Introduction to Random Matrix Theory

I] General Overview and applications of RMT

II] Ensembles of RMT

III] Coulomb gas approach

IV] Local statistics (bulk & edge)

I] Intro.

M : $N \times N$ matrix w. random entries $\underbrace{M_{jk}}_{\text{real or complex}}$

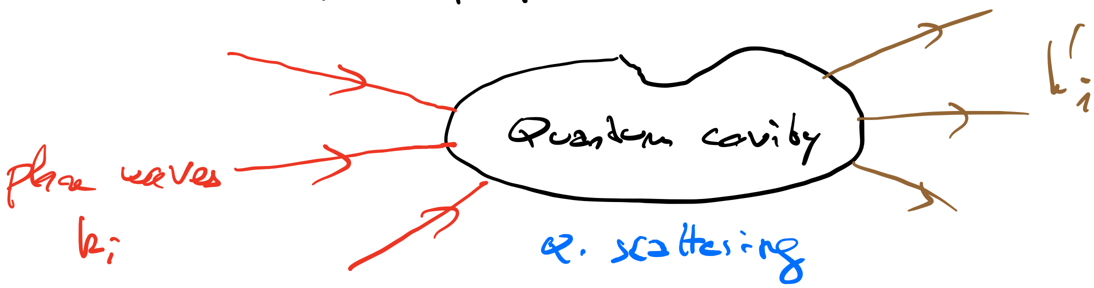
Q : stat. of eigenvalues and eigenvectors
$N \to \infty$

In physics, introduced by Wigner ('50)

↳ study of nuclear physics to model the energy levels of big nuclei.

Since then many applications:

* Mesoscopic physics:



$$\underline{k}' = S\,\underline{k}$$

random matrix

* Random graph : adjacency matrix

* nber theory (Riemann zeta-function)

* combinatorics

* stochastic / fluctuating interfaces
  Kardar-Parisi-Zhang equation

  ↳ related to directed polymers + disorder

In fact RMT was introduced by J. Wishart (1928)

↳ correlation in time series

$$\vec{X}_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{N,t} \end{pmatrix}$$ ← stock price, daily temp.

$$\hat{C}_{jk} = \frac{1}{T} \sum_{t=1}^{T} x_{jt}\, x_{kt}$$

random correlat° matrix

→ applications in data analysis.

2) Ensembles of RMT

→ Matrices w. real spectrum — real symmetric, complex Hermitian

→ Two main categories of rand. matrices:

a) Ens. with independent entries

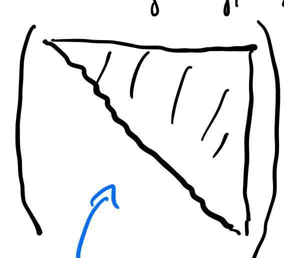
"Wigner matrices"

$$M_{jk} = x_{jk} + i\, y_{jk}$$

Joint proba. density function

$$\rho(M) = \rho(\{M_{jk}\})$$

$M_{ij} = M_{ji}$ $i \neq j$

$M \equiv$ 

fixed by symmetry

$$\rho(M) = \prod_{i=1}^{N} f_i(x_{ii}) \prod_{j<k} f_{jk}^{(R)}(x_{jk})\, f_{jk}^{(I)}(y_{jk})$$

Nber of degrees of freedom $= 1 + 2 + 3 + \dots + N$

$$= \frac{N(N+1)}{2}$$

b) Rotationnally invariant ensembles

M: real symmetric $\Rightarrow \exists (\underline{O}, \underline{\Lambda})$

$$\underline{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}, \quad \lambda_i = \text{eigenvalues of } M$$

w. $OO^t = \mathbb{1}$ such that $M = \underline{O}\,\underline{\Lambda}\,O^{-1}$

Rotationnally ens.: $\mathcal{P}(M) = \mathcal{P}(OMO^{-1})$

↳ the eigenvectors do not play a very important role.

$$\Leftarrow \quad \mathcal{P}(M) \propto e^{-\mathrm{Tr}\, V(M)}$$

$$P(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\sum_i V(\lambda_i)}$$

some function

Porter - Rosenzweigh thm ('60) : the only ensembles that belong to a) and b) are the Gaussian ens.:

$$\mathcal{P}(M) = \frac{1}{Z_N} e^{-a\,\mathrm{Tr}(M^2) - b\,\mathrm{Tr}(M)}$$

• show that this of type a) [Wigner]

$$\mathcal{P}(M) = \frac{1}{Z_N} e^{-a \sum_{j,k} M_{jk} \underbrace{M_{kj}}_{= M_{jk},\ j \neq k} - b \sum_j M_{jj}}$$

$$= \frac{1}{Z_N} e^{-a \sum_j M_{jj}^2 - a \sum_{j \neq k} M_{jk}^2 - b \sum_j M_{jj}}$$

$$= \frac{1}{Z_N} e^{-a \sum_j M_{jj}^2 - b \sum_j M_{jj} - 2a \sum_{j<k} M_{jk}^2}$$

$$= \underbrace{\frac{1}{Z_N} \prod_j e^{-a M_{jj}^2 - b M_{jj}} \prod_{i<j} e^{-2a M_{ij}^2}}_{\hookrightarrow \text{ belongs to a)}}$$

NB: For $b = 0$, the variance of the diag. terms is twice the one of the off-diag. ones.

In the following: set $b = 0$

For M real sym. : Gaussian Orthogonal Ens. GOE

M complex Hermitian : Gaussian Unitary Ens. GUE

c) Joint law of eigenvalues for GOE/GUE

GOE : $M = M^t$ and real.

$$M = \underline{O}\,\underline{\Lambda}\,\underline{O}^{-1}, \quad \underline{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

eigenvectors ($\underline{O}$) eigenvalues ($\underline{\Lambda}$)

$$OO^t = \mathbb{1}$$

Change of variables: $M \equiv \{M_{jk}\} \longrightarrow \{\underline{\Lambda}, \underline{O}\}$

Rk: nber of degrees of freedom:

$$\frac{N(N+1)}{2} = N + \frac{N(N-1)}{2}$$

$$\underset{\substack{\shortparallel \\ e^{-a\,\mathrm{Tr}(M^2)}}}{P(M)} \longrightarrow P(\{\lambda_1, \dots, \lambda_N\}, \underline{O})$$

Related by a Jacobian:

$$P(\{\lambda_1,\dots,\lambda_N\},\underline{O}) = \mathcal{B}(M)\,|\det J|$$

$$J = \text{"}\left\{ \frac{\partial M_{ij}}{\partial \lambda_k}, \frac{\partial M_{ij}}{\partial O_{kl}} \right\}\text{"}$$

of size $\frac{N(N+1)}{2} \times \frac{N(N+1)}{2}$

(GOE) $$\boxed{|\det J| = \prod_{j<k} |\lambda_j - \lambda_k|}$$

indep. of $\underline{O}$

$$\Rightarrow P(\{\lambda_1,\dots,\lambda_N\},\underline{O}) = \frac{1}{Z_N} e^{-a\sum_{i=1}^{N}\lambda_i^2} \underbrace{\prod_{j<k}|\lambda_j-\lambda_k|^\beta}_{\text{level repulsion}}$$

for GOE $\beta = 1$

GUE $\beta = 2$.

- Vandermonde determinant: $x_1 \neq x_2, \dots, x_N$

$$V_N = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_N \\ x_1^2 & x_2^2 & & x_N^2 \\ \vdots & \vdots & & \vdots \\ x_1^{N-1} & x_2^{N-1} & & x_N^{N-1} \end{pmatrix}$$

$\underline{N=2}$: $\begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} = V_2 \implies \det V_2 = x_2 - x_1$.

$$\det V_N = \prod_{i<j} (x_j - x_i)$$

Here we choose:

$$P(\lambda_1, \dots, \lambda_N) = B_N \, e^{-\beta \frac{N}{2} \sum_{i=1}^{N} \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

III.1 Coulomb gas approach (Dyson '62)

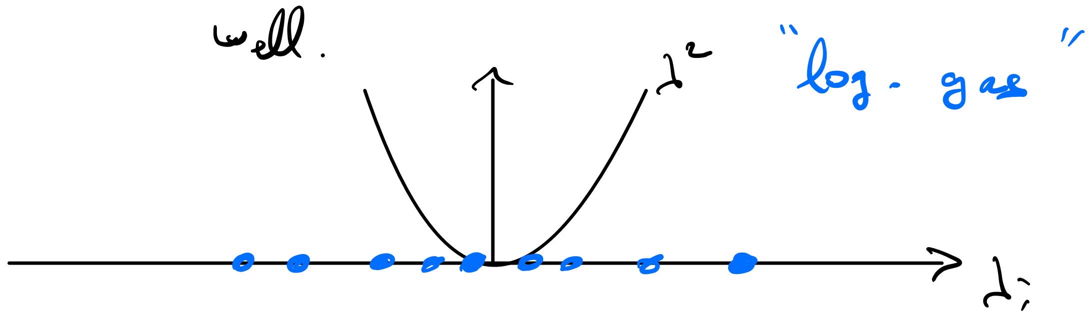
Rewriting:

$$\prod_{i<j} |\lambda_i - \lambda_j|^\beta = e^{\beta \sum_{i<j} \ln |\lambda_i - \lambda_j|} = e^{\frac{\beta}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|}$$

$$\implies P(\lambda_1, \dots, \lambda_N) = B_N \, e^{-\beta E(\{\lambda_i\})}$$

$$E(\{\lambda_i\}) = \frac{N}{2} \sum_{i=1}^{N} \lambda_i^2 - \underbrace{\frac{1}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|}_{\text{repulsive}}$$

$\implies \lambda_i \equiv$ positions of charged particles interacting via the 2d Coulomb interactⁿ confined on a line within a harmoni…



$\Rightarrow$ competition between confinement and the repulsive interactions