A Mathematical Introduction to 3d $\mathcal{N} = 4$ Gauge Theory II

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Last Time

We consider a 3d $\mathcal{N} = 4$ supersymmetric gauge theory:

1. A compact Lie group G
2. A unitary representation $\rho : G \to \mathsf{U}(n)$ acting on $R = \mathbb{C}^n$.

Global symmetries:

- R-symmetry $SU(2)_H \times SU(2)_C$
- Flavour symmetry $G_H \times G_C$

$$G_H = N_{\mathsf{U}(n)}(\rho(G))/\rho(G)$$

$$G_C \supset T_C = \mathsf{Hom}(\pi_1(G), U(1))$$

Central Extensions

The super-Poincaré algebra admits central extensions.

Additional generators that commuting with P_μ, $J^\mu{}_\nu$, $Q_\alpha^{A\dot{A}}$.

The supersymmetry algebra is deformed to

$$\begin{aligned}\{Q_\alpha^{A\dot{A}}, Q_\beta^{B\dot{B}}\} &= \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}(\sigma^\mu)_{\alpha\beta}P_\mu \\ &+ \epsilon_{\alpha\beta}\epsilon^{AB}Z^{\dot{A}\dot{B}} + \epsilon_{\alpha\beta}\epsilon^{\dot{A}\dot{B}}Z^{AB} \\ &+ (\sigma^\mu)_{\alpha\beta}Z_\mu^{AB,\dot{A}\dot{B}}\,.\end{aligned}$$

- Z^{AB}, $Z^{\dot{A}\dot{B}}$ are scalar central charges associated to flavour symmetries.
- $Z_\mu^{AB,\dot{A}\dot{B}}$ is a 1-form central charge associated to domain walls.

Scalar Central Charges

Let us first consider the scalar central charges.

The R-symmetry indices AB, $\dot{A}\dot{B}$ must be symmetric:

- Z^{AB} transforms in the adjoint representation $\mathbf{3}$ of $SU(2)_H$.
- $Z^{\dot{A}\dot{B}}$ transforms in the adjoint representation $\mathbf{3}$ of $SU(2)_C$.

Natural candidates are the generators J_H, J_C of the flavour symmetries, which by definition commute with all P_μ, $J^\mu{}_\nu$, $Q_\alpha^{A\dot{A}}$.

Let us assume they are proportional...

Mass and FI Parameters

The scalar central charges are

$$Z^{AB} = \zeta^{AB} \cdot J_C \qquad Z^{\dot{A}\dot{B}} = m^{\dot{A}\dot{B}} \cdot J_H \,.$$

The constants of proportionality are called mass and FI parameters:

- FI parameters ζ^{AB}
  - Valued in $\mathfrak{t}_C :=$ Lie(T_C)
  - Adjoint representation $\mathbf{3}$ of SU$(2)_H$.
- Mass parameters $m^{\dot{A}\dot{B}}$:
  - Valued in $\mathfrak{t}_H :=$ Lie(T_H)
  - Adjoint representation $\mathbf{3}$ of SU$(2)_C$.

Mass and FI Parameters

These parameters correspond to deformations of the Lagrangian:

- The mass parameters $m^{\dot{A}\dot{B}}$ are vacuum expectation values for scalars $\sigma^{\dot{A}\dot{B}}$ in a vectormultiplet for G_H.
- The FI parameters t^{AB} are vacuum expectation values for scalars σ^{AB} in a twisted vectormultiplet for G_C.

A twisted vectormultiplet is a vectormultiplet with

$$SU(2)_H \leftrightarrow SU(2)_C \, .$$

Many physical and mathematical quantities vary in an interesting way with $m^{\dot{A}\dot{B}}$, t^{AB}...

A Simplification

In what follows, I set

$$\zeta^{11} = 0 \qquad m^{1\dot{1}} = 0$$

and define

$$\zeta := \zeta^{12} \qquad m := m^{1\dot{2}}\,.$$

- This selects a maximal torus $U(1)_H \times U(1)_C$ of the R-symmetry commuting with m, ζ.
- If m, ζ are generic, they will break the flavour symmetry $G_H \times G_C$ to a maximal torus $T_H \times T_C$.

A Simplification

Having broken the R-symmetry to $U(1)_H \times U(1)_C$, we decompose the vectormultiplet and hypermultiplet fields accordingly.

- The vectormultiplet scalars decompose into real and complex components

$$\sigma := \sigma^{1\dot{2}} \qquad \varphi := \sigma^{11}$$

- The hypermultiplet decompose into complex scalars (X, Y)

	G	$\mathrm{U}(1)_H$	$\mathrm{U}(1)_C$
σ	Adj	0	0
φ	Adj	0	1
X	R	$\frac{1}{2}$	0
Y	R^*	$\frac{1}{2}$	0

Supersymmetric Vacua

A fundamental question is to determine the set or moduli space of supersymmetric vacuum states on $\mathbb{R}^3$ and how they depend on the parameters m, ζ.

- Classically, these are configurations of the vectormultiplet fields (A, σ, φ) and hypermultiplet fields (X, Y) preserving all supersymmetries.
- They can be determined by minimizing the classical potential energy or euclidean action.
- Depending on the parameters m, ζ, there may be perturbative and non-perturbative quantum corrections.

Supersymmetric Vacua

The result is that classical supersymmetric vacua are solutions of

$$\mu_{\mathbb{R}} - \zeta = 0 \qquad \mu_{\mathbb{C}} = 0$$
$$(\sigma \cdot J + m \cdot J_H) \cdot X = 0 \qquad (\varphi \cdot J) \cdot X = 0$$
$$(\sigma \cdot J + m \cdot J_H) \cdot Y = 0 \qquad (\varphi \cdot J) \cdot Y = 0$$
$$[\varphi, \varphi^\dagger] = 0 \qquad [\sigma, \varphi] = 0$$

- The top line introduces real and complex moment maps

$$\mu_{\mathbb{R}} : T^*R \to \mathfrak{g}^* \qquad \mu_{\mathbb{C}} : T^*R \to \mathfrak{g}^* \otimes \mathbb{C} \, .$$

- J, J_H denote the generators of G, G_H.
- $\zeta \in \mathfrak{t}_C$ is identified an element of $\mathfrak{g}^*$ through $\mathfrak{t}_C \cong Z(\mathfrak{g}^*) \subset \mathfrak{g}^*$.

Moment Maps

Recall that (X, Y) parametrise a complex vector space $T^*R = R \oplus R^*$.

It has Kähler and holomorphic symplectic forms

$$\omega = dX \wedge d\bar{X} + dY \wedge d\bar{Y} \qquad \Omega = dX \wedge dY \ ,$$

where the natural pairing $R \times R^* \to \mathbb{C}$ is understood.

- $\mu_{\mathbb{R}} : T^*R \to \mathfrak{g}^*$ is the G-moment map associated to ω.
- $\mu_{\mathbb{C}} : T^*R \to \mathfrak{g}^* \otimes \mathbb{C}$ is the G-moment map associated to Ω.

Schematically,

$$\mu_{\mathbb{R}} = \bar{X} J X - Y J \bar{Y} \qquad \mu_{\mathbb{C}} = Y J X$$

Superpotentials

The vacuum equations are the critical points of real and complex superpotentials

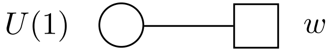
$$h = \sigma \cdot (\mu_{\mathbb{R}} - \zeta) + m \cdot \mu_{H,\mathbb{R}} \qquad W = \varphi \cdot \mu_{\mathbb{C}} \,,$$

with respect to variations of σ, φ, X, Y.

- $\mu_{H,\mathbb{R}} : T^*R \to \mathfrak{g}_H^*$ denotes the real moment map for the G_H action on T^*R.
- For simplicity, I assumed σ, φ have been simultaneously diagonalised to a common Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$
- W is the 3d $\mathcal{N} = 2$ superpotential.

Example

Let us consider the example of supersymmetric QED with w hypermultiplets of unit charge.



- This corresponds to $G = U(1)$ with the unitary representation $R = \mathbb{C}^n$ of weight $\rho = (1, 1, \ldots, 1)$.
- There are neutral vectormultiplet fields σ, φ and hypermultiplet fields (X_j, Y_j) with $j = 1, \ldots, w$ of charge $(+1, -1)$.
- The flavour symmetries are $G_C = U(1)$ and $G_H = PSU(w)$.
- Correspondingly there is an FI parameter $\zeta \in \mathbb{R}$ and mass parameters $(m_1, \ldots, m_w) \in \mathbb{R}^{w-1}$ with $\sum_j m_j = 0$.

Example

The real and complex superpotentials are

$$h = \sum_{j=1}^{w} (\sigma - m_j)|X_j|^2 + \sum_{j=1}^{w} (-\sigma + m_j)|Y_j|^2 - \zeta\sigma$$

$$W = \varphi \sum_{j=1}^{w} X_j Y_j$$

and classic supersymmetric vacua are solutions to

$$\sum_{j=1}^{w} (|X_j|^2 - |Y_j|^2) - \zeta = 0 \qquad \sum_{j=1}^{w} X_j Y_j = 0$$

$$(\sigma - m_j)X_j = 0 \qquad \varphi \cdot X_j = 0$$

$$(-\sigma + m_j)Y_j = 0 \qquad \varphi \cdot Y_j = 0$$

Key Assumption

We make the following assumptions:

1. In the absence of m, ζ the theory flows to an interacting SCFT.
2. For generic m, ζ there are only isolated massive vacua v_α with G is completely broken.

Note that

- This is a constraint on the data (G, R).
- Intuitively, the representation R should be "large enough".
- This is the class of theories that appear most frequently in connection with geometric representation theory.

Example: Supersymmetric QED

Supersymmetric QED with $w > 1$ satisfied the assumption.

- There are a finite number of solutions with $G = U(1)$ completely broken provided $\zeta \neq 0$ and $m_i \neq m_j$ for $i \neq j$.
- Then there are w supersymmetric vacua v_i, $i = 1, \ldots, w$.
- If $\zeta > 0$, then

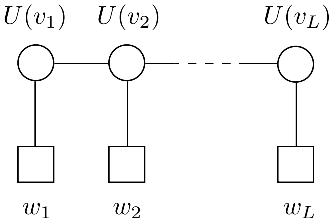
$$v_i \quad : \quad \sigma = m_i \qquad \varphi = 0 \qquad |X_j|^2 = \delta_{ij}\zeta \qquad Y_j = 0$$

- If $\zeta < 0$, then

$$v_i \quad : \quad \sigma = m_i \qquad \varphi = 0 \qquad X_j = 0 \qquad |Y_j|^2 = -\delta_{ij}\zeta$$

Example: quiver

Consider an A_L-type linear quiver.



- Let ω_j and e_j be the fundamental weights and simple roots of $\mathfrak{sl}_{L+1}$.
- $\lambda = \sum_j w_j \omega_j$ is a dominant weight of $\mathfrak{sl}_{L+1}$.
- $\mu = \lambda - \sum_j v_j e_j$ is a weight in the irrep. of highest weight λ.
- Theory satisfies our assumptions if μ is also a dominant weight.

Genericity

What does it mean precisely for m, ζ to be "generic"?

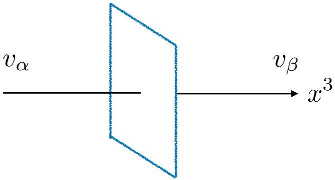
- Consider the parameter space $\mathfrak{t}_H \times \mathfrak{t}_C$ of m, ζ.
- There should be isolated massive vacua on the complement of co-dimension 1 loci in $\mathfrak{t}_H \times \mathfrak{t}_C$.
- On these loci, an extended moduli space of vacua may open up.

We want to determine these loci!

- One approach is to carefully analyse the vacuum equations.
- We'll follow a physical approach using domain walls tunnelling between between vacua.

Domain Walls

Consider a domain wall interpolating between vacua v_α and v_β.



- Domain walls can preserve a 2d $\mathcal{N} = (2,2)$ supersymmetry with vector R-symmetry $U(1)_H$ and axial R-symmetry $U(1)_C$.
- They are solutions of the gradient flow equations

$$\frac{\partial \Phi}{\partial x^3} = g^{\Phi\bar{\Phi}} \frac{\partial h}{\partial \bar{\Phi}}$$

  on $\mathfrak{t} \times T^*R$ parametrised by $\Phi = (\sigma, X, Y)$.
- Interpolate between critical points v_α at $x^3 \to -\infty$ and v_β at $x^3 \to +\infty$.

Domain Wall Tension

The tension $T_{\alpha,\beta}$ of a domain wall is the energy per unit area in the $x^{1,2}$-plane.

It is proportional to the difference between the value of the superpotential h at the two vacua,

$$T_{\alpha,\beta} = |\, h(\alpha) - h(\beta)\,| \,.$$

- This is nothing but the vector central charge $Z_3^{12,\dot{1}\dot{2}}$ in the presence of a domain wall interpolating between vacua α and β in the x^3-direction.

Supersymmetric CS Coupling

In a supersymmetric gauge theory, the values of the superpotential in a vacuum v_α take the form

$$h(\alpha) = \kappa_\alpha(t, m)$$

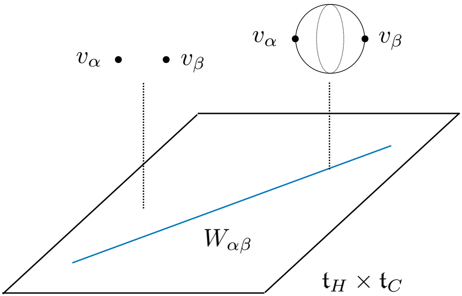
- Bi-linear pairing $\kappa_\alpha : \mathfrak{t}_H \times \mathfrak{t}_C \to \mathbb{R}$.
- Lifts to bi-linear pairing $\kappa_\alpha : \Gamma_H \times \Gamma_C \to \mathbb{Z}$ where Γ_H, Γ_C are the co-character lattices of T_H, T_C.
- This is the effective $\mathcal{N} = 4$ supersymmetric mixed Chern-Simons term between T_H and T_C in the vacuum α.

Walls

Here is the key argument:

- If there is a domain wall between supersymmetric vacua α and β with zero tension, then we can tunnel between them at no cost.
- This indicates a moduli space opening up connecting them.
- This happens along loci in $\mathfrak{t}_H \times \mathfrak{t}_C$,

$$W_{\alpha,\beta} = \{h(\alpha) = h(\beta)\} = \{\kappa_\alpha(m,\zeta) - \kappa_\beta(m,\zeta)\}$$



Walls

Recall that $\kappa_\alpha : \mathfrak{t}_H \times \mathfrak{t}_C \to \mathbb{R}$ is a bi-linear pairing.

- The loci $W_{\alpha,\beta}$ project onto linear hyperplanes through the origin in the vector spaces $\mathfrak{t}_H$, $\mathfrak{t}_C$.
- The connected components of the complement

$$\mathfrak{t}_H \times \mathfrak{t}_C - \bigcup_{\alpha,\beta} W_{\alpha,\beta}$$

  are called chambers.
- This forms a pair of "root systems" associated to each theory satisfying our assumption - first hint of geometric representation theory.

Example

Let us consider the example of supersymmetric QCD with $G = U(1)$ and w hypermultiplets in the fundamental representation.

- For generic parameters there are discrte vacua v_α, $\alpha = 1, \ldots, w$.
- The central charge function is given by

$$h(\alpha) = -\zeta m_\alpha$$

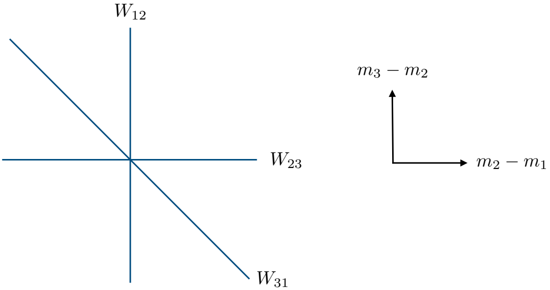
- Domain walls become tensionless when

$$W_{\alpha\beta} = \{\zeta(m_\alpha - m_\beta) = 0\}$$

- This reproduces the conditions $\zeta \neq 0$ and $m_i \neq m_j$ for $i \neq j$ for generic parameters.

Example

The projection of the loci $W_{\alpha,\beta}$ onto hyperplanes in the space of mass parameters $\mathfrak{t}_H \cong \mathbb{R}^{w-1}$ are shown below for $N = 3$.



- There are $w(w-1)/2$ independent hyperplanes with, say, $\alpha < \beta$.
- This is the root system of $\mathfrak{sl}(w)$, where $W_{\alpha\beta}$ corresponds to the root hyperplane associated to the positive root $e_\alpha - e_\beta$.

Next Time

Next time we examine supersymmetric vacua for non-generic mass and FI parameters.

- Higgs X and Coulomb branch $X^!$
- Geometric structures.
- How to recover flavour symmetries T_H, T_C from X, $X^!$.
- How to recover the hyperplane arrangements from X, $X^!$.