A Mathematical Introduction to 3d $\mathcal{N} = 4$ Gauge Theory I

Mathew Bullimore

Durham University

Engineering and Physical Sciences Research Council

Science and Technology Facilities Council

Motivation

There are many connections between supersymmetric quantum field theory, representation theory and enumerative geometry.

Some underlying themes:

- Supersymmetry ensures exact moduli spaces of vacua with mathematical structures: Kähler, hyper-Kähler.
- Symmetries are reflected in isometries and topology.
- Natural appearance of cohomology, K-theory, derived categories of coherent sheaves.

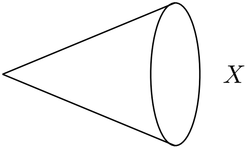
These lectures focus on 3d $\mathcal{N} = 4$ supersymmetry - why?

Why supersymmetry?

Supersymmetric gauge theories with eight supercharges:

$$5\text{d}\ \mathcal{N}=1 \qquad 4\text{d}\ \mathcal{N}=2 \qquad 3\text{d}\ \mathcal{N}=4$$

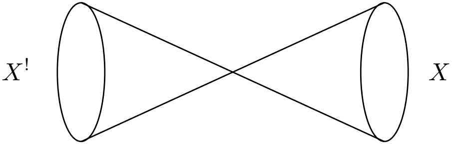
- They have intricate moduli spaces of vacua whose geometric structure depends on dimension.
- They all have one branch - the Higgs branch X - which is a conical hyper-Kähler or holomorphic symplectic space.
- Examples: Nakajima quiver varieties, affine grassmannian slices, nilpotent orbit closures, hyper-toric varieties.



Why three dimensions?

3d $\mathcal{N} = 4$ theories are distinguished by the existence of *two* branches that are hyper-Kähler cones.

- Higgs branch X and Coulomb branch $X^!$.
- There is an infrared duality called 3d mirror symmetry that interchanges X and $X^!$.



Surprising connections between representation theory and enumerative geometry problems associated to X and $X^!$.

Setup and Notation

I will work in three-dimensional euclidean space $\mathbb{R}^3$.

- Coordinates $x^\mu = (x^1, x^2, x^3)$.

The spin group is $\mathrm{Spin}(3) = \mathrm{SU}(2)$.

- Two-component spinors ψ^α transform in the fundamental $\mathbf{2}$.
- Raise and lower indices with invariant tensors $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$.
- Pauli matrices $(\sigma^\mu)^\alpha{}_\beta$.

Momentum and angular momentum generators P_μ, $J^\mu{}_\nu$.

$\mathcal{N}$-extended Supersymmetry

A supersymmetric theory has additional odd generators $Q_\alpha^\mathcal{I}$, $\mathcal{I} = 1, \ldots, \mathcal{N}$, that satisfy

$$\{Q_\alpha^\mathcal{I}, Q_\beta^\mathcal{J}\} = \delta^{\mathcal{I}\mathcal{J}} (\sigma^\mu)_{\alpha\beta} P_\mu$$

- The supercharges $Q_\alpha^\mathcal{I}$ transform as spinors on $\mathbb{R}^3$.
- P_μ, $J^\mu{}_\nu$, $Q_\alpha^\mathcal{I}$ generate the $\mathcal{N}$-extended super-Poincaré algebra.
- The group of outer automorphisms is O$(\mathcal{N})$.
- The connected component SO$(\mathcal{N})$ is known as the R-symmetry.

$$Q_\alpha^I \to R^I{}_J Q_\alpha^J$$

$\mathcal{N} = 4$ Supersymmetry

We are interested in $\mathcal{N} = 4$ supersymmetry.

- The R-symmetry is Spin$(4) \cong$ SU$(2)_H \times$ SU$(2)_C$.
- Introduce spinor indices A, B, ... for SU$(2)_H$.
- Introduce spinor indices $\dot{A}$, $\dot{B}$, ... for SU$(2)_C$.

$$Q_\alpha^{\mathcal{I}} \longrightarrow Q_\alpha^{A\dot{A}}$$

The supersymmetry algebra becomes

$$\{Q_\alpha^{A\dot{A}}, Q_\beta^{A\dot{A}}\} = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}(\sigma^\mu)_{\alpha\beta}P_\mu \, .$$

Supersymmetric Gauge Theories

There are a number of ways to construct quantum field theories with 3d $\mathcal{N} = 4$ supersymmetry.

- Sigma models
- Gauge Theories
- Non-Lagrangian Theories

Different types of theories can be connected by renormalisation group flow and infrared dualities.

Here we will consider supersymmetric gauge theories.

Supersymmetric Gauge Theories

A supersymmetric gauge theory is specified by the following data:

1. A compact Lie group G
2. A linear quaternionic representation $\rho : G \to \mathsf{Sp}(n)$ acting on the vector space $Q = \mathbb{H}^n$.

This data determines the spectrum of supermultiplets and Lagrangian:

1. Vectormultiplet: "Gauge"
2. Hypermultiplet: "Matter"

A Simplification

It is common to use complex numbers rather than quaternions:

- Use the isomorphism

$$\mathsf{Sp}(n) \cong \mathsf{USp}(2n, \mathbb{C}) := \mathsf{Sp}(2m, \mathbb{C}) \cap U(n, n)\,.$$

- A quaternionic representation acts by symplectic-unitary transformations on $Q \cong \mathbb{C}^{2n}$.

Furthermore, we restrict attention to quaternionic representations of “cotangent type”.

- Specify a unitary representation $\rho : G \to U(n)$ on $R \cong \mathbb{C}^n$.
- Then $Q = R \oplus R^*$

Vectormultiplet

The vectormultiplet depends only on G.

It contains the following bosonic fields:

1. A connection A_μ on a principle G-bundle P on $\mathbb{R}^3$.
2. Scalar fields $\sigma^{\dot{A}\dot{B}} = \sigma^{\dot{B}\dot{A}}$ transforming in the adjoint representation $\mathbf{3}$ of $SU(2)_C$ and as sections of $\mathrm{Ad}(P)$.

There are also fermion gauginos that I will omit.

Hypermultiplet

The hypermultiplet depends on the quaternionic representation Q.

It contains scalar fields Φ:

- In the fundamental representation $\mathbf{2}$ of $\mathrm{SU}(2)_H$
- Sections of the associated vector bundle $P \times_G Q$.
- Components Φ^{AI} with $I = 1, \ldots, 2n$,

$$(\Phi^{AI})^\dagger = \epsilon_{AB}\Omega_{IJ}\Phi^{BJ} \,,$$

where Ω_{IJ} is the invariant symplectic form of $USp(2n)$.

Hypermultiplets

Specialise to a cotangent type representation $Q \cong R \oplus R^*$.

We can decompose the $2 \times 2n$ matrix Φ^{AI} into n 2×2 blocks:

$$\Phi^{AI} = (\Phi^1, \dots, \Phi^n) \qquad \Phi^i = \begin{pmatrix} X_i & Y_i \\ \bar{Y}_i & -\bar{X}_i \end{pmatrix}$$

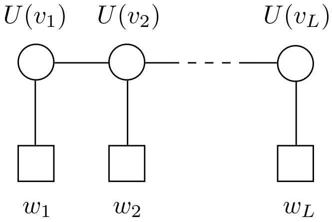
where X, Y transform in unitary representations R, R^*.

In summary, the pair $(X, \bar{Y})$ transform:

- in the fundamental $\mathbf{2}$ of $SU(2)_H$
- in the associated vector bundle $P \times_G R$.

Example

Consider A_L-type unitary quiver gauge theory:



- $G = \mathsf{U}(v_1) \times \cdots \times \mathsf{U}(v_L)$
- Let $V_1, \ldots, V_L$ denote the fundamental representations and $W_1, \ldots, W_L$ denote complex vector spaces of dimensions $w_1, \ldots, w_L$ in the trivial representation.
- Then $R = \bigoplus_{a=1}^{L-1} \mathsf{Hom}(V_a, V_{a+1}) \oplus \bigoplus_{a=1}^{L} \mathsf{Hom}(W_a, V_a)$

Flavour Symmetry

R-symmetries transform the supercharges.

Flavour symmetries commute with the supercharges.

In a 3d $\mathcal{N} = 4$ supersymmetric gauge theory,

- R-symmetry SU$(2)_H \times$ SU$(2)_C$
- Flavour symmetry $G_H \times G_C$.

The two factors G_H,G_C are distinguished by

- how the generators contribute to central extensions of the super-Poincaré algebra - see next lecture!

Higgs Symmetry

The local operators charged under G_H are built from the hypermultiplet fields Φ or X, Y.

It is loosely the symmetries of the hypermultiplet once G is removed.

- For a general quaternionic representation $\rho : G \to \mathsf{Sp}(n)$,

$$G_H = N_{\mathsf{Sp}(n)}(\rho(G))/\rho(G)\,.$$

- For theories of cotangent type specified by a unitary representation $\rho : T \to \mathsf{U}(n)$, there is a simpler formula

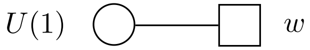
$$G_H = N_{\mathsf{U}(n)}(\rho(G))/\rho(G)$$

Higgs Symmetry Example

For a unitary A_L-type unitary quiver the flavour symmetry is

$$G_H = (U(w_1) \times \cdots \times U(w_L))/U(1)$$

A simple example is supersymmetric QED:



- This corresponds to $G = U(1)$ acting on $R = \mathbb{C}^w$ with weight $\rho = (1, \ldots, 1)$.
- $\rho(\mathrm{U}(1)) \subset \mathrm{U}(w)$ is the diagonal $U(1)$ whose normaliser is $\mathrm{U}(w)$.
- Therefore $G_H = \mathrm{U}(w)/\mathrm{U}(1) = \mathrm{PSU}(w)$.

Coulomb Flavour Symmetry

The Coulomb symmetry is not so straightforward.

- An abelian subgroup $T_C \subset G_C$ is manifest in the UV.
- It is the centre of the Langlands dual of G,

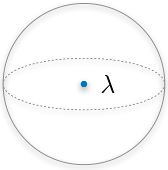
$$T_C = Z(^L G) = \mathrm{Hom}(\pi_1(G), U(1)) \,,$$

  and is independent of the hypermultiplet representation.
- This may be enhanced to a non-abelian group G_C in the IR in way that depends strongly on the hypermultiplet representation.

I will explain how to understand the "topological symmetry" T_C.

Monopole Operators I

A monopole operators at p is defined by excising a small 2-sphere S_p^2 and imposing boundary conditions of a Dirac monopole:



$$A_+ = \frac{\lambda}{2}(+1 - \cos\theta)d\phi$$

$$A_- = \frac{\lambda}{2}(-1 - \cos\theta)d\phi$$

where $A_\pm$ denote the connection on patches $U_\pm \subset S_p^2$.

- $\lambda : S^1 \to G$ is a 1-valued transition function.
- This is element of the co-character lattice $\text{Hom}(\text{U}(1), T)$ modulo Weyl transformations.
- Monopole operators labelled by dominant co-characters.

Monopole Operators II

The monopole operator indices a G-bundle on S_p^2 with connection A.

- Its topological degree $d \in \pi_1(G)$ is the homotopy class of the transition function $\lambda : U(1) \to G$.
- The topological symmetry T_C counts the topological degree.
- This means the character lattice of the topological symmetry is

$$\mathsf{Hom}(T_C, U(1)) \cong \pi_1(G)\,,$$

which implies $T_C \cong \mathsf{Hom}(\pi_1(G), \mathsf{U}(1))$.

Coulomb Symmetry Example

Here are some examples:

- The topological symmetry for $G = \mathsf{U}(v)$ is $T_C = \mathsf{U}(1)$.
- In a unitary quiver gauge theory with $G = \mathsf{U}(N_1) \times \cdots \times \mathsf{U}(N_L)$, $T_C = \mathsf{U}(1)^L$, which may be enhanced to a non-abelian subgroup $G_H \subset SU(L+1)$.
- For $G = \mathsf{SU}(v)$ there is no topological symmetry.
- The topological symmetry can also be discrete, for example, when $G = \mathsf{SO}(3)$, $T_C = \mathbb{Z}_2$.

Summary

A 3d $\mathcal{N} = 4$ supersymmetric gauge theory is labelled by

1. A compact Lie group G
2. A linear quaternionic representation $\rho : G \to \mathsf{Sp}(n)$ acting on the vector space $Q = \mathbb{H}^n$.

It has the following global symmetries:

- R-symmetry $SU(2)_H \times SU(2)_C$
- Flavour symmetry $G_H \times G_C$ depending on the data (G, ρ).

Next Lecture

In the next lecture:

1. Central extensions of the super-Poincaré algebra.
2. Mass and FI parameters.
3. Supersymmetric vacua.
4. Domain walls.
5. Walls and chambers.