Universality of dimers via Imaginary geometry

Gourab Ray
University of Victoria

(ongoing) Joint project with

- N. Berestycki (U. Vienna)
- B. Laslier (U. Paris-Diderot)

- Plan

Lecture 1.

- Brief Intro to dimer model.
- The new perspective, connection to spanning trees.
- Overview of the results and future works.

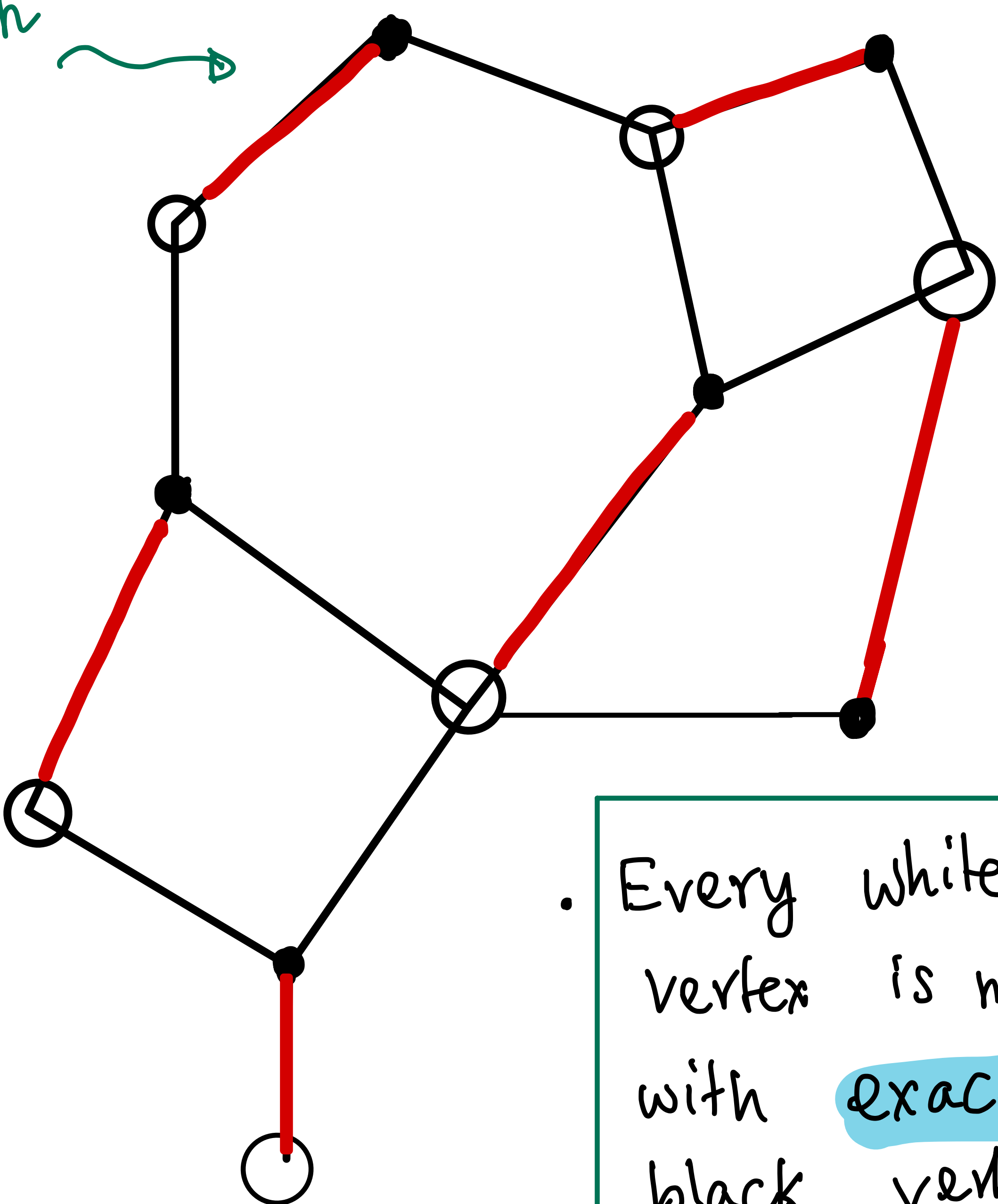
Lecture 2.

Detailed proof outline of the main theorem.

- A dimer model is a model of

$\boxed{\text{uniform perfect matching}}$

Bipartite graph



- Every white vertex is matched with exactly one black vertex

Dimer model

G: A bipartite graph.

m: A perfect matching.

$\mathcal{M}$: Set of all perfect matchings.

$$\boxed{\mu(m) = \frac{1}{|\mathcal{M}|}}$$

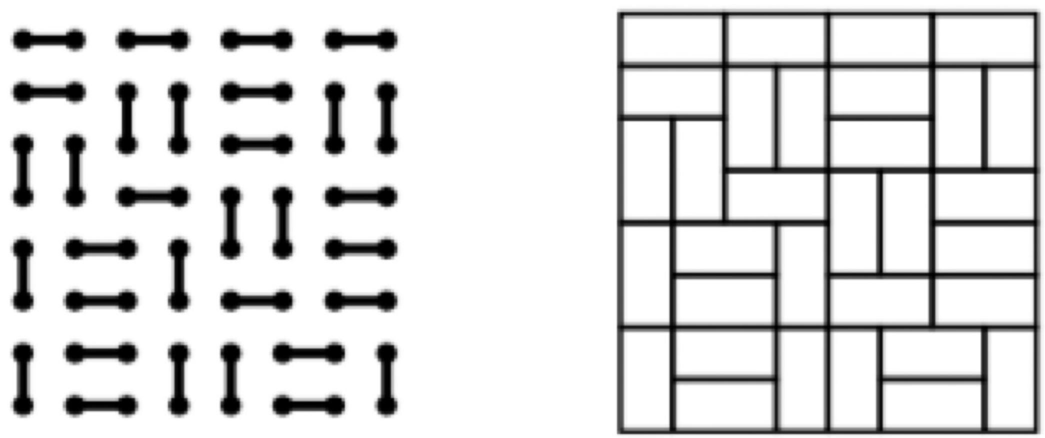
partition function.

μ: Prob. measure on perfect matchings.

Remark: - One can also add weights to edges.

- We will be mostly concerned with planar bipartite graphs

- Some special cases (with visual aspects)



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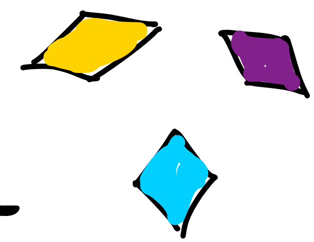
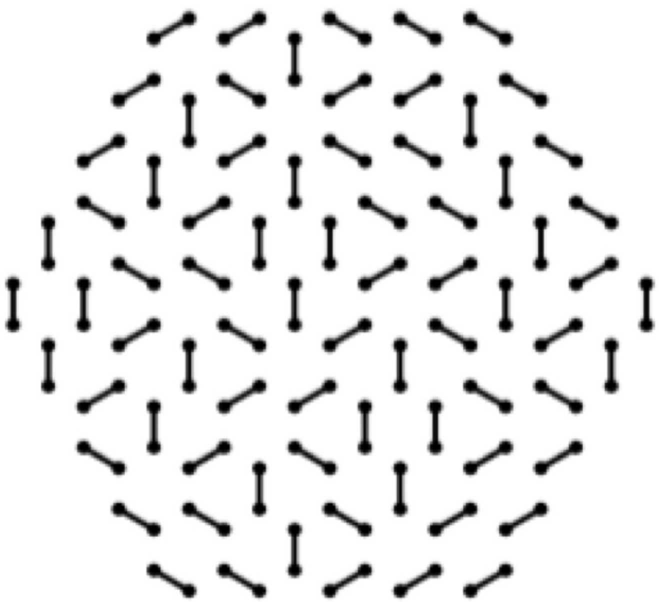
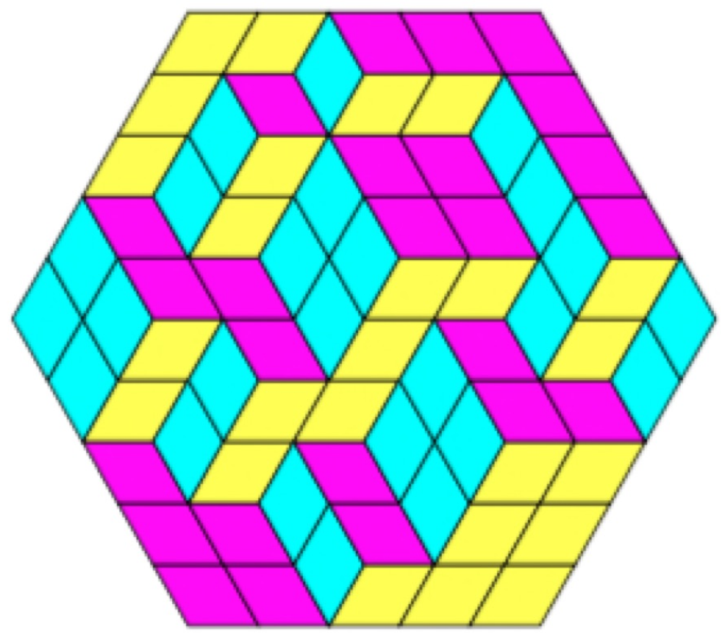
dimers on square lattice = Domino tilings.

- Some special cases (with visual aspects)

Lozenges







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Dimers on hexagonal lattice = Lozenge tilings

Some background

- This is a classical model in statistical mechanics, studied by
  - Kenyon, Propp, Lieb, Okounkov, Sheffield, Dubédat, De-Tiliére .... and many more.
- Usual approach.
  - Study Kasteleyn matrix. K (similar to the adjacency matrix).

- Z = $\det(K)$

(partition function)

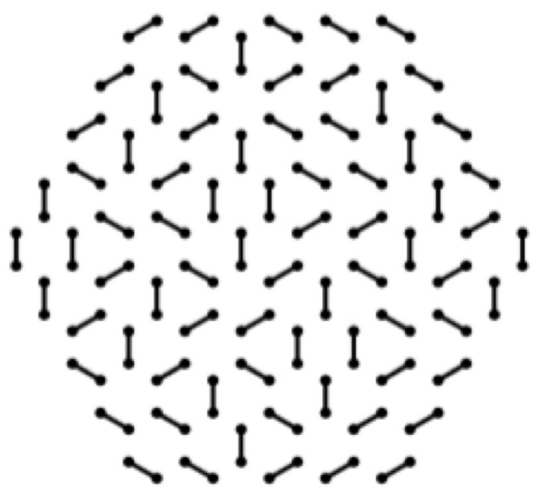
Kasteleyn matrix.

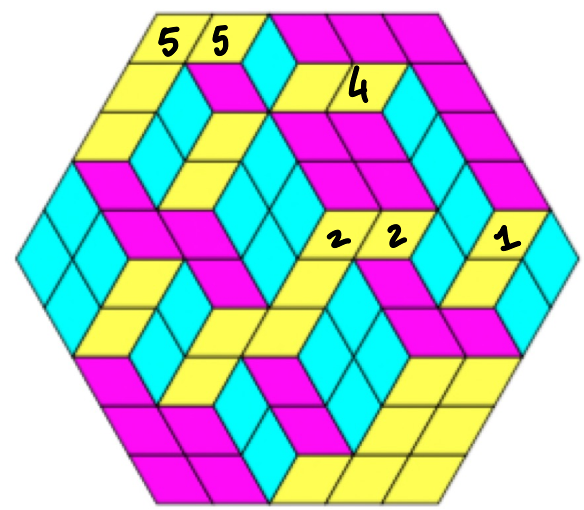
- (Exact solvability).

$$\boxed{Z_{m,n} = \prod_{j=1}^{m} \prod_{k=1}^{n} \left| 2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right) \right|^{1/2}}$$

dimers on $\mathbb{Z}_m \times \mathbb{Z}_n$ (m×n torus, square lattice).

- Dimers on planar bipartite graphs can also be encoded by height functions (introduced by Bill Thurston).





height function = height of cubes.

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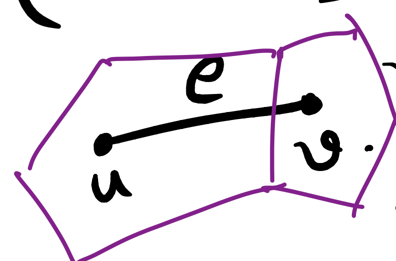
This coding is particularly visual for Lozenge tilings, ≡ stack of cubes

- A general way to define height function of dimers.

(For general bipartite planar graphs)

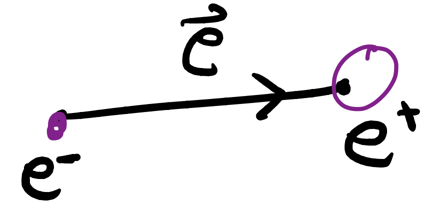
- It is a function $h : \text{Faces}(G) \mapsto \mathbb{R}$

- It is defined through its (discrete) gradient, ($h(v) - h(u)$ for 

)

(so defined upto global additive constant unless some value is fixed).

$\vec{E}$: oriented edges.



- Define $w(\vec{e}) = 1$ if $e^- = \bullet$, $e^+ = \circ$

dimer present.

(flow from black to white)

$= 0$ if dimer absent.

dimer flow

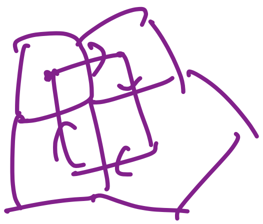
$$w(\vec{e}) = -w(-\vec{e}) \quad \text{(antisymmetric)}$$

- Notice $\displaystyle\sum_{e^- = u} w(\vec{e}) = 1$ if $u = \bullet$

$= -1$ if $u = \circ$

- Let $w_0 : \vec{E} \mapsto \mathbb{R}$ be **any** function $\Big\}$ reference flow

with
$$w_0(\vec{e}) = \begin{cases} 1 & \text{if } u = \bullet \\ -1 & \text{if } u = \circ \end{cases}$$

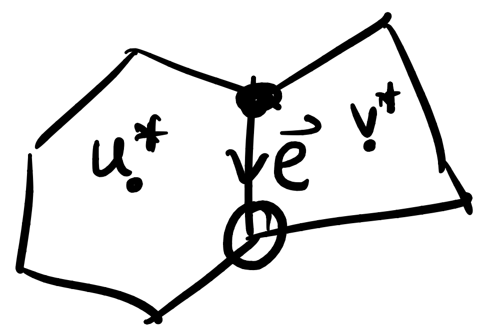


$$w_0(\vec{e}) = -w_0(-\vec{e}).$$

Then $\nabla = \boxed{w - w_0}$ is a **gradient flow**,

ie, $\displaystyle\sum_{e^- = u} \nabla(\vec{e}) = 0$ for any u.

Define **height function**



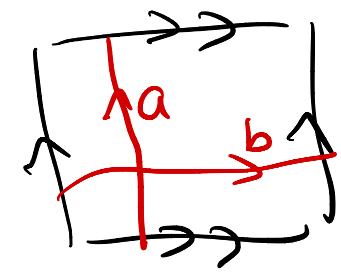
$$\boxed{h(v^*) - h(u^*) = \nabla(\vec{e})}$$

- What happens on multiply connected domains? (finite handles & holes).

$$\nabla = \underbrace{\nabla f}_{\text{an honest function on faces.}} + \underbrace{h}_{\text{harmonic 1-form.}}$$

(A version of hodge decomposition)

e.g.: On 2d. torus : $h \equiv \underbrace{(a, b)}$



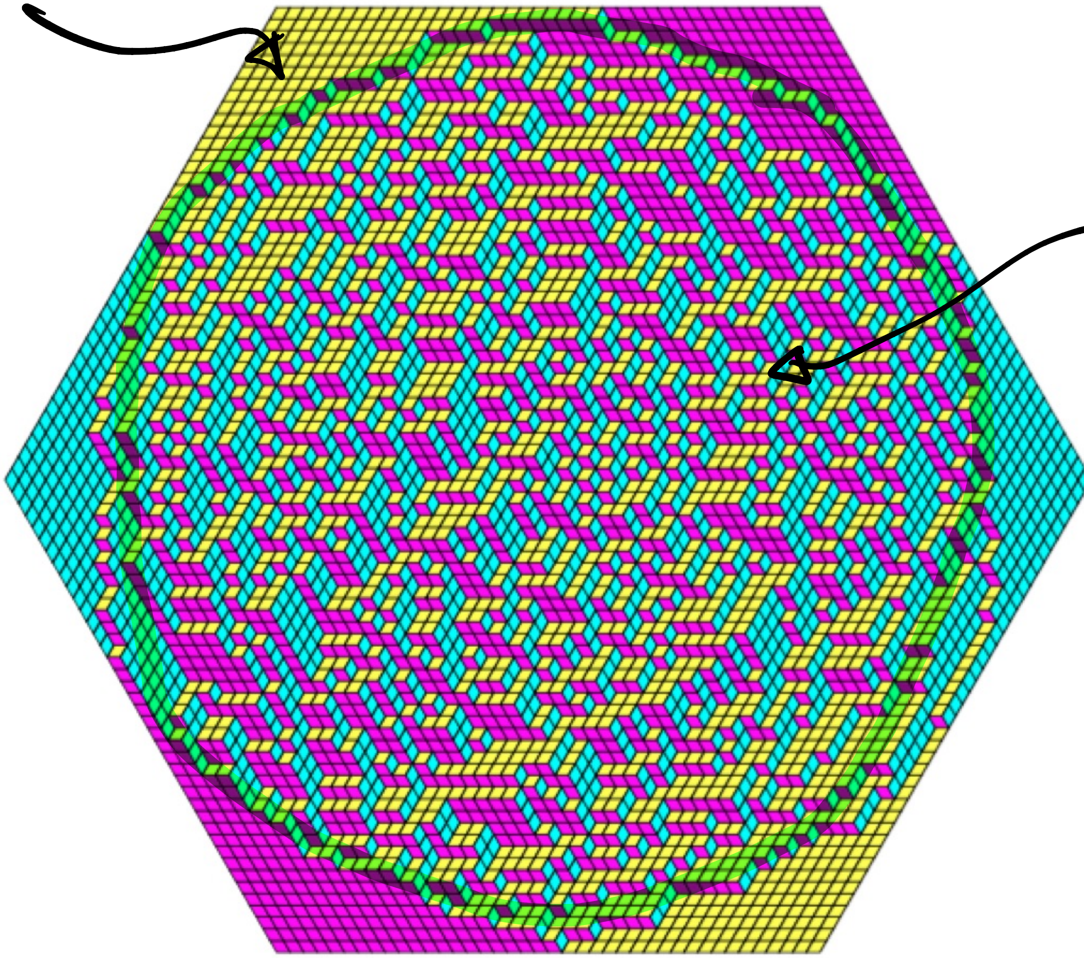
Goal: To understand large scale behaviour of $\boxed{\bar{h} = h - E(h)}$

$\underbrace{\qquad\qquad\qquad}_{\text{fluctuation}}$

E: Expectation/mean.

- Note $\bar{h}$ does NOT depend on the choice of reference flow.
- Fixing the boundary height can have a drastic effect (everything could be frozen).

Frozen



Liquid

(Kenyon, Okounkov, Sheffield: 'Dimers & amoeba' Ann. Math '06)

(Arctic circle phenomenon).

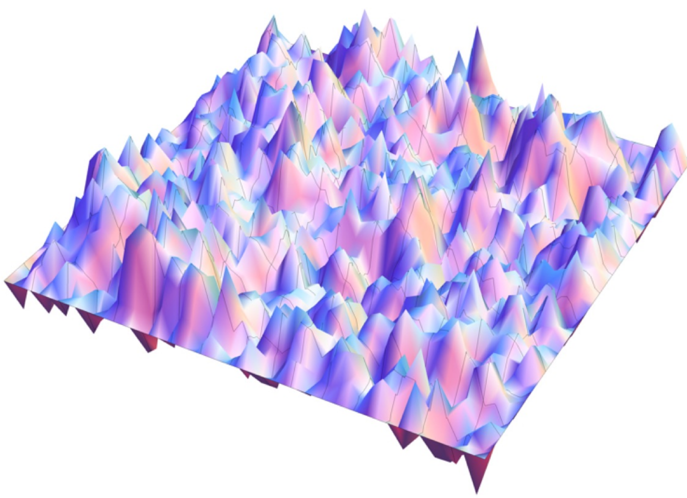
Conjecture: In the liquid region, the height function converges to the Gaussian free field in the scaling limit. (Dirichlet/zero boundary).

$\boxed{\text{GFF}}$ (in $D \subset \mathbb{C}$)

- Gaussian process $(h_x)_{x \in D}$
- Conformally invariant
- Random distribution

$$(h, f) \sim N\left(0, \int_{D^2} f(x)\, G^D(x,y)\, f(y)\, dx\, dy\right)$$

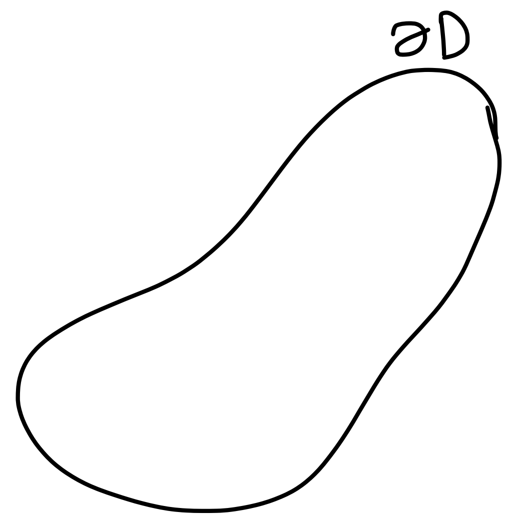
$G^D = -\frac{1}{2\pi} \Delta^{-1}$: Green's function in D.



NOT a random function!

GFF with boundary condition

$(u_x)_{x \in \partial D}$:



Zero boundary GFF

$+$ harmonic extension

of $(u_x)_{x \in \partial D}$.

Problem: Finding asymptotics of Kasteleyn matrix is hard for graphs with microscopic irregularities.

(e.g.: • A GFF scaling limit was shown by Kenyon in $\mathbb{Z}^2$.

• "Isoradial graphs": De Tilière

• Some related works by Bufetov, Gorin, Petrov for Lozenge tilings.

Our Goal

- Extend the convergence result to graphs satisfying
- Invariance principle:

$$\text{Random walk} \xrightarrow[\text{limit}]{\text{scaling}} \text{Brownian motion.}$$

- Equivalently, discrete harmonic $\longrightarrow$ cont. harmonic.
- Kasteleyn approach: "derivative of discrete harmonic $\longrightarrow$ "Derivative" of cont. harmonic

- Unfortunately invariance principle is not enough as

$$h^{\delta} - \mathbb{E} h^{\delta} \xrightarrow{\text{should}} \text{GFF}$$ without any scaling

$$\frac{X_1 + \dots + X_n - \mathbb{E}(X_1 + \dots + X_n)}{\sqrt{n}\sqrt{\mathrm{var}(X_1 + \dots + X_n)}}$$

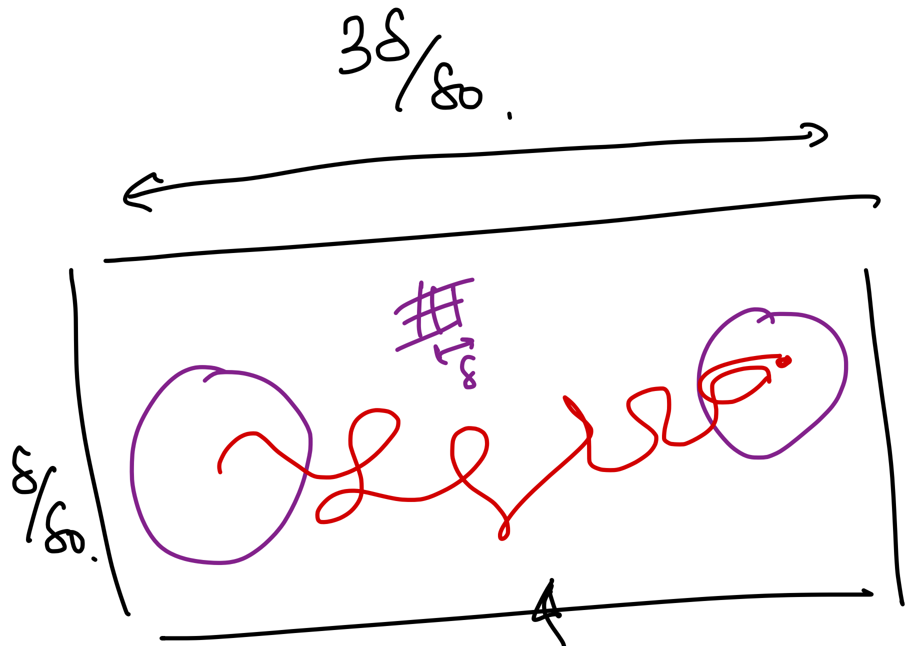
So we need to essentially control the microscopic scales as well.

- In the simply connected case, we will even get some information on $\mathbb{E} h^{\delta}$.

Russo-Seymour-Welsh assumption (for microscopic scales)

- Key idea in 2d-Statistical physics models.

- $\exists c, \delta_0 > 0$

$\mathbb{P}\Big($ 

$\Big) > c$

Random walk crosses the rectangle without exiting.

Key Idea: Use a well known correspondence between dimers and Uniform Spanning trees

- Combine this with → ◦ Scaling limit result of UST (branches are SLE_2).

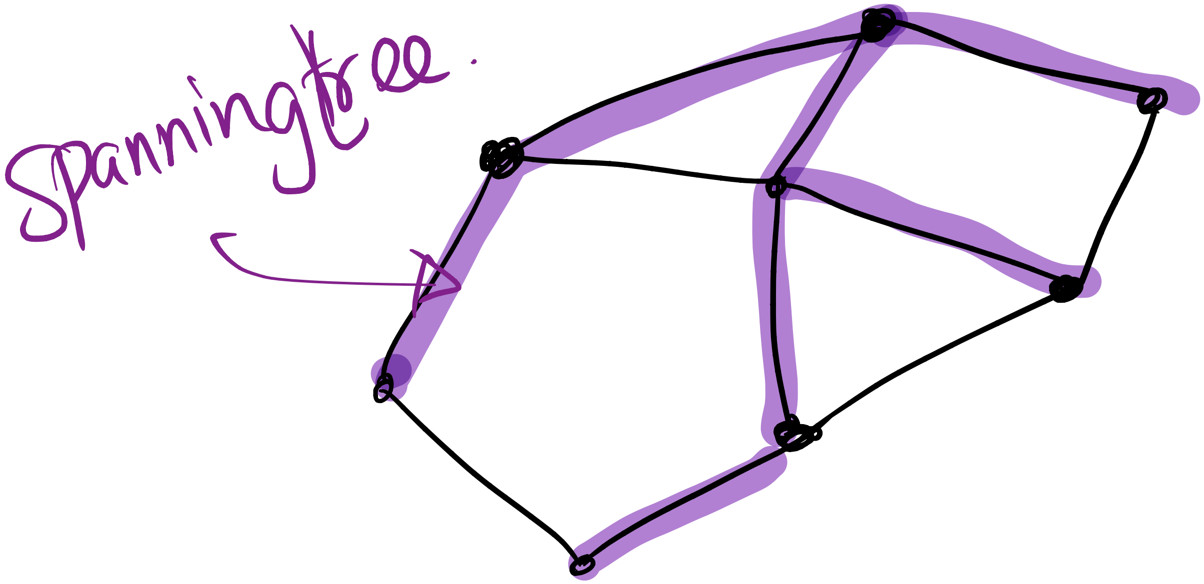
  ◦ Imaginary geometry

  (coupling between GFF and SLE via "winding")

Uniform spanning tree:

uniformly picked.

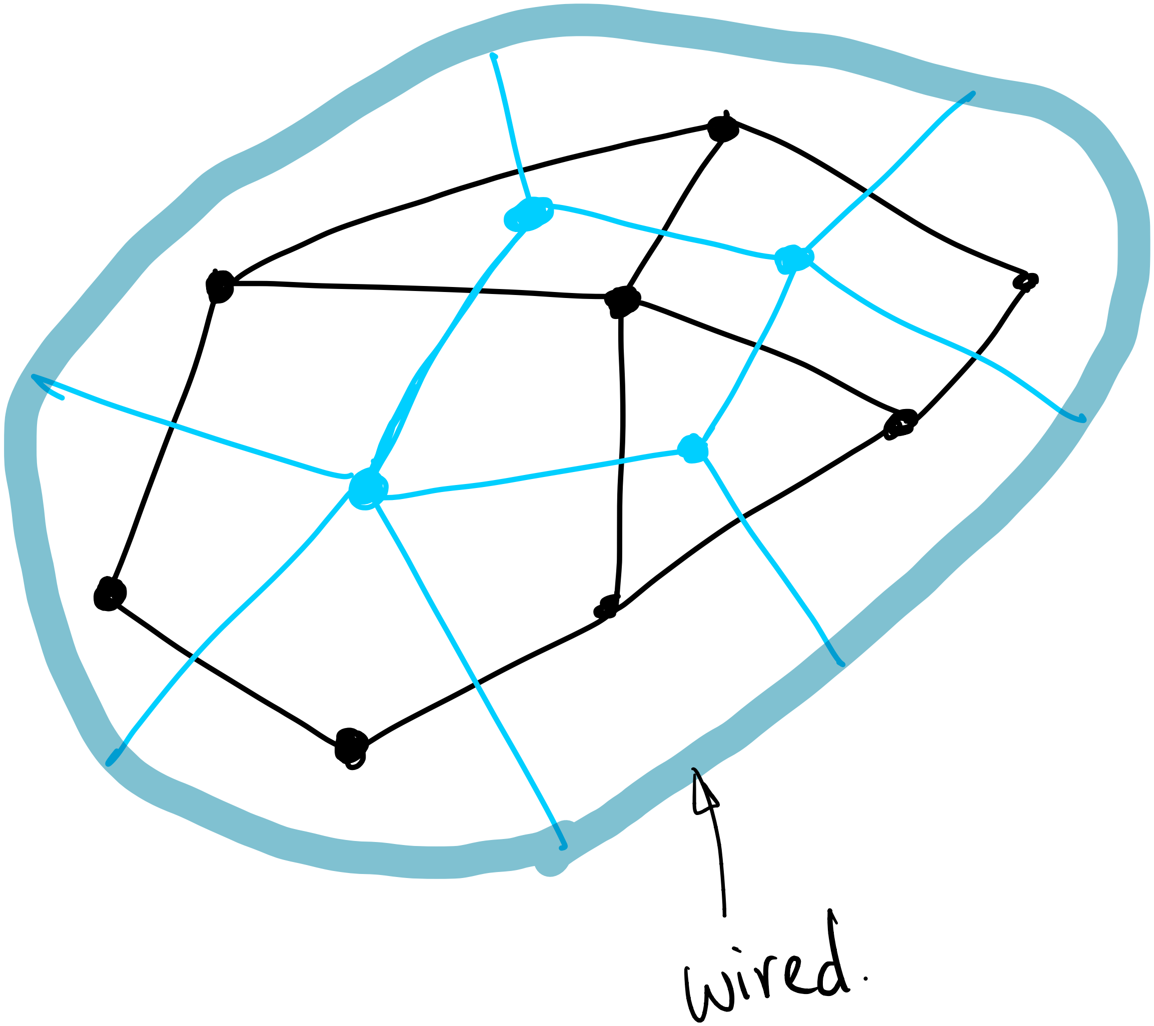
spanning tree



- Dimers and spanning trees can be related in various ways.

  - Temperley-Fisher Correspondense
  - T-graphs (Sheffield-Kenyon: Dimers, Tilings and trees)

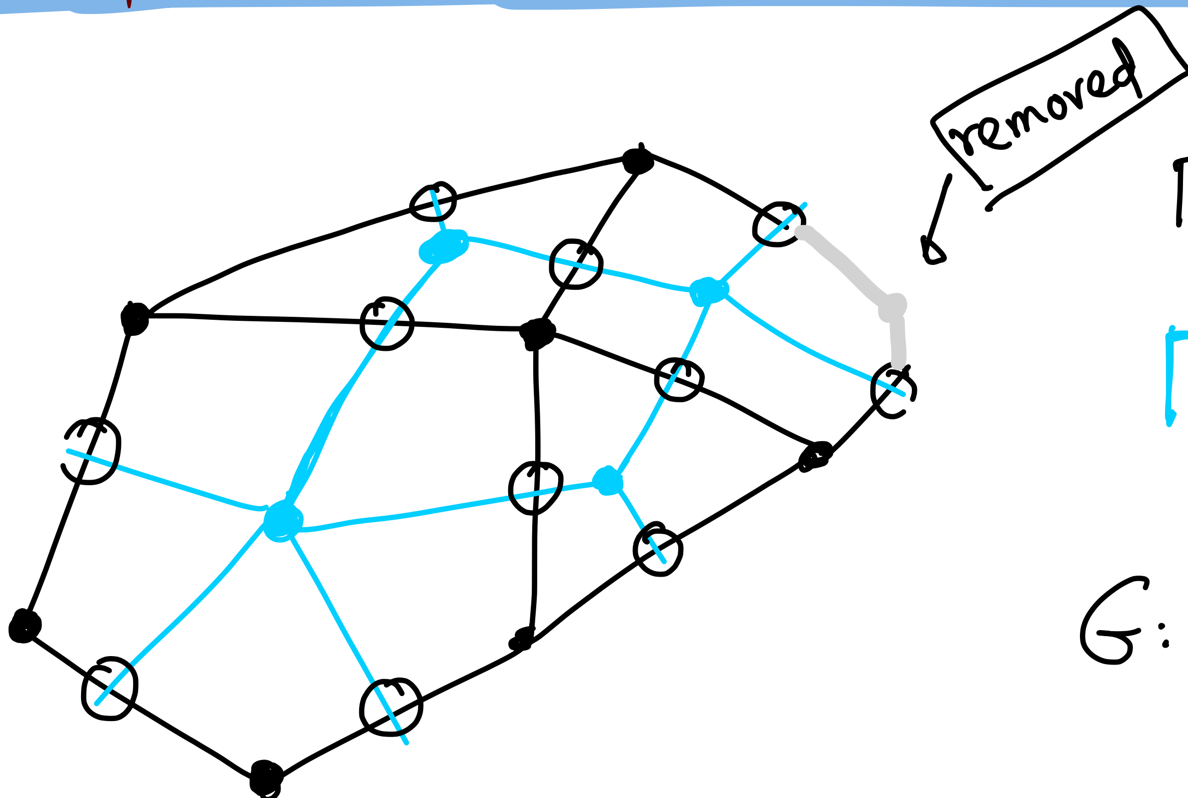
Temperley Fisher bijection



Γ: black graph.

Γ^{+}: Dual graph.

Temperley Fisher bijection



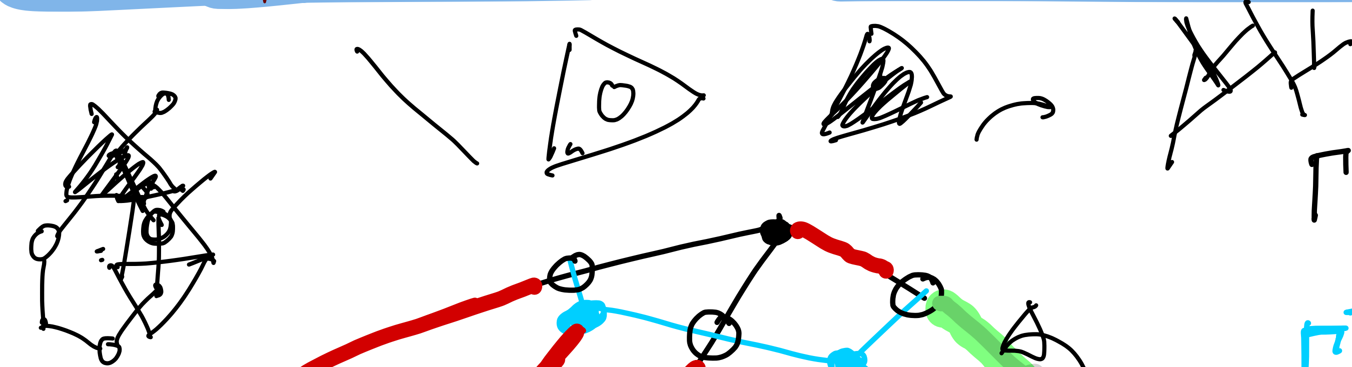
Γ: black graph.

Γ^+: Dual graph.

G: $\Gamma \cup \Gamma^+ \cup$ white vertices.

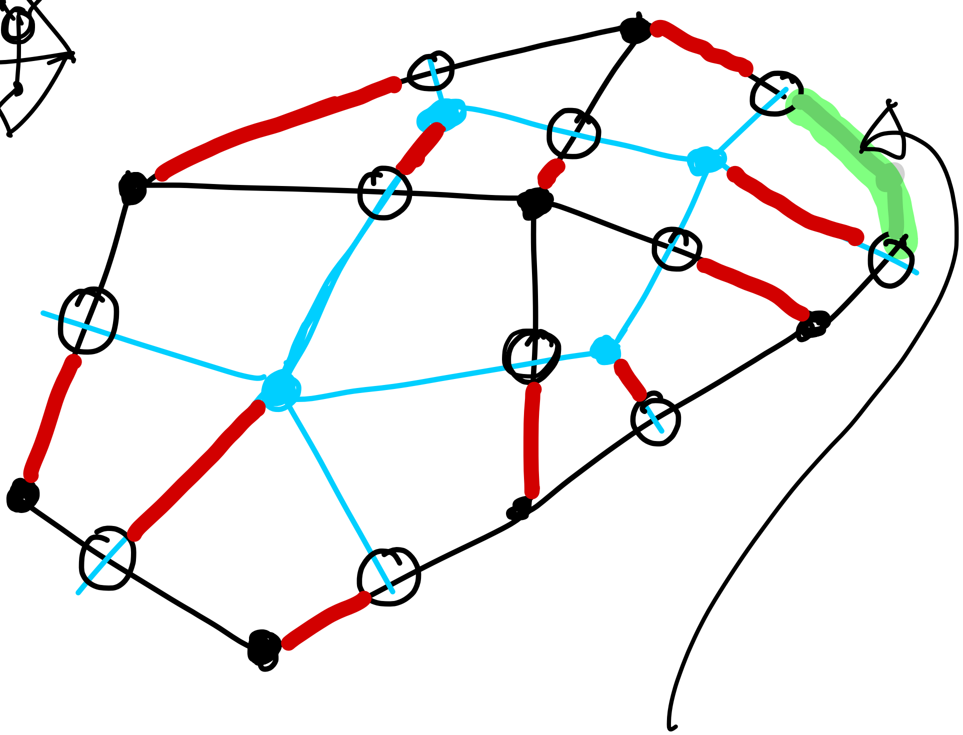
$\setminus$ {one grey vertex} on boundary.

Temperley Fisher bijection



Γ: black graph.

$\Gamma^\dagger$: Dual graph.



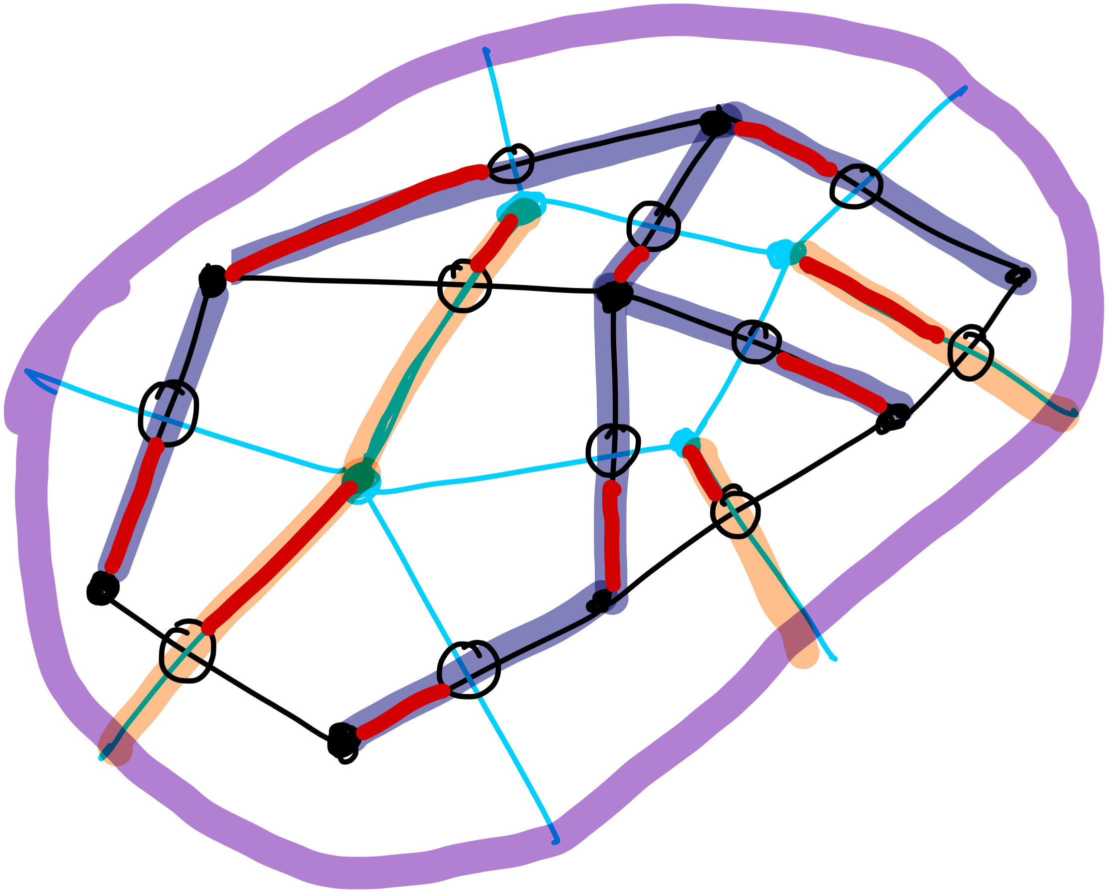
REMOVE ONE BLACK VERTEX.

G: $\Gamma \cup \Gamma^\dagger \cup$ white vertices. $-$ {one black vertex} on boundary

(bipartite)

— : dimer on G.

Temperley Fisher bijection



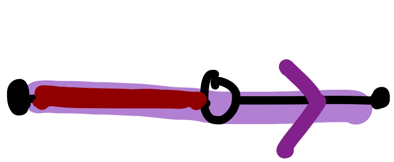
Γ: black graph.

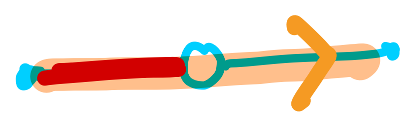
Γ^+: Dual graph.

G: $\Gamma \cup \Gamma^+ \cup$ white vertices.

(bipartite)

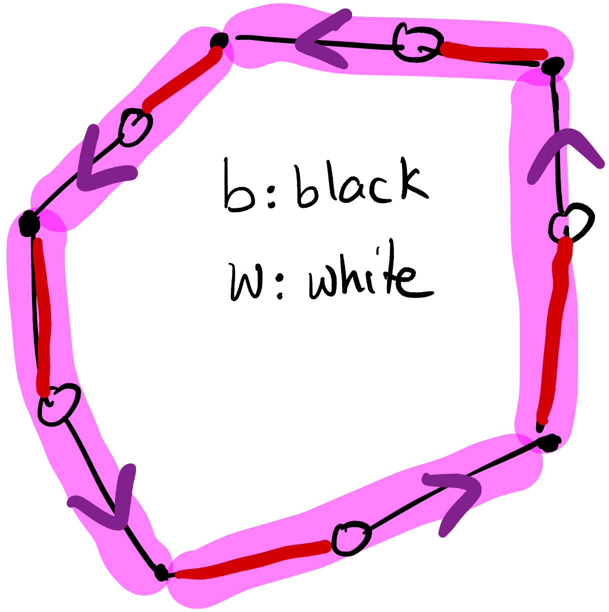
Local map:





spanning tree of Γ + spanning tree of Γ^+

Why a tree?



#w: # edges. (in Γ)

$$\#V + \#F \underbrace{-1}_{\text{external face.}} = \#b.$$

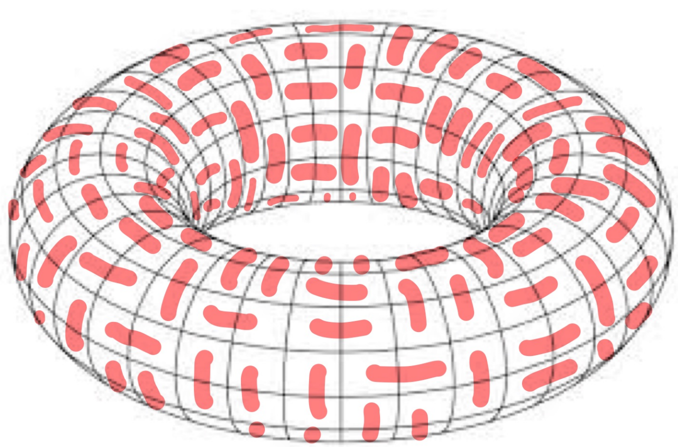
Euler: $\#V - \#E + \#F = 2$

$$\#b + 1 - \#w = 2.$$

but we need $\boxed{\#b = \#w.}$ $\Rightarrow$ No cyles.

- This is why we need to remove a black vertex from the boundary.

Extension to Multiply Connected domains.

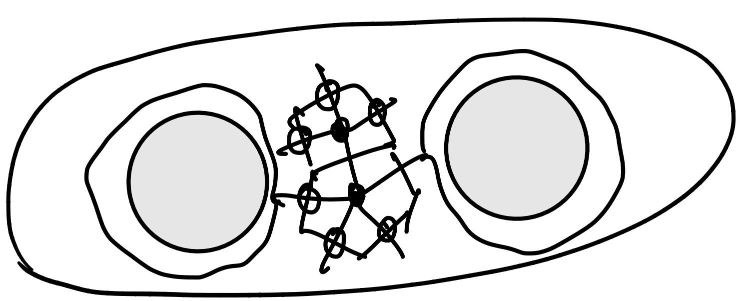


$$\#V + \#F = \#b.$$

$$\#E = \#W.$$

<u>Euler</u>: $\#V - \#E + \#F = 2 - 2g - b$

$$\#b - \#W = 2 - 2g - b$$



$\#b = \#W$ iff $2g + b = 2$

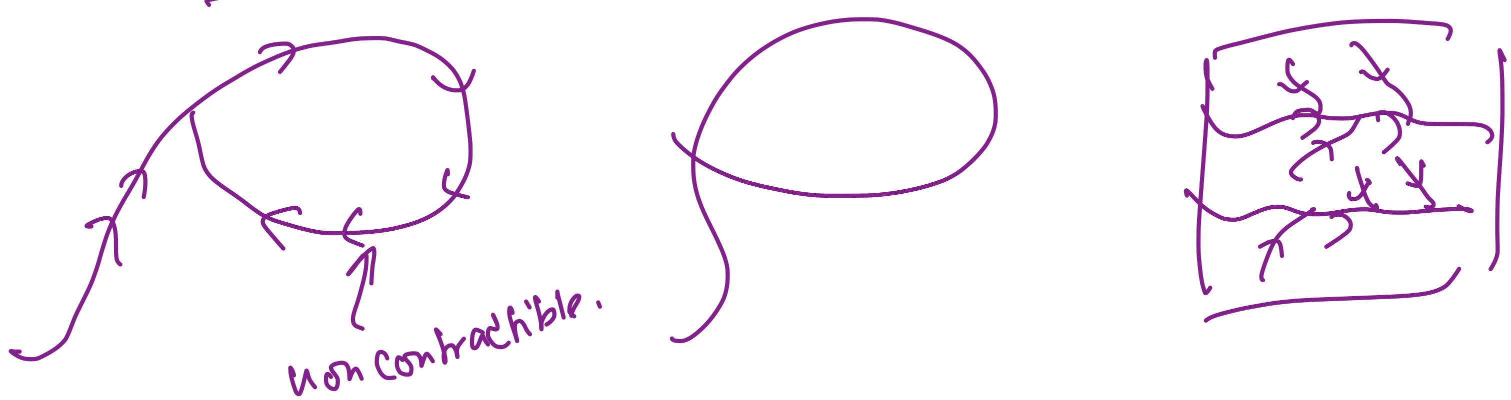
$\Rightarrow g = 1,\ b = 0$

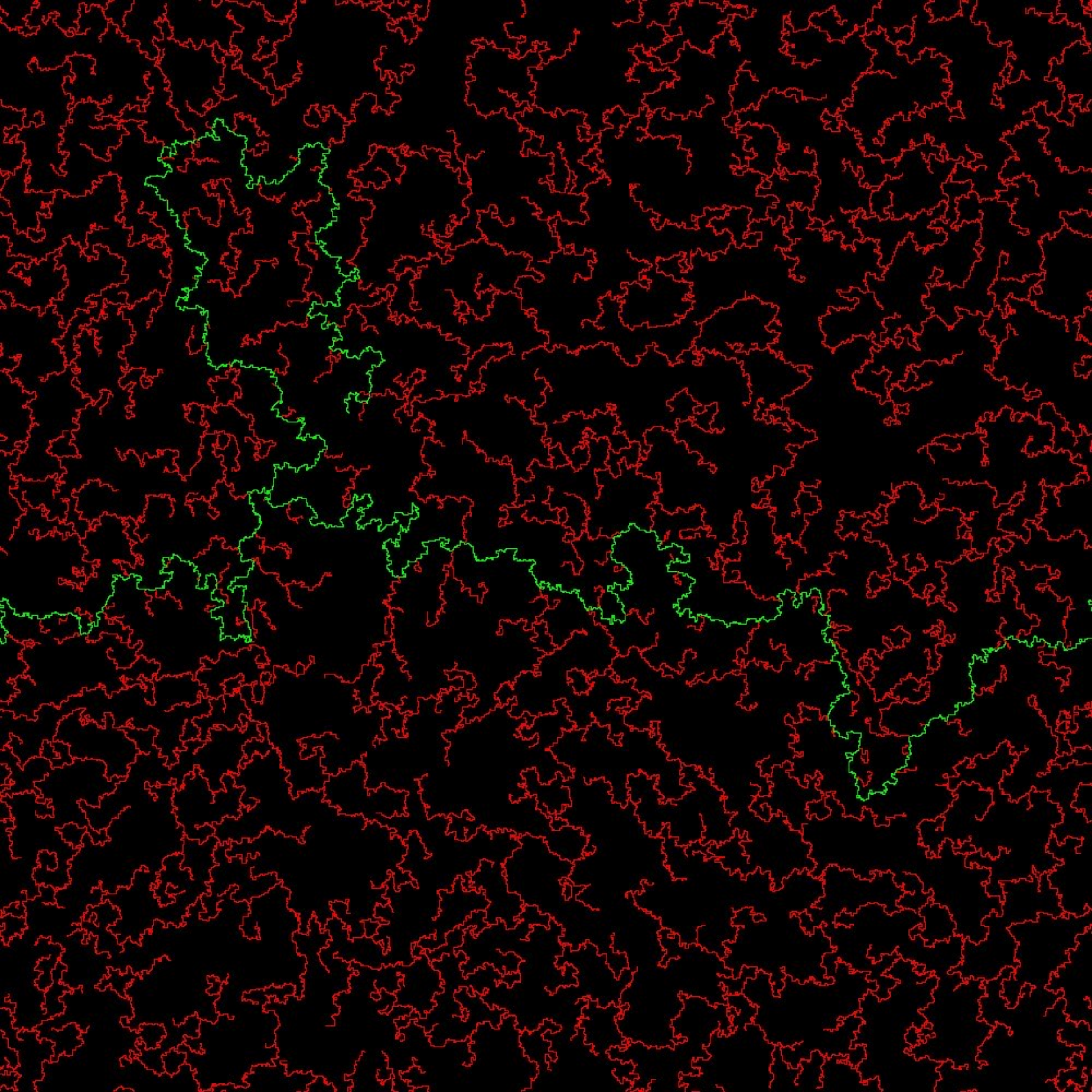
OR $b=2, g=0$. (torus/annulus).

- In other cases, we need to remove $2g+b-2$ edges (ie white vertices) to make it dimerable

(we call them puncture).

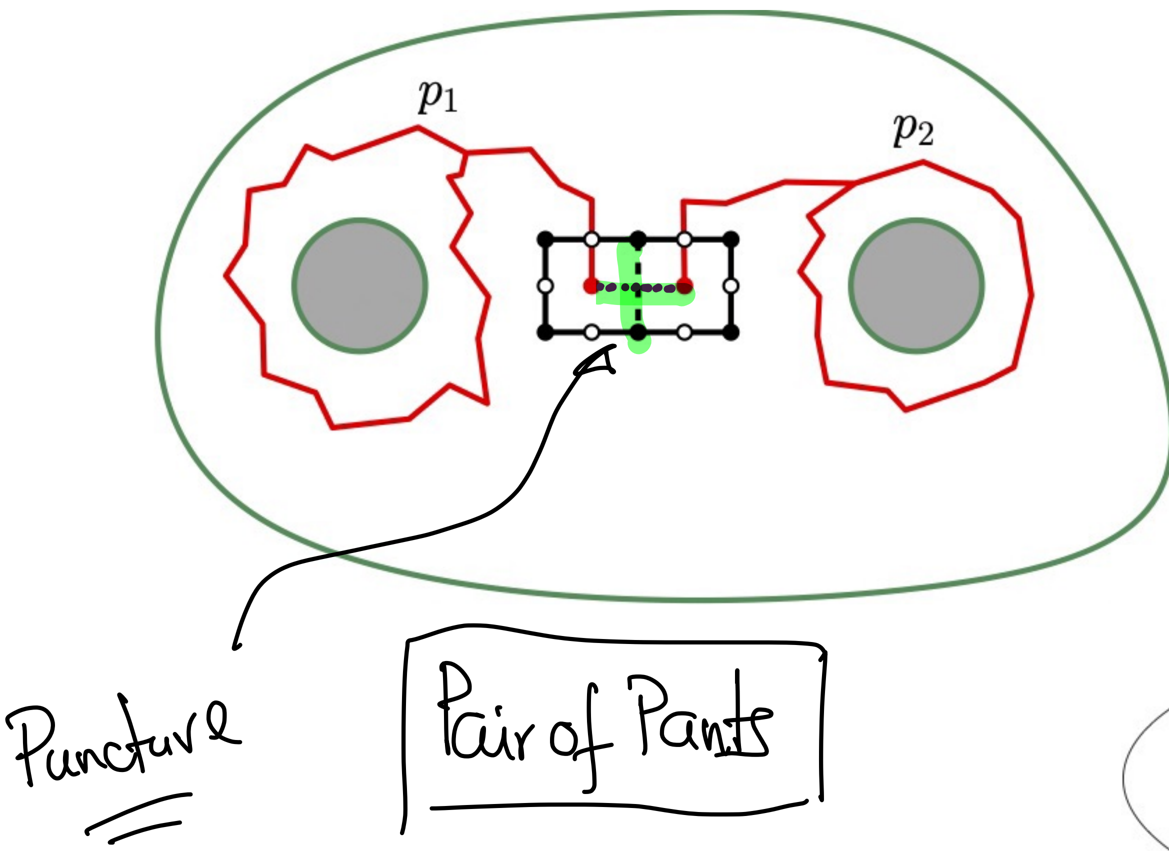
- On torus, Temperleyan bijection gives cycle rooted spanning forests.

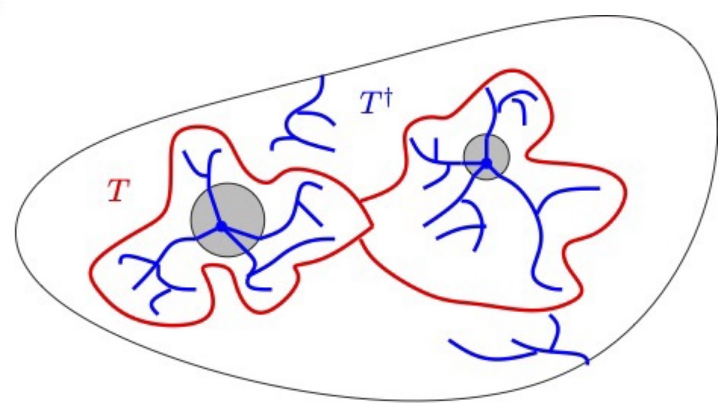




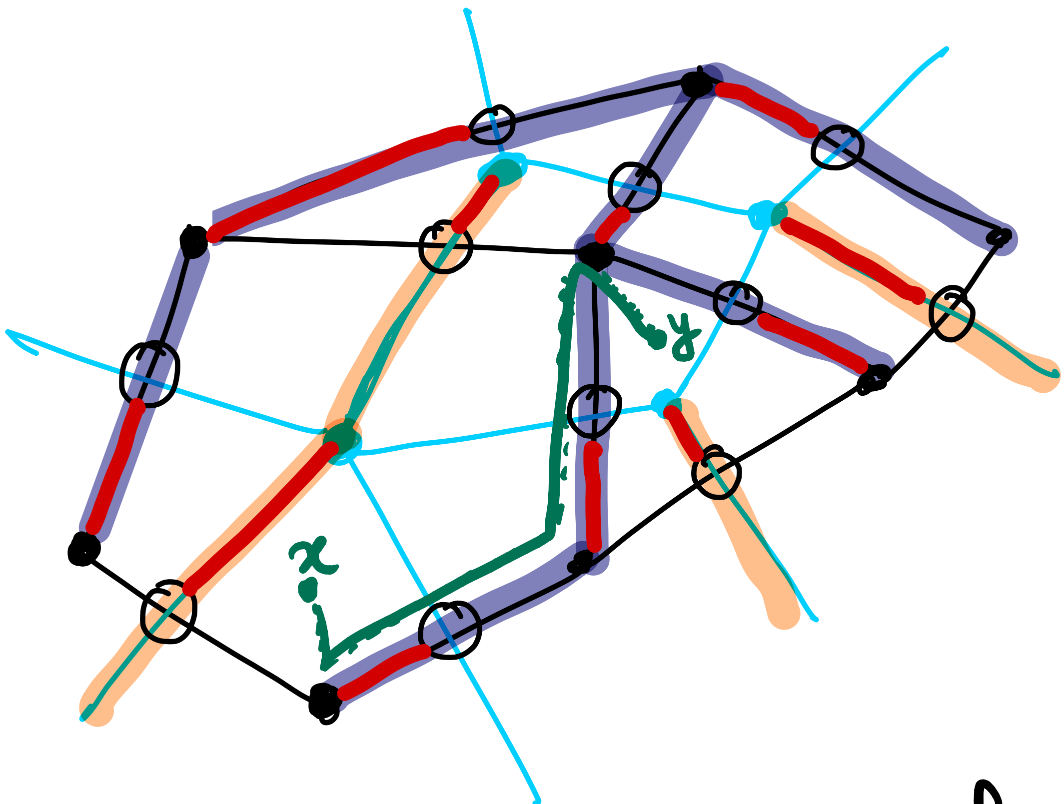
• In general, dimers are not in bijection with cycle rooted spanning forests but a special subclass:

cyles from punctures must divide the manifold into annuli





(Remarkable) Observation by Benjamini.

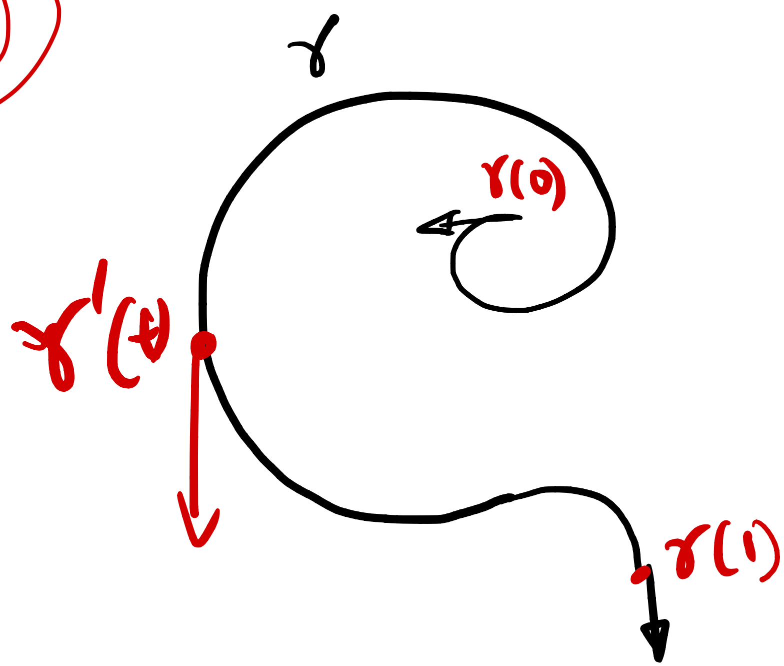


$h_x - h_y$

= "Winding" of the green path.

for a well chosen reference flow.

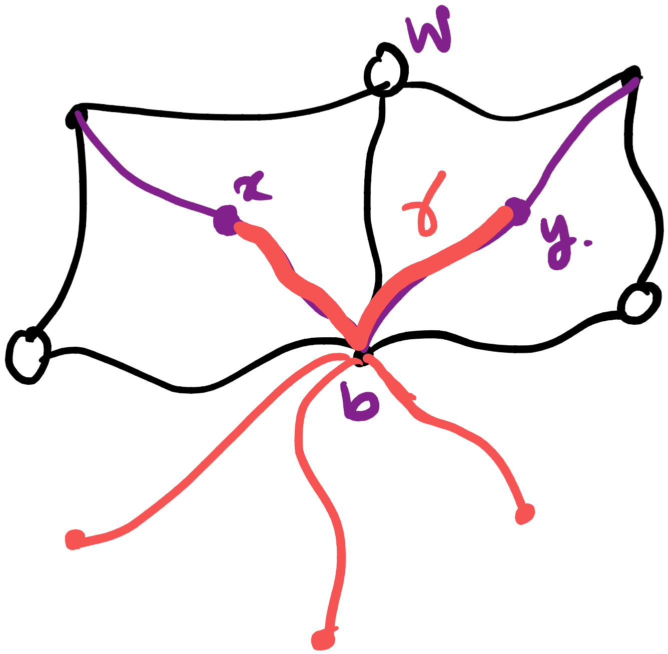
- Winding (for smooth curves) is the "amount a curve has turned"



$$\text{Winding}(\gamma) = 3\pi - \frac{\pi}{2} = \frac{5\pi}{2}.$$

$$\gamma'(t) = r(t)e^{i\theta(t)} \longrightarrow \begin{cases} = \int_0^1 \theta(t)\,dt \\ = \int_0^1 \arg(\gamma'(t))\,dt. \end{cases}$$

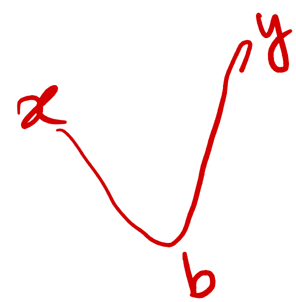
Our version of this reference flow

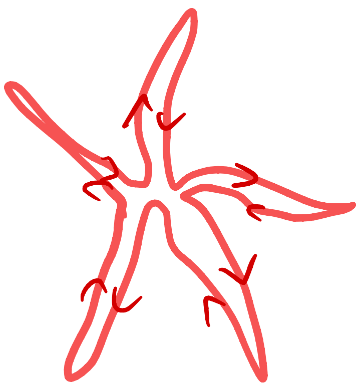


$$\omega_{ref}(b \to w) = \frac{1}{2\pi}\underbrace{\left(W(\gamma) - \pi\right)}$$

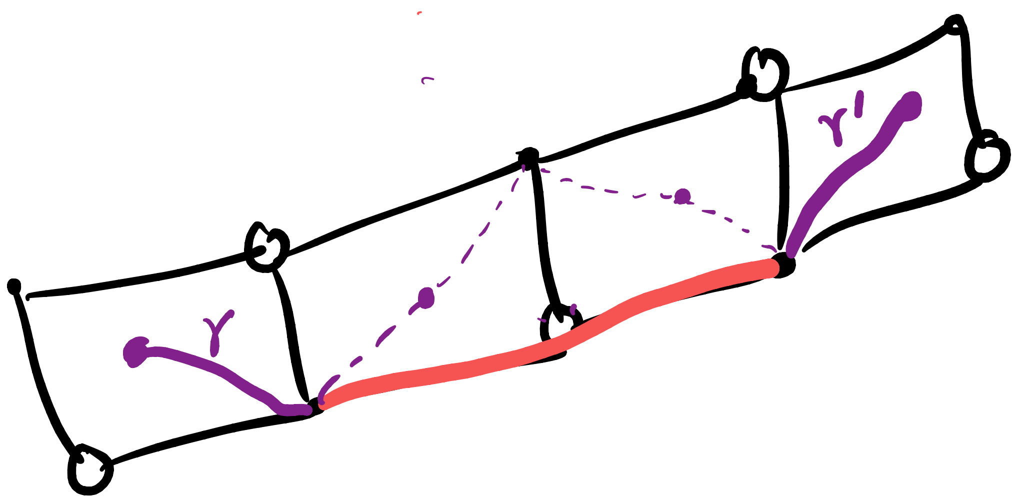
$$\|$$

winding of





$$\underbrace{\sum_{w \sim b} \omega_{ref}(b \to w)} = -1.$$

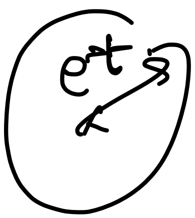


Winding = height function.

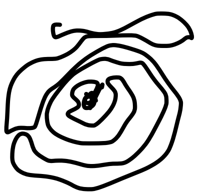
white and black alternately add $\pm\pi$.

Overall:

Dimers $\xleftrightarrow{1\text{-}1}$ spanning tree + dual spanning tree pair.



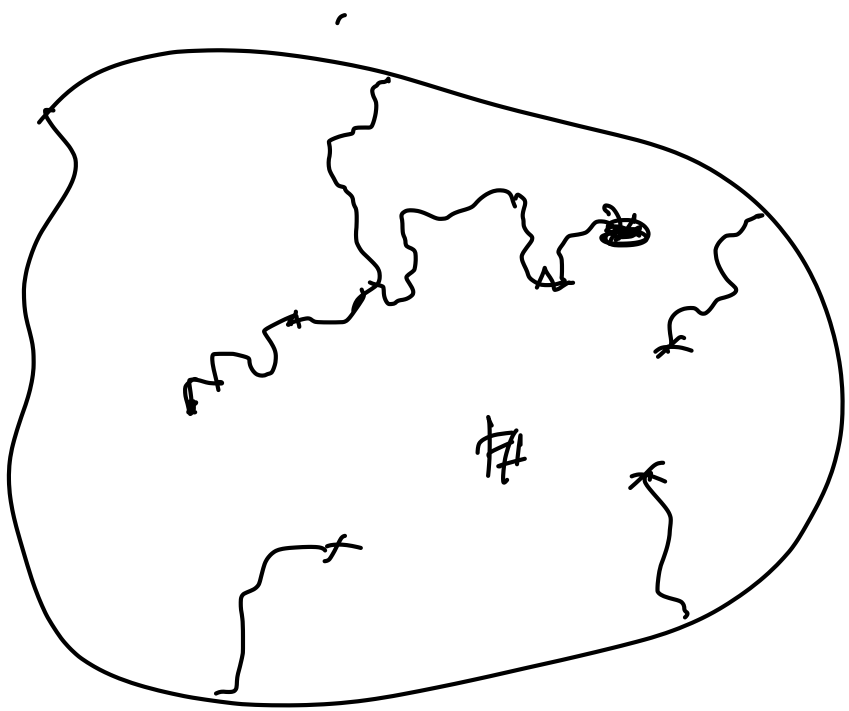
$\Leftrightarrow$ Uniform dimer $\xleftrightarrow[\text{measure preserving bijection.}]{}$ Uniform spanning tree.



height change of dimer $\longleftrightarrow$ winding of tree.

- Connection between UST / Random walk : Wilson's algorithm.

- Loop erased random walk.



LERW $\to$ SLE_2.

Radial SLE$_\kappa$

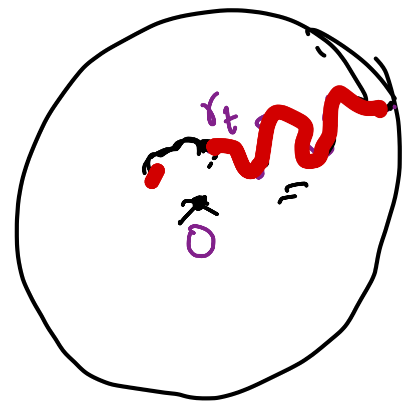
$\boxed{\text{From } 1 \text{ to } 0}$ in $\mathbb{D}$ is described by

- A family of continuous curves $(\gamma_t)_{t\geq 0}$
- a unique family of conformal maps $\boxed{(g_t)}_{t\geq 0}$ such that

$$g_t : \mathbb{D}\setminus\gamma_t \xrightarrow[\text{conf.}]{} \mathbb{D}$$

$$g_t(0)=0$$

$$g_t'(0)>0.$$



- g_t satisfies the Loewner equation

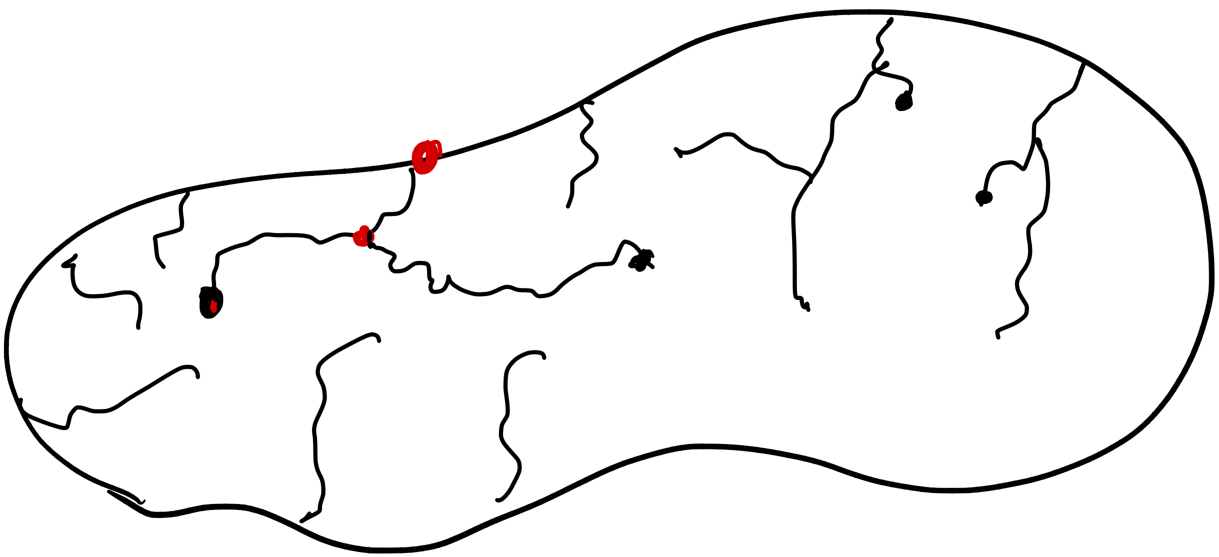
$$\frac{\partial g_t(z)}{\partial t} = g_t(z)\,\frac{e^{i\sqrt{\kappa}B_t}+g_t(z)}{e^{i\sqrt{\kappa}B_t}-g_t(z)}$$

$\forall z\in \mathbb{D}\setminus\gamma[0,t].$

- Thm (Lawler-Schramm-Werner / Yadin-Yehudayoff).

Loop erased random walk $\longrightarrow SLE_2$.

- Wilson's algorithm in continuum



Thm (Yadin-Yehudayoff).

If Random walk $\longrightarrow$ Brownian motion

then branches of UST $\longrightarrow$ SLE_2 curves

$\downarrow$

$\boxed{\text{fractal curves, Hausdorff dim. } 5/4. \text{ (Beffara)}}$

Schramm-Loewner Evolution

$$\text{height function} \xleftarrow{\text{winding}} \text{Unif. Spanning tree}$$

$$\text{height function} \overset{?}{\dashrightarrow} \text{GFF}$$

$$\text{Unif. Spanning tree} \xrightarrow{\text{scaling lim.}} \text{Cont. Spanning tree}$$

$$\text{GFF} \overset{?}{\dashleftarrow} \text{Cont. Spanning tree}$$

Problem: Each branch winds ∞-often in both positive and negative direction

(as it should since it should NOT be a random function).

- **Program**

Step 1: Show that winding of UST

$\longrightarrow$ Conf. invariant "continuum winding field".

Step 2: Identify the limit using

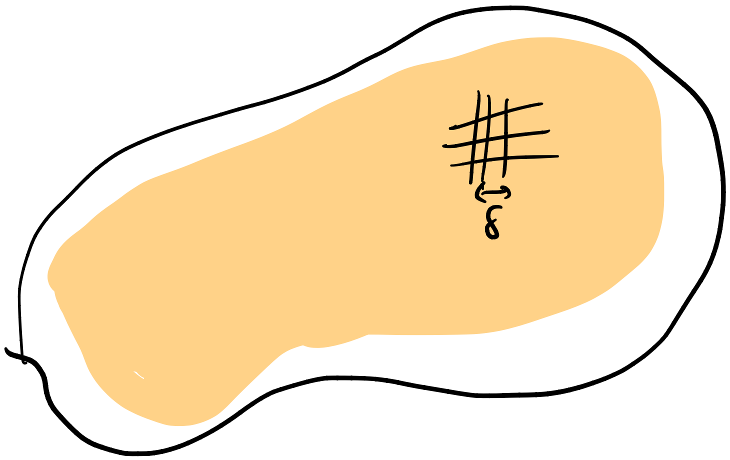
Imaginary geometry

(Miller-Sheffield).

Theorem: (Berestycki, Laslier, R '20)

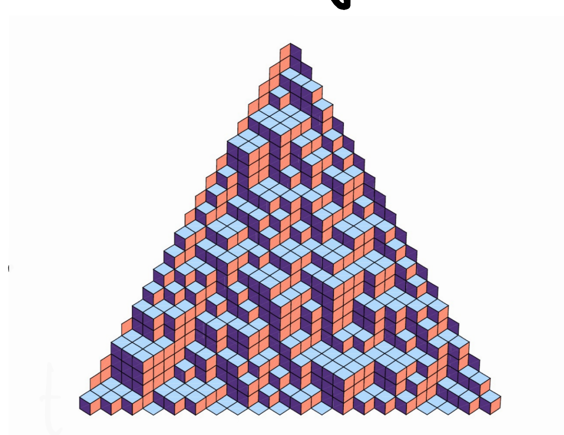
(Simplest version)

- G^δ : Sequence of graphs with mesh size δ. Then.



Winding of (wired) UST (Dimer height function). $\to$ GFF with winding boundary condition.

- Temperleyan graphs.
- Lozenge tiling with slope.
- Dimers in random environment.



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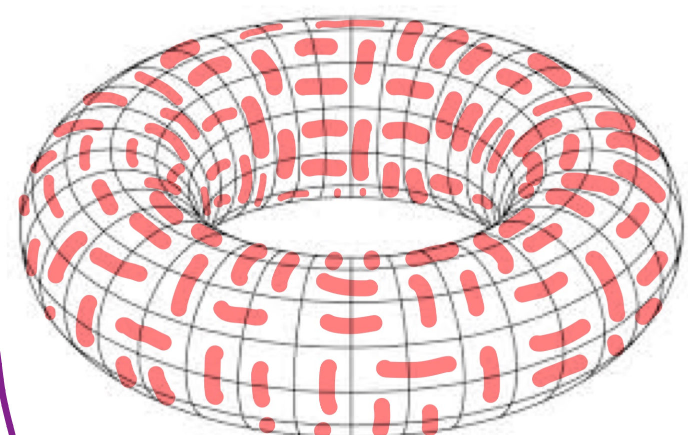
Theorem: (Berestycki, Laslier, R' 20)

G^δ : sequence of Temperleyan graphs on a Riemann surface with finitely many handles and holes. satisfying

- invariance principle
- RSW crossing property.

- height one form has a scaling limit which is conformally invariant and universal.

[In torus the limit is identified as a compactified GFF (Kenyon/Dubédat)]



Imaginary geometry

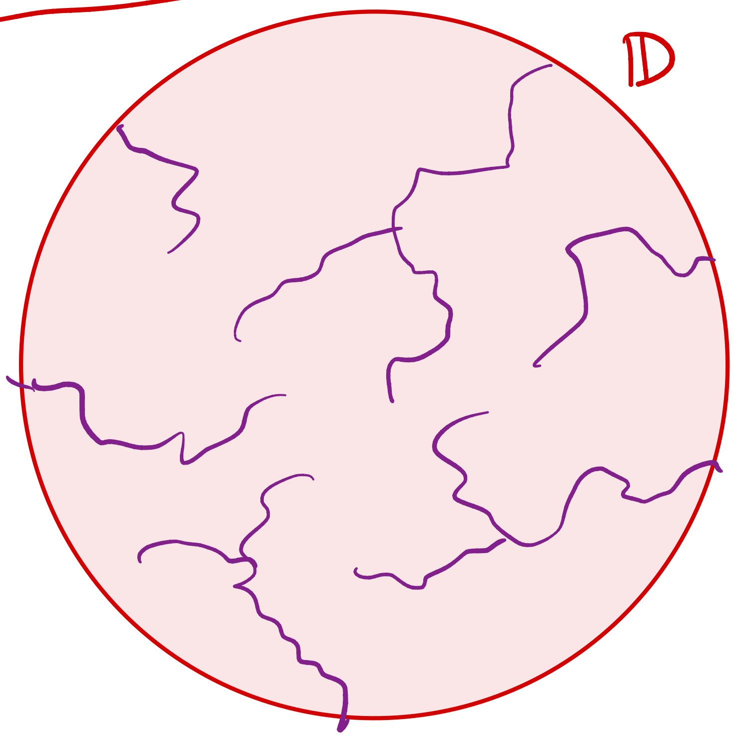
Idea: $\left(e^{ich_x}\right)_x \approx$ vector field.

$\rightarrow$ flow lines $\approx$ SLE_κ.

SLE_2: UST branches.

Imaginary Geom coupling ($\kappa=2$).

(Dubédat, Miller-Sheffield)



$\exists$ coupling between UST and GFF on $\mathbb{D}$ such that:

A: Finite collection of UST branches.

$g_A : \mathbb{D} \setminus A \xrightarrow{\text{Conf.}} \mathbb{D}$

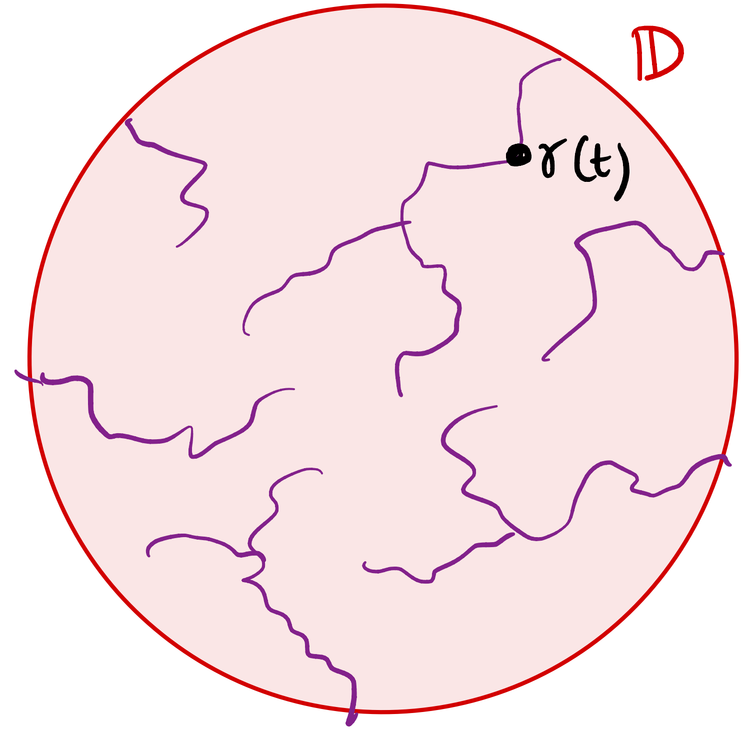
Given A, law of GFF.

$$\mathrm{GFF}^0(\mathbb{D}\setminus A) + \frac{1}{\sqrt{2}} \arg\left(g_A'(\cdot)\right)$$

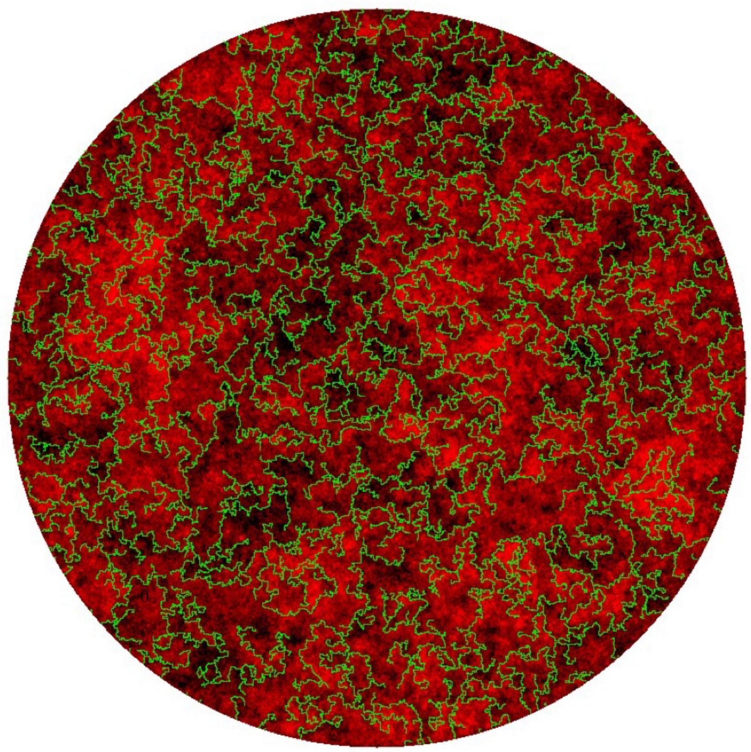
- $\arg\left(g_A'(\gamma(t))\right)$

  $\approx \arg\left(\gamma'(t)\right)$

- Does not make direct sense as curves are rough.



UST / GFF imaginary geometry coupling

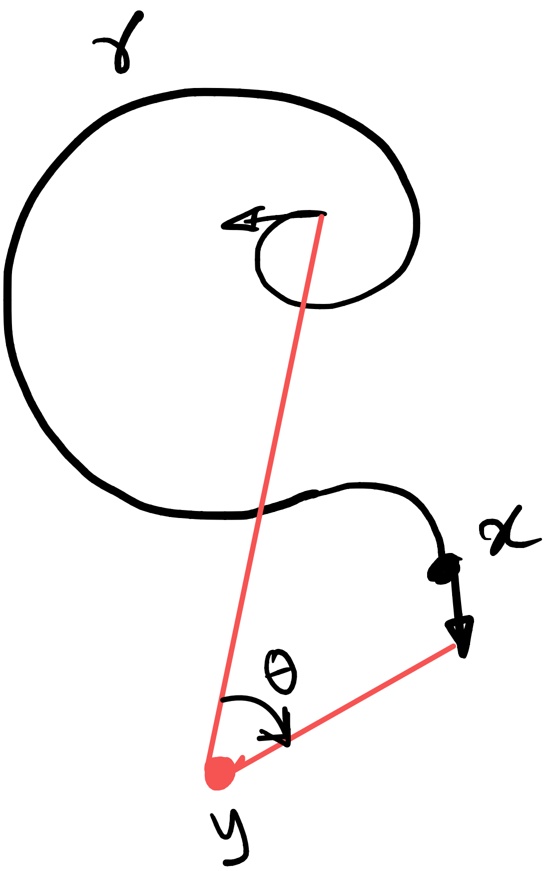


Remark:

UST is a function of GFF in this coupling.

Issue: Imaginary geometry does not calculate winding "hands on".

- We do this directly to connect with the height functions.

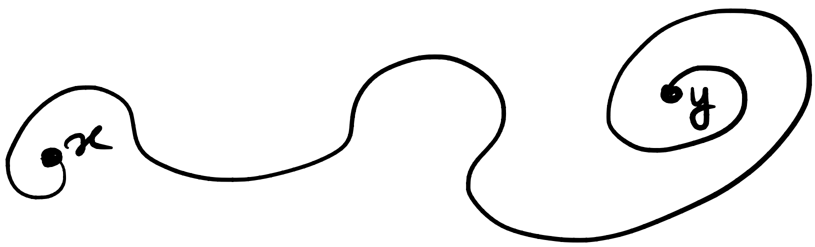


Key (but elementary) idea

"Calculate winding seen from an external point". y.

Observation: This makes the winding "continuous" as long as $x \neq y$.

Lemma



$$W(\gamma, x) + W(\gamma, y) = W(\gamma)$$

winding seen from x

winding seen from y.

winding

approximation:

Cut off the branch.

$\varepsilon = e^{-t}$

Regularization of winding field.

$$\int h^{\delta}(z) f(z) dz = \int \left[h^{\delta}_t(z) + \epsilon^{\delta}_t(z) \right] f(z) dz$$

$h_t^\delta(z) \approx$ continuous part.

$\epsilon_t^\delta(z) \approx$ microscopic part.

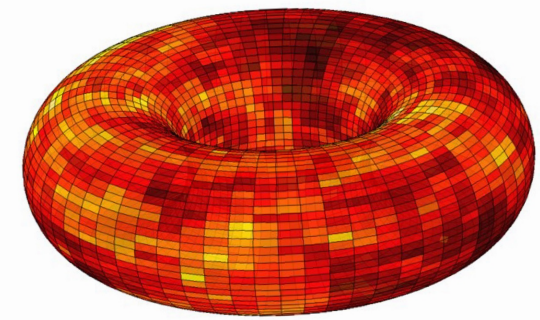
- <u>Continuous argument + Imaginary geometry</u>

$$\lim_{t\to\infty} \lim_{\delta\to 0} h_t^\delta(z) = \text{GFF}$$

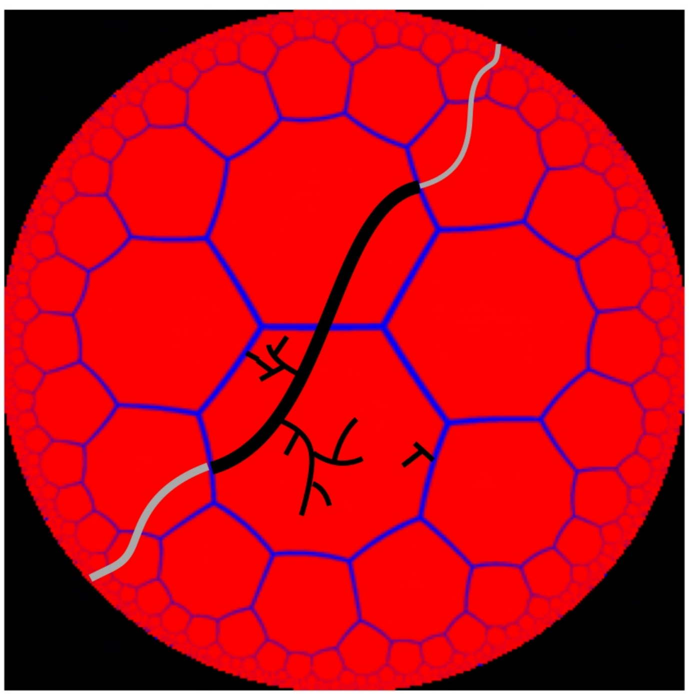
- <u>Discrete argument.</u> (Multiscale coupling)

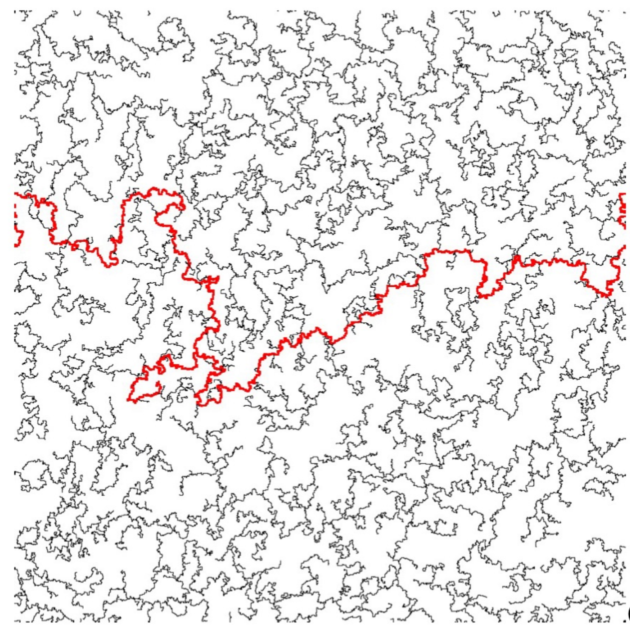
ϵ_t^δ : "Independent" for different z

- For Riemann surfaces:
  Lift to the universal cover
  and use similar ideas.



- No imaginary geometry!
  (No identification yet).





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Future:

(1) Extend Imaginary geom. to other geometries.

(2) Interacting dimers (?) other SLE curves.?

- Smaller goals

- Say something more about the height function in a pair of pants.

- Dimers in non-elliptic random environment (percolation/ Voronoi tiling etc).