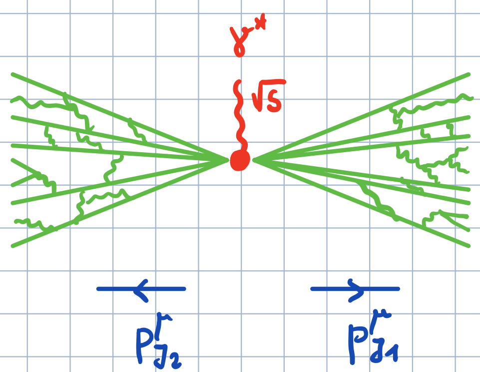
III. Construction of Soft-Collinear Effective Theory

A long-standing problem in QCD is how to systematically account for long-distance effects (including power corrections) in processes which do not admit a local OPE.

→ OPE provides rigorous framework for an expansion in powers and logarithms of the large scale for euclidean processes

→ processes involving energetic light particles pose new challenges: p^μ has some large components, but $p^2 \approx 0$

Consider $e^+e^- \to 2$ jets:



highly collimated jets of particles: large energy along jet axis, small invariant mass

$$p^\mu_{J_1} = \left(E_1,\, 0,\, 0,\, \sqrt{E_1^2 - m_{J_1}^2}\,\right)$$
$$p^\mu_{J_2} = \left(E_2,\, 0,\, 0,\, -\sqrt{E_2^2 - m_{J_2}^2}\,\right) \quad ; \quad E_i \approx \frac{\sqrt{s}}{2}, \quad m_{J_i}^2 \ll s$$

↳ intrinsically Minkowskian process

↳ what is there to integrate out ?

Introduce small expansion parameter:

$$\boxed{\lambda \sim \frac{m_J}{Q} \ll 1} \qquad (Q = \sqrt{s})$$

Define two light-like reference vectors along jet directions:

$$n^\mu = (1,0,0,1)\,, \quad \bar{n}^\mu = (1,0,0,-1)$$

$$n^2 = 0\,, \quad \bar{n}^2 = 0\,, \quad n\cdot\bar{n} = 2$$

Decompose 4-vectors in a <u>light-cone basis</u> spanned by n^μ, $\bar{n}^\mu$ and two perpendicular directions:

$$\boxed{p^\mu = (n\cdot p)\,\frac{\bar{n}^\mu}{2} + (\bar{n}\cdot p)\,\frac{n^\mu}{2} + p_\perp^\mu}$$

$$\Rightarrow \begin{cases} n\cdot p_{J_1} = E_1 - \sqrt{E_1^2 - m_{J_1}^2} \simeq \dfrac{m_{J_1}^2}{2E_1} \simeq \dfrac{m_{J_1}^2}{\sqrt{s}} \sim \lambda^2\, Q \\ \bar{n}\cdot p_{J_1} = E_1 + \sqrt{E_1^2 - m_{J_1}^2} \simeq 2E_1 \simeq \sqrt{s} = 1\cdot Q \\ p_{J_1}^\perp = 0 \end{cases}$$

Similarly: $\quad n\cdot p_{J_2} \sim 1\cdot Q\,, \quad \bar{n}\cdot p_{J_2} \sim \lambda^2\, Q\,, \quad p_{J_2}^\perp = 0$

Individual partons inside the jets can carry momenta with the same scaling rules, but these can also have transverse components as long as $p_\perp^2 < m_{J_i}^2$. We thus find:

partons inside jet 1: $(n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (\lambda^2, 1, \lambda)\, Q$

↳ collinear (n-collinear) particles

partons inside jet 2: $(n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (1, \lambda^2, \lambda)\, Q$

↳ anti-collinear ($\bar{n}$-collinear) particles

↳ note: $p_i^2 = (n\cdot p_i)(\bar{n}\cdot p_i) + p_{\perp,i}^2 \sim \lambda^2 Q^2$ (both cases)

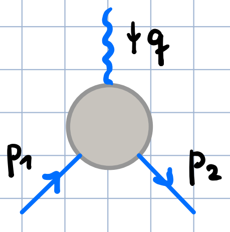
These collinear particles have virtualities much less than the hard scale $Q^2 = s$. But in virtual diagrams we can also exchange hard particles with:

$$p_i^\mu \sim Q\,, \text{ i.e. } \quad (n\cdot p_i, \bar{n}\cdot p_i, p_i^\perp) \sim (1,1,1)\, Q$$

↳ hard particles

We could try to integrate out these hard quantum fluctuations and construct an EFT built out of collinear and anti-collinear particles. However, one finds that this is not the whole story.

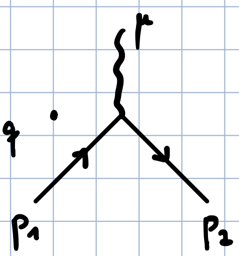
It is instructive at this point to consider a concrete example, the (off-shell) Sudakov form factor:



$|q^2| \gg |p_i^2|$

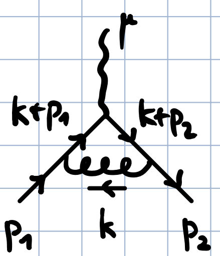
$(m_i = 0)$

One-loop calculation:

$Z_q \cdot$ 

$= Z_q \; \bar{u}(p_1)\,\gamma^\mu\, u(p_2)$; $\quad Z_q = 1 + \mathcal{O}(\alpha_s)$

WFR $\qquad$ (off-shell WFR factor)



from coupling renormalization

$$= -i\, C_F\, g_s^2\, \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D}\, \frac{\bar{u}(p_1)\,\gamma^\alpha (\not{k}+\not{p}_2)\,\gamma^\mu (\not{k}+\not{p}_1)\,\gamma_\alpha\, u(p_1)}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[(k+p_2)^2+i0\right]}$$

$\boxed{D = 4-2\epsilon}$

use Feynman gauge $\xi = 1$

For our purposes, it suffices to focus on the scalar loop integral:

$$I = i\,\pi^{-D/2}\,\mu^{2\epsilon} \int d^D k\; \frac{2\,p_1\cdot p_2}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[(k+p_2)^2+i0\right]}$$

$$= \underbrace{\ln\frac{Q^2}{P_1^2}\,\ln\frac{Q^2}{P_2^2}}_{\text{Sudakov double Log}} + \frac{\pi^2}{3} + \mathcal{O}\!\left(\frac{P_i^2}{Q^2}\right) \qquad \text{finite for } \epsilon \to 0$$

with:

$$Q^2 = -q^2 - i0 = -s - i0 < 0 \quad \to \text{ non-zero imag. part}$$

$$P_i^2 = -p_i^2 - i0 > 0 \quad (\text{off-shell} \to \text{IR regulators})$$

↪ note that $Q^2 = -(p_2-p_1)^2 \simeq 2\,p_1\cdot p_2$ up to $\mathcal{O}(\lambda^2)$ terms

In the CMS:

$$(n\cdot q,\ \bar{n}\cdot q,\ q_\perp) = (1, 1, 0)\,\sqrt{s} \qquad \text{hard}$$

$$(n\cdot p_1,\ \bar{n}\cdot p_1,\ p_1^\perp) \sim (\lambda^2, 1, 0)\,\sqrt{s} \qquad \text{collinear}$$

$$(n\cdot p_2,\ \bar{n}\cdot p_2,\ p_2^\perp) \sim (1, \lambda^2, 0)\,\sqrt{s} \qquad \text{anti-collinear}$$

We will now decompose this result into a sum of contributions each depending on only a single scale. The hard contribution (depending on Q^2) arises from hard quantum fluctuations. Contributions from lower scales will later be associated with modes in the EFT.

Method of regions: (Smirnov 1990s)

Systematic method for performing "Taylor expansions" of Feynman graphs F_Γ by decomposing them into "regions":

$$F_\Gamma \sim \sum_\gamma M_\gamma \, F_\Gamma$$

Taylor expansion in variables that are small in γ

sum over sets γ of subgraphs

Practical procedure (roughly):

1. determine large and small scales in the graph

2. introduce factorization scales μ_i and divide the loop integrals into regions in which each loop momentum is related to one of these scales

3. perform a Taylor expansion in parameters that are small in a given region

4. after the expansion, ignore the factorization scales and integrate over the entire loop integration domain in each region

For step 4 to be valid, it is essential that F_Γ is defined in <u>dimensional regularization</u>!

↳ follows from vanishing of scaleless integrals:

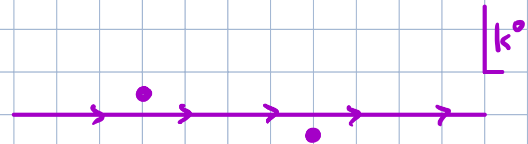
$$\boxed{\int d^D k \, \frac{1}{(k^2)^a} \equiv 0}$$

<u>Comment:</u>

A more rigorous treatment uses the notion of "singular surfaces":

- loop integrals (in dim. reg.) are contour integrals in the complex momentum plane

- non-zero contributions result from pinch singularities, where contours cannot be deformed so as to avoid the poles:



- this happens when some propagators go on shell (happens only when some loop momenta match external scales)

- other, off-shell propagators can be expanded about the singular surfaces (no need for hard cutoffs)
- graph F_Γ is then equal to the sum of all singular subgraphs

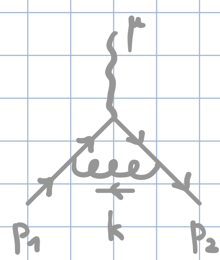
(see e.g. Beneke, Smirnov 1997)

We will see how this procedure works in concrete examples. Every region we identify (hard, collinear, anti-collinear, ...) will be associated with either a Wilson coefficient (hard region) or a field in the low-energy EFT.

Region analysis of the Sudakov form factor:

We now decompose the scalar one-loop integral

$$I = i\pi^{-D/2}\mu^{2\epsilon}\int d^Dk \, \frac{2p_1\cdot p_2}{(k^2+i0)\left[(k+p_1)^2+i0\right]\left[(k+p_2)^2+i0\right]}$$



into regions where the loop momentum k^μ is hard, collinear, and anti-collinear.

a) hard region: $k^\mu \sim (1,1,1)\,Q$

Expansions of propagators:

$$(k+p_1)^2 = k^2 + 2k\cdot p_1 + p_1^2$$

$$= \underset{1}{k^2} + \underset{1}{2k}\cdot\left(\underset{\lambda^2}{(n\cdot p_1)}\frac{\bar{n}}{2} + \underset{1}{\bar{n}\cdot p_1}\frac{n}{2} + \underset{\lambda}{p_1^\perp}\right) + \underset{\lambda^2}{p_1^2}$$

$$= k^2 + (n\cdot k)(\bar{n}\cdot p_1) + \mathcal{O}(\lambda) \simeq (k+p_{1-})^2$$

drop

$$(k+p_2)^2 = k^2 + (\bar{n}\cdot k)(n\cdot p_2) + \mathcal{O}(\lambda) \simeq (k+p_{2+})^2$$

with:

$$p_{1-}^\mu = (\bar{n}\cdot p_1)\frac{n^\mu}{2}\,, \qquad p_{2+}^\mu = (n\cdot p_2)\frac{\bar{n}^\mu}{2} \qquad \text{(null vectors)}$$

This gives:

Taylor expansion of $2p_1\cdot p_2$

$$\boxed{\begin{aligned} I_h &= i\,\pi^{-D/2}\mu^{2\epsilon}\int d^Dk\,\frac{2p_{1-}\cdot p_{2+}}{(k^2+i0)\left[(k+p_{1-})^2+i0\right]\left[(k+p_{2+})^2+i0\right]} \\ &= \Gamma(1+\epsilon)\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right] \end{aligned}}$$

↪ appearance of double and single poles in ϵ (IR divergences, since integral is UV-finite)

↪ result depends on hard scale Q^2 only (and on the factorization scale μ)