A short recap of the previous lecture I

✓ I] General Overview and applications of RMT

✓ II] Ensembles of RMT (Wigner vs rot. inv. ens.)

III] Coulomb gas approach

IV] Local statistics (bulk & edge)

Let $M = (M_{jk})$ be a $N \times N$ $\left\{ \begin{array}{l} \text{real sym. } (\beta=1) \\ \text{complex Hermitian } (\beta=2) \end{array} \right.$ matrix

$$\underbrace{P(\lambda_1, \ldots, \lambda_N)}_{\text{joint proba. density Function of the eigenvalues}} = B_N \, e^{-\beta \frac{N}{2} \sum_{i=1}^{N} \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

joint proba. density Function of the eigenvalues

$$\Rightarrow P(\lambda_1, \ldots, \lambda_N) = B_N \, e^{-\beta E(\{\lambda_i\})}$$

$$E(\{\lambda_i\}) = \frac{N}{2} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|$$

$\Rightarrow$ Dyson's log-gas

$\Rightarrow$ competition between confinement and the repulsive interactions

- Typical scale of λ_i's $\equiv \lambda_{typ}$

Potential energy : $\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2 \sim \frac{N}{2}\times\lambda_{typ}^2\times N$

$$\sim O(N^2 \times \lambda_{typ}^2)$$

suppose that $\frac{1}{2}\sum_{i\neq j}|\lambda_i-\lambda_j| \sim O(N\times N^2)$

Interaction energy : $\frac{1}{2}\sum_{i\neq j}\ln|\lambda_i-\lambda_j| = O\left(\frac{N(N-1)}{2}\right)$

$$= O(N^2)$$

$\Rightarrow$ balancing the two terms:

$$N^2\lambda_{typ}^2 \sim N^2 \Rightarrow \lambda_{typ} = O(1) = O(N^0)$$

• Empirical eigenvalue distribution:

$$\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

such that $\int_{-\infty}^{\infty} d\lambda \, \rho_N(\lambda) = 1, \ \forall N$.

For GOE, GUE

$$\lim_{N\to\infty} \rho_N(\lambda) = \langle \rho_N(\lambda) \rangle$$

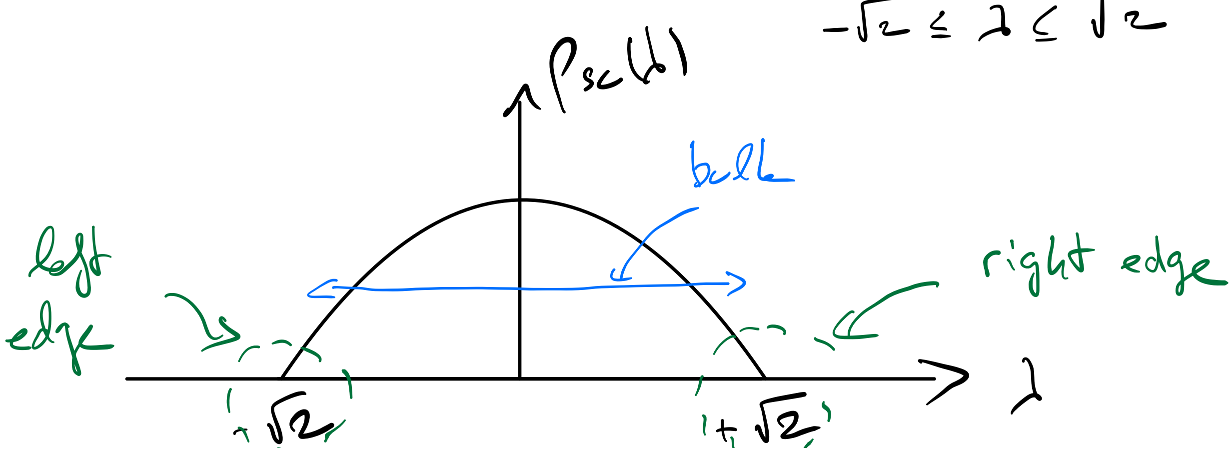
$\langle \dots \rangle$ w. respect to $P(\lambda_1, \dots, \lambda_N)$

self-averaging for λ in the bulk.

Wigner semi-circle:

$$\rho_N(\lambda) \xrightarrow[N\to\infty]{} \rho_{sc}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$

$$-\sqrt{2} \le \lambda \le \sqrt{2}$$



• Universality of the Wigner semi-circle:

* Wigner matrices: for independent entries w. $\langle M_{jk} \rangle = 0$, $\langle M_{jk}^2 \rangle < \infty$

then $\rho_N(\lambda) \xrightarrow[N\to\infty]{} \rho_{sc}(\lambda)$, + scale transformation

* For rot. invariant ensemble

$$P(\lambda_1, \dots, \lambda_N) = \tilde{B}_N \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N \sum_{i=1}^{N} V(\lambda_i)}$$

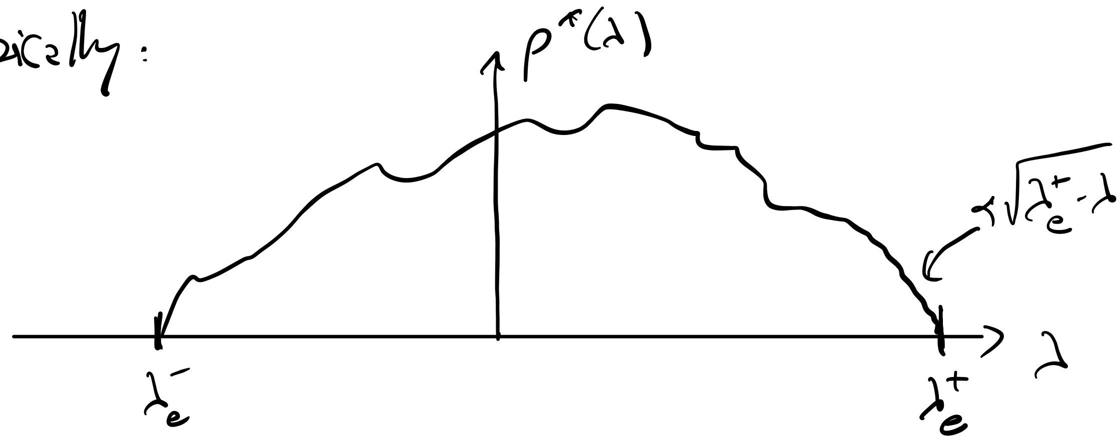
$$P(M) \propto e^{-N \operatorname{Tr} V(M)}$$

IF $\dfrac{V(\lambda)}{\ln \lambda} \xrightarrow[\lambda\to\infty]{} +\infty$

then $\rho_N(\lambda) \xrightarrow[N\to\infty]{} \rho^*(\lambda)$ which has a finite support.

But in gal $\rho^*(\lambda) \neq \rho_{sc}(\lambda)$

Typically:

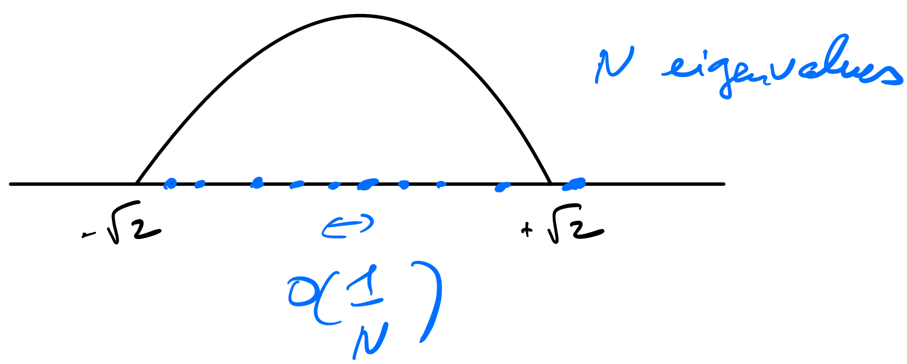


4.1 Local statistics

Bulk vs edge

a) Bulk

* Spacing between eigenvalues



$$\lambda_1 < \lambda_2 < \dots < \lambda_N$$

$$s = \lambda_{i+1} - \lambda_i = O\left(\frac{1}{N}\right)$$

Distribution of s ?

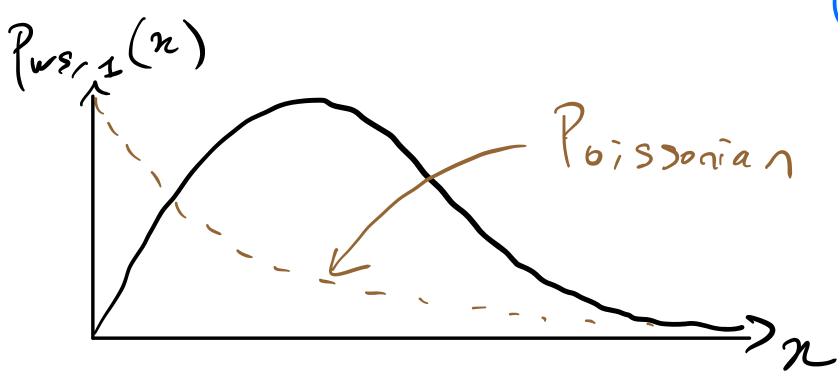
$P_N(s)$: proba. density function of 's'

$$P_N(s) \xrightarrow[N\to\infty]{} \frac{1}{\langle s_N\rangle} P_\beta\left(\frac{s}{\langle s_N\rangle}\right)$$

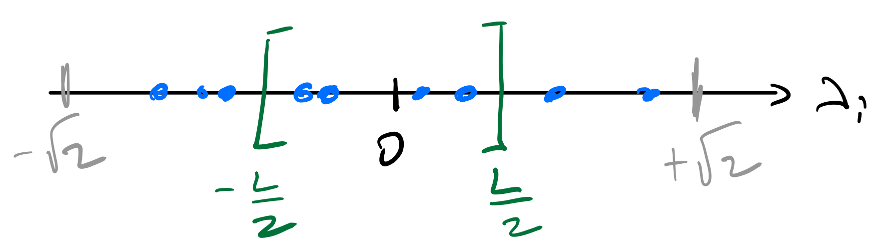
$P_\beta(x)$ is well approximated by the Wigner surmise

$$P_\beta(x) \approx P_{WS,\beta}(x) = a_\beta \underline{x^\beta} e^{-b_\beta x^2}$$

level repulsion



* Nber variance :



N_L = nber of eigenvalues in $\left[-\frac{L}{2}, \frac{L}{2}\right]$

$$\langle N_L \rangle \simeq \int_{-L/2}^{L/2} dx \; \rho_{sc}(x) = O(L)$$

$$\langle \mathrm{Var}\, N_L \rangle \simeq \frac{2}{\beta \pi^2} \ln(NL) \ll \langle N_L \rangle$$

$\frac{L}{2} < \sqrt{2}$

$\Rightarrow$ rigidity of the spectrum

(more rigid than a Poisson point process)

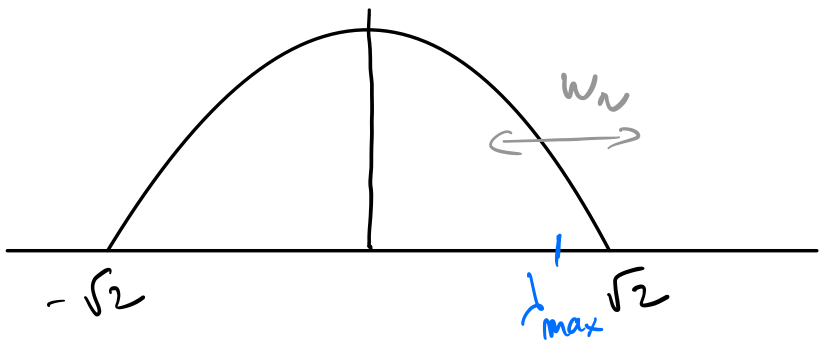
<u>Rk</u> : $\mathbb{P}(N_L = 0) \equiv$ gap proba.

$\rightarrow$ Wigner surmise

b) Edge

Width of the edge : w_N

Define $\lambda_{max} = \max\limits_{1 \leq i \leq N} \lambda_i$



when $N \to \infty$: $\lambda_{max} \longrightarrow \sqrt{2}$

Argument from extreme value statistics:

Nber of eigenvalues in $[\sqrt{2} - w_N, \sqrt{2}] \sim o(1)$

$$N \times \int_{\sqrt{2} - w_N}^{\sqrt{2}} \rho_{sc}(x)\, dx = o(1)$$

$\sim \sqrt{\sqrt{2} - x}$

$$\Rightarrow \quad N \times w_N^{3/2} = o(1)$$

$$\Rightarrow \boxed{w_N = O(N^{-2/3})}$$

$\Rightarrow$ Tracy-Widom distributions ('94, '96)
(TW)

$$\lambda_{max} = \sqrt{2} + \frac{1}{\sqrt{2}} N^{-\frac{2}{3}} \underbrace{\chi_\beta}_{\text{TW random variable}}$$

For $\beta = 1, 2$, $\mathcal{F}_\beta(x) = \mathbb{P}(\chi_\beta \le x)$

Painlevé II : $q''(s) = 2q^3(s) + s q(s)$

$s \in \mathbb{R}$ $\qquad q(s) \underset{s \to \infty}{\sim} Ai(s)$

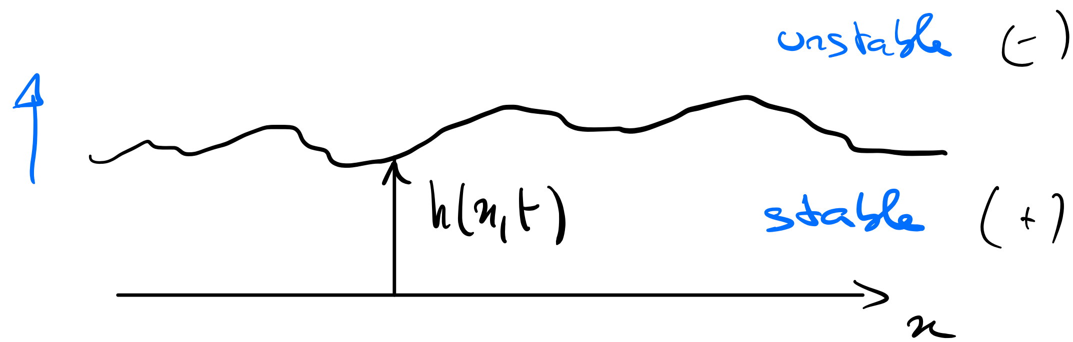
Airy-Functn

$$\beta = 2: \quad \mathcal{F}_2(x) = \exp\left[-\int_x^\infty ds\,(s-x)\,q^2(x)\right]$$

$$\mathcal{F}_2'(x) \sim \begin{cases} \exp\left(-\frac{\beta}{24}|x|^3\right) & x \to -\infty \\ \exp\left(-\frac{2\beta}{3} x^{3/2}\right) & x \to +\infty \end{cases}$$

→ Connection w. Kardar-Parisi-Zhang equation (KPZ)

1+1 dim. fluctuating interface

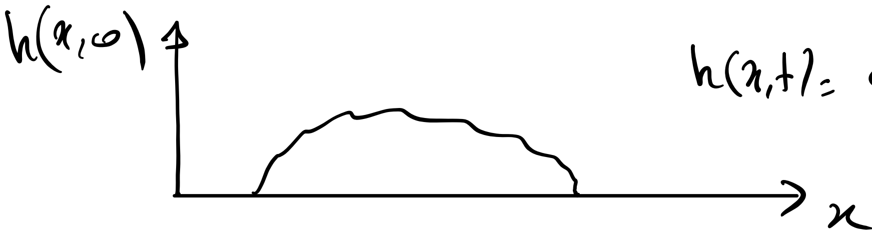


$$\partial_t h(x,t) = \nu \Delta h(x,t) + \frac{\lambda}{2}(\nabla_x h)^2 + \sqrt{D}\underbrace{\xi}_{\text{white noise}}$$

• Flat initial condition : $h(x,0) = 0, \ \forall x$

$$h(x,t) = c_F t + (\Gamma_F t)^{1/3} \underbrace{\chi_1}_{\text{Tracy-Widom distribution } \beta = 1 \text{ (GOE)}}$$

• Curved initial profile



$$h(x,t) = c_c t + \underbrace{(\Gamma_c t)^{\frac{1}{3}} \chi_2}_{\text{TW for } \beta = 2}$$

Rk: Ising model + mag field