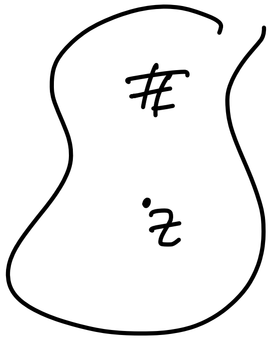
$$\boxed{\text{Lecture 2}}$$

$\boxed{\text{Theorem:}}$ G^δ $\quad \delta$: mesh size $\to 0$.

T^δ: Wired UST



$\gamma^\delta(z)$: Branch of UST from z.

For any test function $f : C_c^\infty(D) \to \mathbb{R}$.

Let $W(\gamma^\delta(z))$ = "winding" of $\gamma^\delta(z)$.

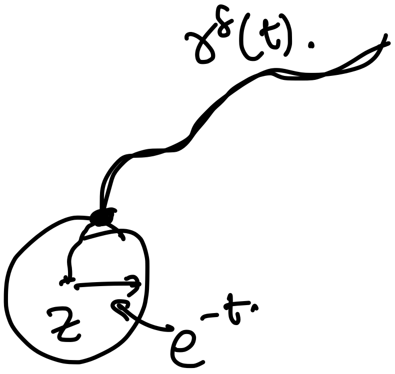
Want to show

$$\int_D W(\gamma^\delta(z))\, f(z)\,dz \xrightarrow{(d)} \int_D G^{FF}(z) f(z)\,dz.$$

GFF with "winding" boundary condition.

$$W(\gamma^\delta(z)) = \underbrace{W(\gamma^\delta_t(z))} + \underbrace{\varepsilon^\delta_t(z)}$$

winding of remaining part.



$W(\gamma^\delta_t) \longrightarrow$ Continuous theory (SLE) to deal with it.

$\varepsilon^\delta_t \longrightarrow$ Use RSW assumption to show

$\varepsilon^\delta_t(z)$ is "independent" for different z.

- We will show that

$$\mathbb{E}\left(\left(\int W(\gamma^\delta(z)) f(z)\right)^k\right) \longrightarrow \mathbb{E}\left[\left(\int \mathrm{GFF}(z) f(z)\right)^k\right]$$

(Moment method).

Windings of curves

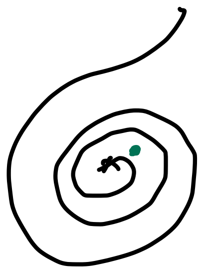
In the beginning assume $\gamma : [0,1] \mapsto \mathbb{R}^2$ some curve which is smooth.

- (Intrinsic Winding).

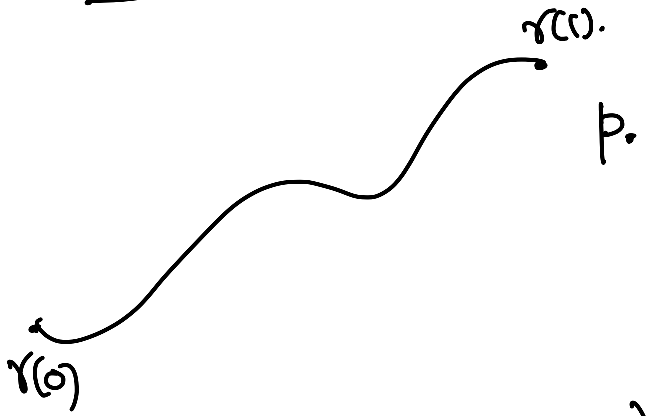
$$\gamma'(t) = r_{int}(t)\, e^{i\theta_{int}(t)}.$$

θ_{int} taken continuously.

$$W_{int}(\gamma) = \int_0^1 \theta_{int}(t)\,dt.$$



- (Topological winding) $\quad p \notin \gamma[0,1]$.



$$\gamma(t) - p.$$

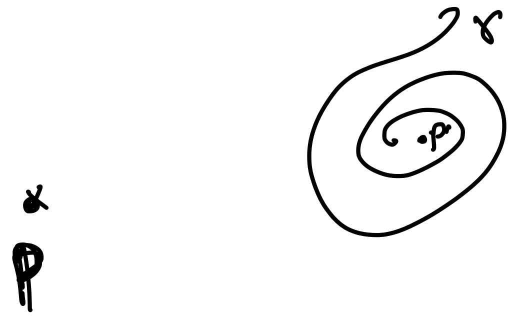
$$= r(t)\, e^{i\tilde{\theta}(t)}.$$

$\tilde{\theta}$ taken continuously

$$W(\gamma, p) = \int_0^1 \tilde{\theta}(t)\,dt.$$

If $p \notin \gamma^{\delta}[0,1]$ $\forall\ \delta$.

then $W(\gamma^{\delta}, p)$ is cont. in δ.



$|W(\gamma, p')| > |W(\gamma, p)|$.

Lemma Take γ: smooth curve.
$[0,1] \mapsto \mathbb{R}$.

$$W_{int}(\gamma) = W(\gamma, \gamma(0)) + W(\gamma, \gamma(1)).$$

Here

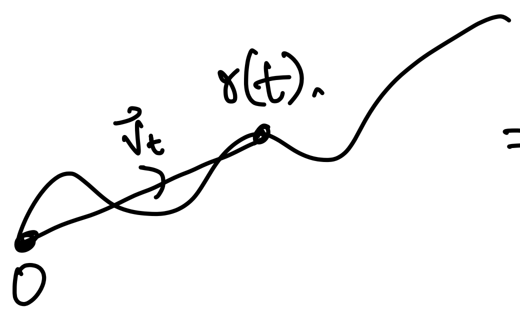
$$W(\gamma, \gamma(0)) = \lim_{t \to 0+} W(\gamma(t,1), \gamma(0)).$$

$$W(\gamma, \gamma(1)) = \lim_{t \to 1} W(\gamma[0,t], \gamma(1)).$$

Pf: Take $\mathrm{Arg} \in [-\pi, \pi]$.

Let $\widetilde{W} = W \mod 2\pi$.

$$\widetilde{W}[\gamma[0,t], \gamma(t)]$$



$$= -\vec{V}_t - (\mathrm{Arg}(\gamma'(t) + \pi)$$

$$= -\vec{V}_t + \mathrm{Arg}(\gamma'(t)).$$

$$\tilde{W}[\gamma[0,t], \gamma(0)] = \vec{V}_t - \operatorname{Arg}(\gamma'(0))$$

$$\tilde{W}(\gamma(0,t), \gamma(t)) + \tilde{W}[\gamma[0,t], \gamma(0)]$$

$$= \operatorname{Arg}(\gamma'(t)) - \operatorname{Arg}(\gamma'(0)).$$

$$= \arg(\gamma'(t)) - \arg(\gamma'(0)) + 2\pi \underbrace{K_t}_{\in \mathbb{Z}.}$$

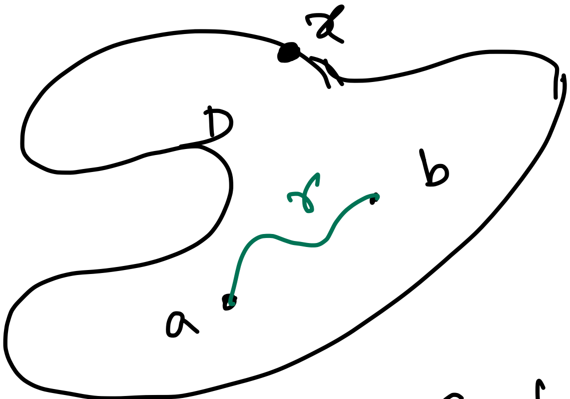
Notice for small t, $K_t = 0$

By continuity $K_t \equiv 0$.

Let $t \to 1-$ to conclude $\blacksquare$

Winding seen from boundary of a domain.

$x \in \partial D$.



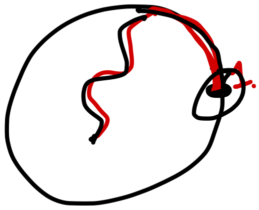
We can define a function $\arg_{D,x}$.

Such that.

$$\arg_{D,x}(a) - \arg_{D,x}(b) = \operatorname{Im}\Big(\underbrace{\int_\gamma \frac{dz}{z-x}}\Big).$$

$\downarrow$

Independent of γ.

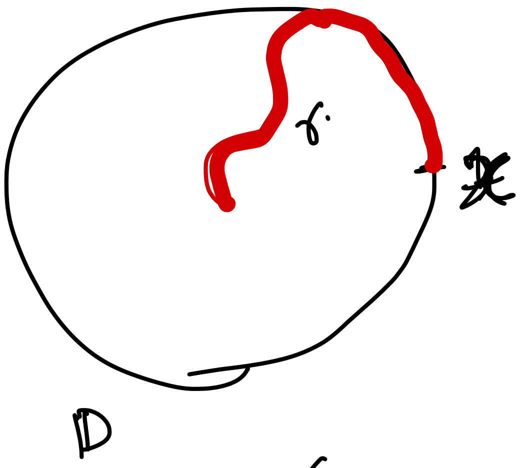


e.g.: we can define $\arg_{\mathbb{D},1} \in [0, 2\pi]$.

such that $\arg_{\mathbb{D},1}(0) = \pi$

- If we have a curve γ, with $\gamma(0) = x$, then

$$\arg_{D,x}(\gamma'(0)) = \lim_{\varepsilon \to 0^+} \arg_{D,x}(\gamma(\varepsilon)).$$



$W(\gamma, x)$

$= \arg_{D,x}(\gamma(1))$

$- \arg_{D,x}(\gamma'(0)).$

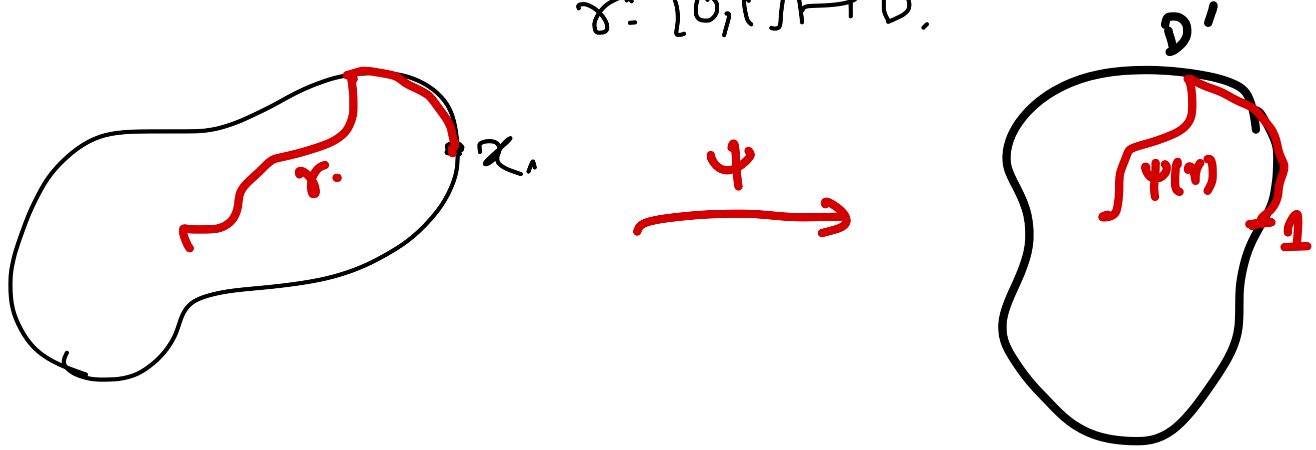
For smooth γ.

$$W_{int}(\gamma) = W(\gamma, \gamma(1)) + \arg_{D,x}(\gamma(1)) - \arg_{D,x}(\gamma'(0)).$$

Change of winding under conformal mapping

Let $\psi : (D, x) \longmapsto (D', 1)$

$\gamma : [0,1] \longmapsto D.$



$$W_{int}(\psi(\gamma)) = \int_0^1 \arg((\psi \circ \gamma)'(t))dt$$

$$= \int_0^1 \arg(\psi'(\gamma(t)))dt + \underbrace{\int_0^1 \arg(\gamma'(t))dt}$$

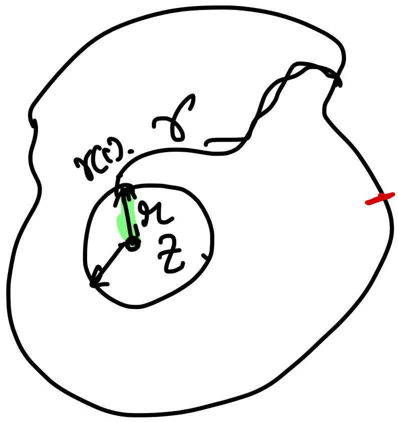
$$= \arg(\psi'(\gamma(1))) - \arg(\psi'(\gamma(0)))$$

$$+ W_{int}(\gamma).$$

We can extend this change of winding formula for top. windings as well.

Lem:

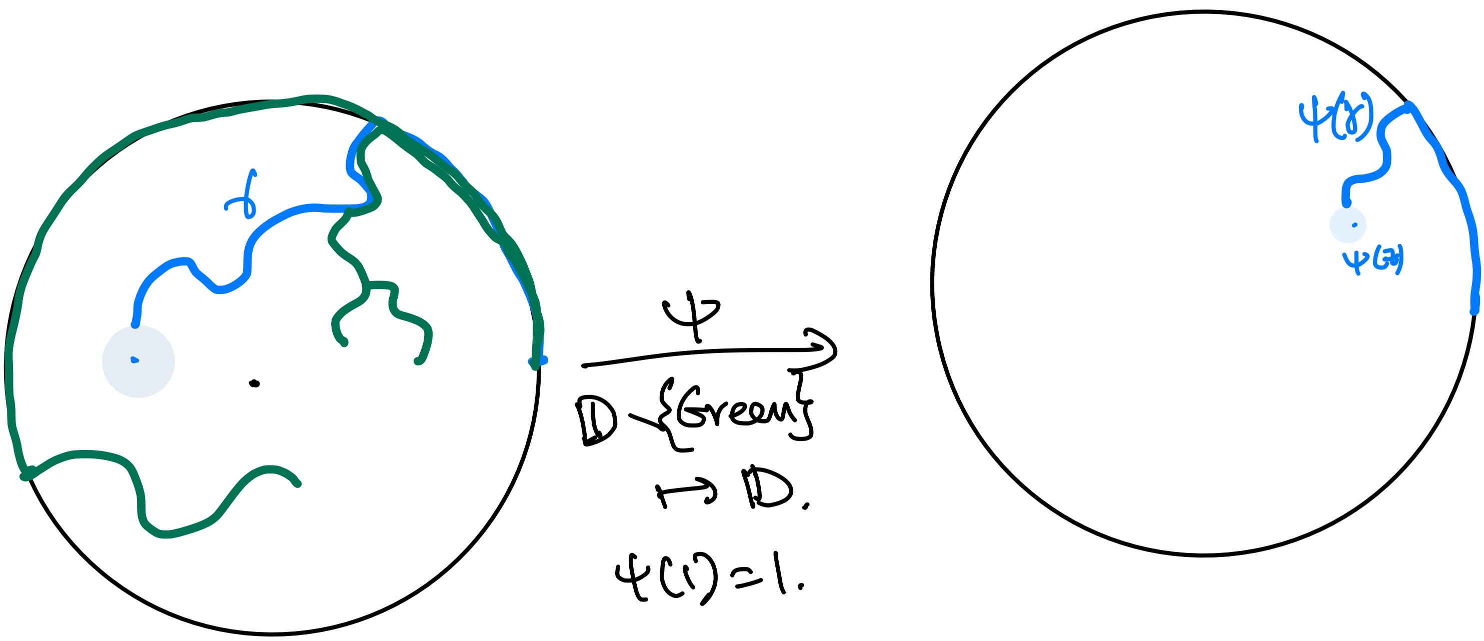
$$W(\psi(\gamma), \psi(z)) - W(\gamma, z) = \boxed{\arg_{\psi'(0)}(\psi'(z))} + \underline{\arg_{D,x}}(z) - \underline{\arg_{D',x'}}(\psi(z)) + \underline{O\left(\frac{r}{R}\right)}.$$

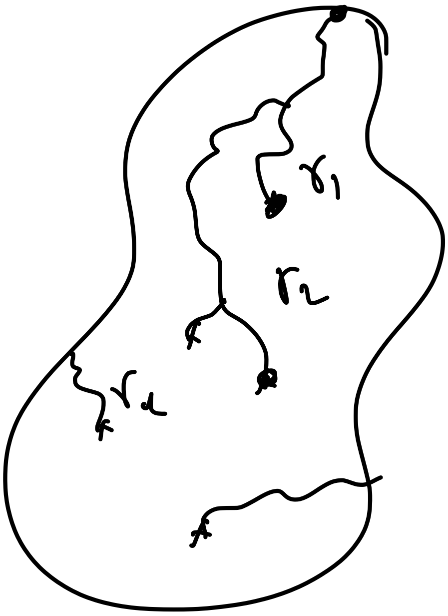


$\psi'(0) \neq 0$. $\arg_{\psi'(0)}$ defined upto global shift by $\mathbb{R}$.

where $R =$ conf radius of D seen from z.

Remark: This Lemma works even if γ is fractal. as long as $z \notin \gamma(1)$



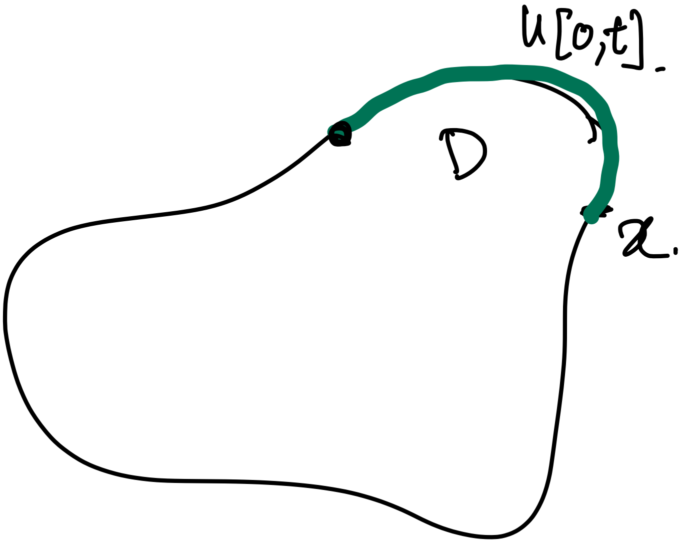


Given $\gamma_1, \dots, \gamma_k$,

The rest of the branches is a UST in $D \setminus \{\gamma_1, \dots, \gamma_k\}$.

"winding boundary condition"

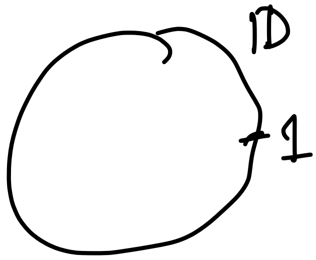
(D, x)



u_D : winding bdry cond.

= harmonic extension $\text{Wint}(u[0,t])$.

$\varphi : (D, x) \xrightarrow[\text{conf.}]{} (\mathbb{D}, 1)$



$$u_{D,x} = u_{\mathbb{D},1} \circ \varphi - \arg_{\varphi'(D)}(\varphi'(\cdot)).$$

$$u_{\mathbb{D},1}(z) \underset{\substack{\text{Calc.} \\ \text{using} \\ \text{Mobiusmaps.}}}{=} 2\arg_{\mathbb{D},1}(z).$$

Imaginary Geometry

(Dubedat / Miller - Sheffield).

<u>Idea:</u> h : GFF

view $\text{"}(e^{i c h_x})_{x \in D}\text{"}$: vector field.

Flow lines of this vector field "=" SLE_κ $\kappa = f(c)$.

<u>Theorem:</u> $\chi = \frac{1}{\sqrt{2}} = \left(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}\right)$. $\kappa = 2$.

$\exists$ a coupling between.

$h = h_D^0 + \chi\, u_{(D,x)}$ ← winding bdry cond.

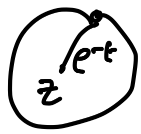
and UST

so that given $\bar{\gamma} = (\gamma_1, \dots, \gamma_k)$.

the law of h in $D \setminus \bar{\gamma}$

$$h^0_{D\setminus\bar{\gamma}} + \chi \underbrace{u_{(D\setminus\gamma, x)}}_{\text{winding bdry cond}^n\text{ in slitted domain.}}$$

Also, h is <u>measurable</u> w.r.t. $\mathcal{T}$.

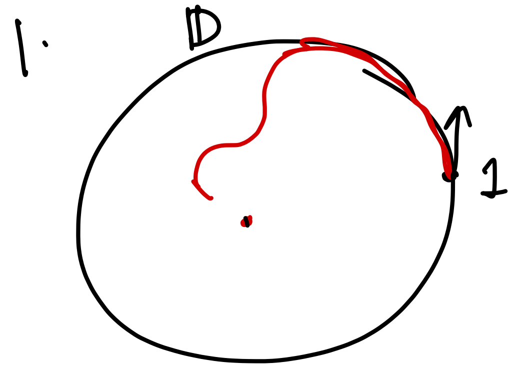


<u>Def</u>: $$h_t^D(z) = W(\underbrace{\gamma_z[t]}_{\text{branch upto capacity } t \text{ from } z.}, z) + \arg_{D,x}(z) - \arg_{(D,x)}(\gamma'(1)).$$

<u>Thm</u>: $$\left(h_t^D(z)\right)_{z\in D} \longrightarrow h^D_{GFF}$$

h^D_{GFF} : Intrinsic winding GFF $+\frac{\pi}{2}$.

1.



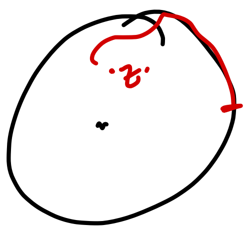
$$\mathbb{E}(h_t(0))$$

$$= \underbrace{\mathbb{E}(W(\gamma_0(t),0))} + \underbrace{\arg_{\mathbb{D},1}(0)} - \arg_{\mathbb{D},1}(\gamma'(0))$$

Symmetry

$$= \pi + \pi - \frac{\pi}{2}$$

$$= \frac{3\pi}{2}.$$



• $\mathbb{E}(h_t(z))$. use $\psi : \mathbb{D} \mapsto \mathbb{D}$

$\psi(1) = 1.$

$\psi(z) = 0.$

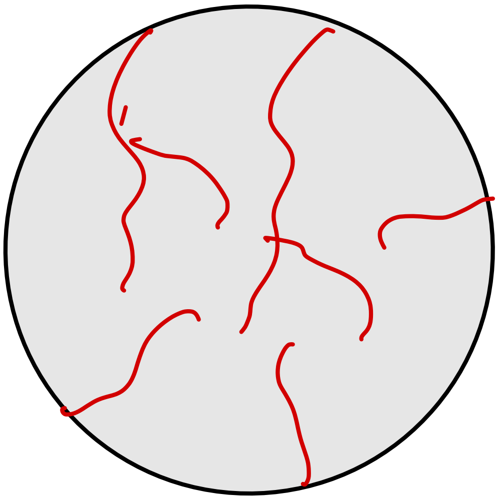
Use change in winding formula.

$$h_t(z) - h_t(0) = \underbrace{-\arg_{\psi'(\mathbb{D})}(\psi'(z))} + \underbrace{\varepsilon(t)}_{} \quad \varepsilon(t) = O(e^{-c(t)}).$$

$$\xrightarrow{\text{calculation.}} \lim_{t \to \infty} \mathbb{E}(h_t(z)) = \underbrace{2\arg_{(\mathbb{D},1)}(z) - \frac{\pi}{2}}.$$

winding boundary at z

$+\frac{\pi}{2}.$



Let A: lot of branches.

Want $\mathbb{E}(h_t(z) \mid A)$.

(Recall Imaginary geom: $h_{GFF} = \frac{1}{\chi} h^0_{\mathbb{D}} + u_{\mathbb{D},1}$.

h_{GFF} cond on A:

$= \frac{1}{\chi} h^0_{\mathbb{D}\setminus A} + u_{\mathbb{D}\setminus A,1}$.)

$g_A : \mathbb{D}\setminus A \xrightarrow{\text{conf}} \mathbb{D}$, $g_A(1) = 1$.

$$\mathbb{E}(h_t(z) \mid A) = \mathbb{E}\Big(\tilde{h}_t^{\mathbb{D}}(g_A(z)) - \arg_{g_A'(\mathbb{D}\setminus\gamma)}(g_A'(z))$$

$$\xrightarrow{t\to\infty} \underbrace{u_{\mathbb{D},1}(g_A(z)) + \frac{\pi}{2}}_{u_{\mathbb{D}\setminus A,1}(z).}$$

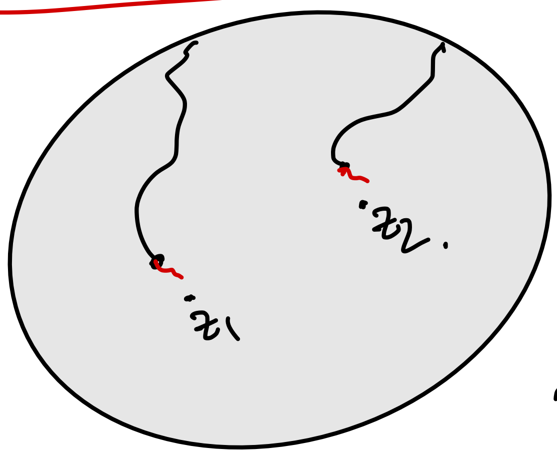
Conditional expectation of the winding field

= " " " " Imaginary geom GFF.

Now if we show that

$$\mathrm{Var}\left(\int \overline{h_t(z)\,dz}\,(A)\right) \approx 0.$$

k- point functions



Lem

$$\lim_{t\to\infty} \mathbb{E}\left(h_t(z_1)\cdots h_t(z_k)\right) \text{ exists.}$$

Idea: red parts are independent.

. Regularity of limit $\lim_{t\to\infty} \mathbb{E}\left(h_t(z_1)\cdots h_t(z_k)\right)$

$$\lesssim \log^{\alpha}\left(\frac{1}{r}\right).$$

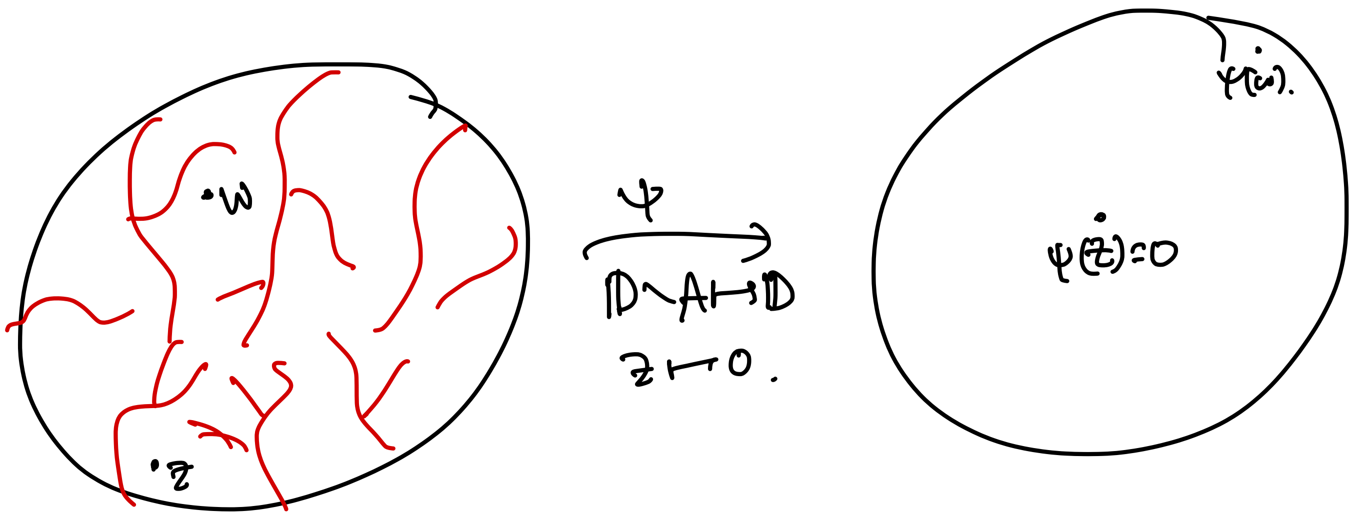
$$r = \min\{|z_i - z_j|\}$$

$$\mathbb{E}\left[\left(\int h_t(z) f(z)\,dz\right)^k\right] \xrightarrow{t\to\infty}$$

Given a lot branches. A

$$\int_{\mathbb{D}\times\mathbb{D}} \mathbb{E}\left(\overline{h_t}(z)\, \overline{h_t}(w) \,|\, A\right) dz\, dw$$

$\approx$ Small.



Given this we show. $\exists$ a coupling

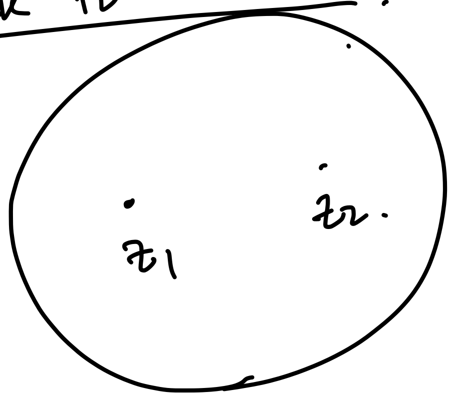
$$\left(h_t(z),\ h^{D}_{GFF} \right).$$

$\uparrow$.

Im. geom. GFF

As $t\to\infty$ $\quad h_t(z) \xrightarrow{\mathbb{P}} h^{D}_{GFF}$

Back to discrete.



$$\mathbb{E}\left(\overline{h^{\delta}_{\cdot}}(z_1)\, \overline{h^{\delta}}(z_2) \right).$$

$$= \mathbb{E}\Big(\big(\overline{h^{\delta}_t}(z_1) + \overline{\varepsilon^{\delta}_t}(z_1) \big) \big(\overline{h^{\delta}_t}(z_2) + \overline{\varepsilon^{\delta}_t}(z_2) \big) \Big).$$

Want to show. at discrete level.

$\mathcal{E}_t^\delta(z_1)$, $\mathcal{E}_t^\delta(z_2)$ are "independent"

and $\mathcal{E}_t^\delta(z_1) \overset{\text{indpt.}}{\sim} h_t^\delta(z_2)$.

$\mathcal{E}_t^\delta(z_2) \overset{\text{indpt}}{\sim} h_t^\delta(z_1)$.

<u>Theorem</u> Given z_1, z_2 $\exists$ a coupling between UST $\mathcal{T}$ and independent USTs $\mathcal{T}_1$ & $\mathcal{T}_2$.

such that

$$\mathcal{T} \cap B(z_1, R) = \mathcal{T}_1 \cap B(z_1, R)$$

$$\mathcal{T} \cap B(z_2, R) = \mathcal{T}_2 \cap B(z_2, R).$$

$$\mathbb{P}(R \leq r\varepsilon) \leq C\varepsilon^{c'}$$

$r = |z_1 - z_2|$. (Multiscale coupling argument/ Schramm's lemma).