Lecture #3

Setting: $E/\mathbb{Q}$ elliptic curve, cond $= N$

$p \nmid 2N$ Eisenstein prime:

$$E[p]^{ss} \cong \mathbb{F}_p(\phi) \oplus \mathbb{F}_p(\psi) \quad \text{as } G_{\mathbb{Q}}\text{-modules,}$$

with $\underset{\substack{\| \\ \omega\phi^{-1}}}{\phi, \psi} : G_{\mathbb{Q}} \longrightarrow \mathbb{F}_p^\times$

Last time:

Theorem A (C.–Grossi–Skinner)

Suppose $\phi|_{G_{\mathbb{Q}_p}} \neq \mathbb{1}, \omega$.

Let $K/\mathbb{Q}$ imaginary quadratic field s.t.:

- Heegner hyp. holds.
- $p\mathcal{O}_K = v\bar{v}$ splits in K.

Then $X_v(E/K_\infty^-)$ is Λ^--torsion, with

$$\mathrm{char}_{\tilde{\Lambda}^-}\left(X_v(E/K_\infty^-)\right) = \left(\mathcal{L}_v^{BDP}(E/K)^2\right).$$

(Here $\Lambda^- := \mathbb{Z}_p[\![\Gamma^-]\!] \subset \tilde{\Lambda}^- := \mathbb{Z}_p^{ur}[\![\Gamma^-]\!]$ anti-cyclotomic Iwasawa algebras)

Today's goal

Theorem B (C.–Grossi–Skinner).

Suppose $\phi|_{G_{\mathbb{Q}_p}} \neq \mathbb{1}, \omega$.

Then $X(E/\mathbb{Q}_\infty)$ is Λ-torsion, with

$$\mathrm{char}_\Lambda X(E/\mathbb{Q}_\infty) = (\mathcal{L}_p(E/\mathbb{Q})).$$

Hence Mazur's Main Conj. holds.

Previous results on Mazur's MC for Eisenstein primes:

- Rubin : proof in the CM case.
- Kato : $X(E/\mathbb{Q}_\infty)$ is Λ-torsion, with

$$\mathrm{char}_\Lambda X(E/\mathbb{Q}_\infty) \supset (\mathcal{L}_p(E/\mathbb{Q})) \text{ in } \Lambda[\tfrac{1}{p}].$$

- Wüthrich : $\mathcal{L}_p(E/\mathbb{Q}) \in \Lambda$ & above divisibility holds in Λ.

- Greenberg–Vatsal: proof for "half" of the cases:

$$\phi = \text{either} \begin{cases} \text{unr at } p \ \& \text{ odd, or} \\ \text{ramif. at } p \ \& \text{ even.} \end{cases}$$

$$\left(\underset{\substack{\text{Mazur-Wiles} \\ + \text{ Ferrero-Washington}}}{\overset{}{\Longrightarrow}} \quad \mu\big(X(E/\mathbb{Q}_\infty)\big) = \mu\big(\mathcal{L}_p(E/\mathbb{Q})\big) = 0 \right).$$

<u>Remark</u>. Our Thm. B also gives a new proof of Greenberg–Vatsal's result.

§2. Comparing Iwasawa invariants

After Kato's divisibility, to prove Mazur's MC enough to show

$X(E/\mathbb{Q}_\infty)$ & $\mathcal{L}_p(E/\mathbb{Q})$ have the same μ & λ invariants.

The Greenberg-Vatsal method can show this provided $\mu=0$,

but in general $\mu>0$.

Greenberg: If ϕ is ramif. at p & odd, (or unr at p & even)

then $\mu(X(E/\mathbb{Q}_\infty))>0$.

In fact, if $\exists\, C\subset E[p^\infty]$ cyclic, $\#C=p^m$ with $G_{\mathbb{Q}}$-action ramif. at p & odd

then $\mu(X(E/\mathbb{Q}_\infty))\geqslant m$.

Expectaction: $\mu =$ max'l such m.

Stevens: similar results/expectation for $\mu(\mathcal{L}_p(E/\mathbb{Q}))$.

We'll use a different method.

Choose $\alpha : \Gamma^- \to R^\times$ anti-cyclotomic character,

$\alpha \equiv 1 \pmod{\varpi^M}$, $M \gg 0$

s.t. $\mathcal{L}_v^{BDP}(E(\alpha)/K)(0) \neq 0$ (Rec. Know $\mathcal{L}_v^{BDP}(E/K) \neq 0 \in \tilde{\Lambda}^-$)

<u>Theorem</u> (refinement of $\left.\begin{matrix}\text{Lei}\\ \text{Kings}\end{matrix}\right\}$-Loeffler-Zerbes)

(Burungale-Skinner-Tian, Bertolini-Darmon-Venerucci & C.-Grossi-Skinner.

Let $E_\bullet :=$ elliptic curve in the isogeny class of E s.t.

$$\begin{array}{ccc} H^1_{\acute{e}t}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(1)) & \twoheadrightarrow & V_f \\ \cup & & \cup \\ H^1_{\acute{e}t}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \twoheadrightarrow & T_f \cong T_p E_\bullet \end{array}$$

max'l quotient where $T_\ell = a_\ell(E)$ $\forall \ell$

shown by Wüthrich

Then $\exists$ 2-variable Beilinson–Flach class

$$BF_\alpha \in H^1_{Iw}(K_\infty, T_pE_\bullet(\alpha)).$$

$$\left.\begin{tikzcd}[column sep=small]
& K_\infty \arrow[dl, dash] \arrow[dr, dash] & \\
K_\infty^+ = K\mathbb{Q}_\infty \arrow[dr, dash, "\Gamma"'] & & K_\infty^- \arrow[dl, dash, "\Gamma^-"] \\
& K &
\end{tikzcd}\right\} \Gamma_K \cong \mathbb{Z}_p^2$$

together with two <u>explicit reciprocity laws</u>:

$$\begin{tikzcd}
\text{(ERL 1)} \quad \mathrm{res}_v(BF_\alpha) \arrow[r, leftrightarrow] & \mathcal{L}_p^{PR}(E_\bullet(\alpha)/K) \arrow[d, mapsto, "\text{for } \alpha = \mathbb{1}"'] & \in \mathbb{Z}_p[[\Gamma_K]] \arrow[d, two heads] \\
& \mathcal{L}_p^{PR}(E_\bullet/K)^+ \underset{\substack{\uparrow \\ \text{up to } \Lambda^\times}}{=} \mathcal{L}_p(E_\bullet/\mathbb{Q}) \cdot \mathcal{L}_p(E_\bullet^K/\mathbb{Q}) & \in \Lambda
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
(\mathrm{ERL2}) \quad \mathrm{res}_{\bar{v}}(\mathrm{BF}_\alpha) \arrow[r, leftrightarrow] & \mathcal{L}_v^{\mathrm{Gr}}(E_\bullet(\alpha)/K) \arrow[d, mapsto] & \in & \mathbb{Z}_p^{\mathrm{ur}}[[\Gamma_K]] \arrow[d, two heads] \\
& \mathcal{L}_v^{\mathrm{Gr}}(E_\bullet(\alpha)/K)^- \underset{\substack{\uparrow \\ \text{up to } (\tilde{\Lambda}^-)^\times}}{=} \mathcal{L}_v^{\mathrm{BDP}}(E_\bullet(\alpha)/K)^2 & \in & \tilde{\Lambda}^-
\end{tikzcd}$$

<u>Key ingredient</u>. Beilinson-Flach classes over $\mathbb{Q}_\infty$ assoc. to $f \otimes g$ ← canonical CM Hida family with $g_1 = \mathrm{Eis}_1(\mathbb{1}, \eta_{K/\mathbb{Q}})$.

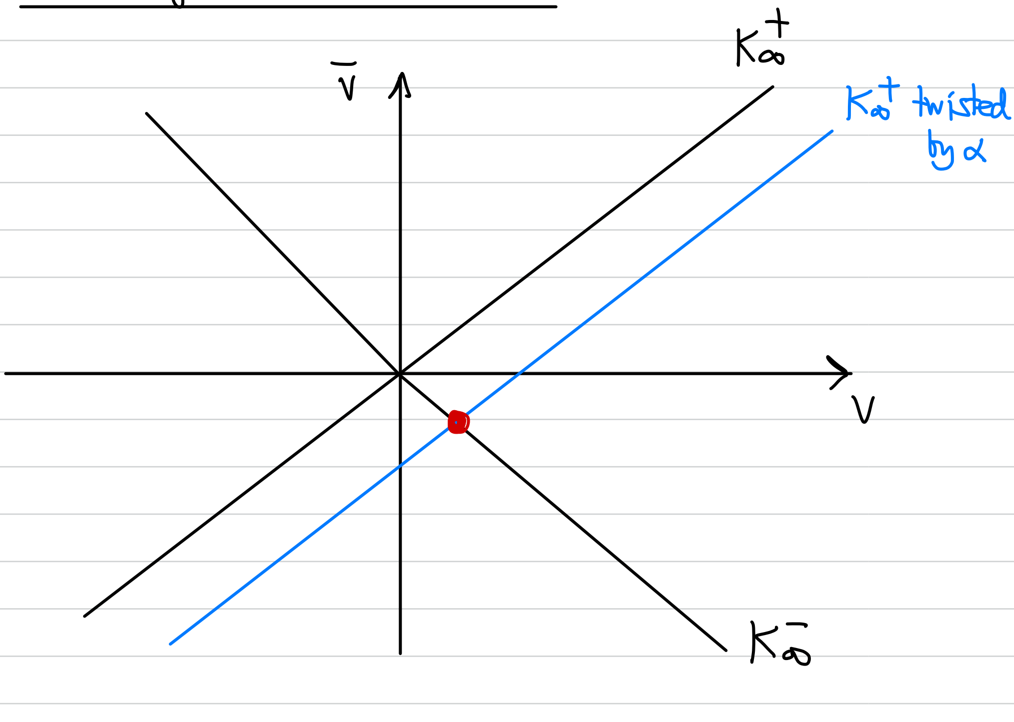
↑ quadr. character assoc. to $K/\mathbb{Q}$

In particular, this gives an <u>equivalence</u> between 3 different cyclotomic Main Conj.:

$$\begin{array}{ccc} & \text{IMC for } BF_\alpha^+ & \\ \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow \\ \text{IMC for } X(E_0(\alpha)/K_\infty^+) & & \text{IMC for } X_v(E_0(\alpha)/K_\infty^+) \end{array}$$

proj. of BF_α to $H^1_{Iw}(K_\infty^+, T_pE_0(\alpha))$

§3. Sketch of Thm A $\Rightarrow$ Thm B



By Euler system argument for BF_α^+:

$$\underbrace{\mathrm{char}_\Lambda X(E_\bullet(\alpha)/K_\infty^+)}_{\substack{!!\\ (\mathcal{F}_\alpha)}} \supseteq \left(\mathcal{L}_p^{PR}(E_\bullet(\alpha)/K)^+\right) \text{ in } \Lambda.$$

while Thm. A twisted by α and specialized at $T=0$ + ER1-2 gives

$$\mathcal{F}_\alpha(0) \sim_p \mathcal{L}_p^{PR}(E_\bullet(\alpha)/K)(0) \neq 0$$

possible by Rohrlich + congruence argument

$$\Rightarrow (\mathcal{F}_\alpha) = \left(\mathcal{L}_p^{PR}(E_\bullet(\alpha)/K)\right)$$

$$\mathrm{char}_\Lambda X(E_\bullet/K_\infty^+) \equiv (\mathcal{F}_\alpha) \pmod{\varpi^M}, \qquad \left(\mathcal{L}_p^{PR}(E_\bullet(\alpha)/K)\right) \equiv \mathcal{L}_p^{PR}(E_\bullet/K) \pmod{\varpi^M}.$$

using $\alpha \equiv 1 \pmod{\varpi^M}$

$$\Longrightarrow \mathrm{char}_\Lambda X(E_\bullet/K_\infty^+) = \left(\mathcal{L}_p^{PR}(E_\bullet/K)^+\right)$$

+ divisibility with $\alpha = \mathbb{1}$

$$\Longrightarrow \mathrm{char}_\Lambda X(E_\bullet/\mathbb{Q}_\infty) = \left(\mathcal{L}_p(E_\bullet/\mathbb{Q})\right)$$

use Kato to separate $E/\mathbb{Q}$ & $E^K/\mathbb{Q}$

$\Longrightarrow$ Mazur's MC holds for our given E $\boxtimes$

invariance under isogenies