look for solutions with $Y_j = 0$,

$$\sum_{j=1}^{w} |X_j|^2 = \zeta\,.$$

- This describes complex projective space $\mathbb{CP}^{w-1} = S^{2w-1}/U(1)$ as a quotient along fibers of the Hopf fibration.
- Including Y_j gives the cotangent bundle, $X = T^*\mathbb{CP}^{w-1}$.

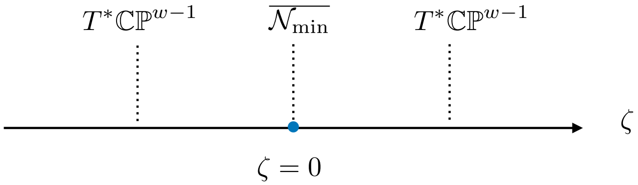
Example: Supersymmetric QED

The FI parameter space is $\mathfrak{t}_C = \mathbb{R}$ with hyperplane at $\{\zeta = 0\}$

- When $\zeta = 0$, X is the closure of a minimal nilpotent orbit

$$X = \overline{\mathcal{N}_{\text{min}}} \subset \mathfrak{sl}(w, \mathbb{C})\,.$$

- This is parametrised by the meson matrix $M_{ij} = X_i Y_j$.
- Turning on ζ gives a holomorphic symplectic resolution.



Global Symmetries

Turning on a generic FI parameter ζ, the supersymmetric gauge theory flows to a sigma model with target space X.

We would like to understand global symmetries from this perspective:

- R-symmetries
- Flavour symmetries

We also want to re-introduce the mass parameters m and recover the hyperplane arrangement from this perspective.

R-symmetry

If we regard X is a hyper-Kähler manifold:

- $SU(2)_H$ acts by isometries of the hyper-Kähler metric on X, rotating the $\mathbb{CP}^1$ family of complex structures.

If we regard X as a holomorphic symplectic manifold:

- $U(1)_H$ acts by isometries of X, induced by

$$(X,Y) \to (e^{i\theta/2}X, e^{i\theta/2}Y)$$

- It is a Hamiltonian isometry of ω.
- It transforms Ω with weight $+1$: $\Omega \to e^{i\theta}\Omega$.

Flavour Symmetries

The flavour symmetries T_H, T_C are also encoded in geometry and topology of X.

Under the assumptions in lecture 2:

1. The topological symmetry T_C is recovered from topology of X,

$$\mathfrak{t}_C = H_2(X, \mathbb{R}) \,.$$

2. The flavour symmetry G_H is the group of Hamiltonian isometries of X that leave invariant the holomorphic symplectic form Ω.

Example: Supersymmetric QED

Recall that for supersymmetric QED, $X = T^*\mathbb{CP}^{w-1}$

- Topological symmetry: $H_2(X,\mathbb{R}) = \mathbb{R}$ generated by the class $[\mathbb{CP}^{w-1}]$, in agreement with $T_H = U(1)$.
- Hamiltonian isometries preserving the holomorphic symplectic form on $X = T^*\mathbb{CP}^{w-1}$ are precisely $G_H = PSU(w)$.

Mass Parameters

Let us now re-introduce the mass parameters $m \in \mathfrak{t}_H$.

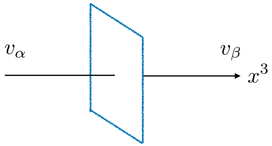
From the perspective of a sigma model to X, this is a deformation by a real superpotential

$$h_m = m \cdot \mu_{H,\mathbb{R}}$$

- $h_m : X \to \mathbb{R}$ is the moment map for the $U(1)_m \subset T_H$ Hamiltonian isometry generated by m.
- Induces a potential proportional to $|dh_m|^2$.
- The supersymmetric vacua are now critical points of h.
- They are equivalently fixed points of the $U(1)_m$ action.

Domain Walls and Hyperplanes

If m is generic the critical locus consists of isolated points $v_\alpha \in X$.



- We can again study 2d $\mathcal{N} = (2,2)$ domain walls.
- They are now solutions to gradient flow for h_m on X.
- The tension is $|h_m(\alpha) - h_m(\beta)|$.
- The critical locus jumps when $\{h_m(\alpha) - h_m(\beta)\} = 0$.
- This reproduces gauge theory answer: $h(\alpha) = h_m(\alpha)$.

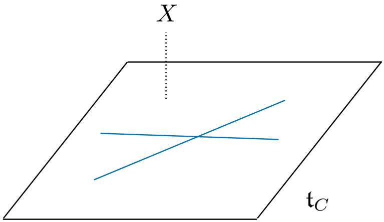
Coulomb Branch $X^!$

Coulomb Branch

To access the Coulomb branch,

- $\zeta = 0$
- m generic

Generic now means in the complement of hyperplanes $W_{\alpha\beta}$ in $\mathfrak{t}_H$.



Classical Description

The equations for classical supersymmetric vacua become:

$$\mu_{\mathbb{R}} = 0 \qquad \mu_{\mathbb{C}} = 0$$
$$(\sigma \cdot J + m \cdot J_H) \cdot X = 0 \qquad (\varphi \cdot J) \cdot X = 0$$
$$(\sigma \cdot J + m \cdot J_H) \cdot Y = 0 \qquad (\varphi \cdot J) \cdot Y = 0$$
$$[\varphi, \varphi^\dagger] = 0 \qquad [\sigma, \varphi] = 0$$

Under our assumptions, for generic m:

- Only solutions have $X, Y = 0$.
- σ, φ in a common Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$.
- For generic σ, φ, gauge symmetry G is broken to maximal torus T.

Classical Description

What about the gauge connection A?

- If the gauge symmetry is broken to a maximal torus $T \subset G$, the connection can be dualised to a *periodic* scalar field γ called the dual photon,

$$*dA_a = d\gamma_a \qquad a = 1, \ldots, \mathrm{rk}(G)\,.$$

- Near generic values of the vectormultiplet scalars, the Coulomb branch therefore looks like $(\mathbb{R}^3 \times S^1)^{\mathrm{rk}(G)}$, parametrised by $(\sigma_a, \varphi_a, \gamma_a)$.
- This classical picture receives dramatic quantum corrections!

Quantum Corrections

The Coulomb branch $X^!$ receives quantum corrections.

- When G is abelian, there are only 1-loop quantum corrections, which were completely understood 25 years ago [1].
- When G is non-abelian, there are also non-perturbative corrections.
- Many predictions from mirror symmetry and brane constructions.
- There has been much recent progress using tools from algebraic geometry [2] - see lectures of Hanany and Nakajima.

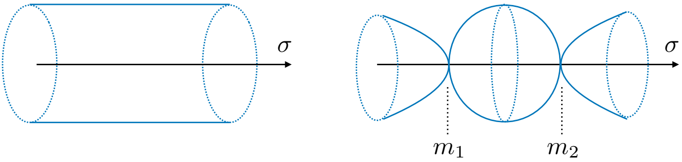
[1] Seiberg, Seiberg-Witten, Seiberg-Intriligator

[2] Cremonesi-Hanany-Zaffaroni, Nakajima, Nakajima-Braverman-Finkelberg, MB-Dimofte-Gaiotto, and many others

Example: Supersymmetric QED

In supersymmetric QED, $G = U(1)$.

- The classical result would be $\mathbb{R}^3 \times S^1$.
- In fact, it is the resolution of A-type singularity, $X = \widetilde{\mathbb{C}^2/\mathbb{Z}_w}$.
- The classical and quantum cross-section $\varphi = 0$ are illustrated below for $w = 2$.



General Structure

The general structure of $X^!$ is the same as X with the R-symmetries and flavour symmetries interchanged.

- For generic m, it is a smooth holomorphic symplectic manifold of complex dimension
$$\text{dim}_{\mathbb{C}} X^! = 2\text{rk}(G)\,.$$
- Becomes holomorphic symplectic cone as $m \to 0$.
- Flavour symmetries:
  - $\mathfrak{t}_H = H_2(X, \mathbb{R})$.
  - $\mathfrak{t}_C =$ Hamiltonian isometries of $X^!$ preserving holomorphic symplectic form.

Towards Symplectic Duality

Comparisons

To a supersymmetric gauge theory, we have assigned two spaces X, $X^!$.

Features of the theory can be understood from the perspective of both.

For example, flavour symmetries

- $\mathfrak{t}_H = \{$ Hamiltonian isometries of $X\} = H_2(X^!, \mathbb{R})$
- $\mathfrak{t}_H = \{$ Hamiltonian isometries of $X^!\} = H_2(X, \mathbb{R})$

and supersymmetric vacua

- Vacua $v_\alpha = \{T_H$-fixed points of $X\} = \{T_C$-fixed points of $X^!\}$.
- Supersymmetric CS levels / vector central charge:

$$h(\alpha) = h_m(\alpha) = h^!_t(\alpha)$$

Symplectic Duality

Pairs of holomorphic symplectic spaces X, $X^!$ with these properties are known as symplectic dual [3].

- There are non-trivial equivalences between mathematical structures associated to X, $X^!$.
- The idea is to find observables in a supersymmetric gauge theory that can be equivalently described in sigma models to X and $X^!$.
- Important implications in representation theory.

[3] Braverman-Licata-Proudfoot-Webster

Mirror Symmetry

Mirror Symmetry

A related but distinct phenomenon is 3d mirror symmetry.

This is an infrared duality:

- Two theories $\mathcal{T}$, $\mathcal{T}^\vee$ that flow to the same superconformal fixed point with flavour and R-symmetries exchanged,

$$SU(2)_H \leftrightarrow SU(2)_C \qquad G_H \leftrightarrow G_C \, .$$

- This exchanges the Higgs and Coulomb branch

$$X(\mathcal{T}) \leftrightarrow X^!(\mathcal{T}^\vee) \qquad X(\mathcal{T}^\vee) \leftrightarrow X^!(\mathcal{T})$$

Brane constructions provide a plethora of mirror pairs!