Lecture #1

§0. Introduction

$E/\mathbb{Q}$ elliptic curve, cond $= N$

$p \nmid 2N$ good ordinary prime

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) \to Ш(E/\mathbb{Q})[p^\infty] \to 0$$

Consider the following (1) – (3):

(1) p-part of BSD formula in rk 0

$$L(E,1) \neq 0 \implies \mathrm{ord}_p\left(\frac{L(E,1)}{\Omega_E}\right) = \mathrm{ord}_p\left(\frac{\#Ш(E/\mathbb{Q})}{(\#E(\mathbb{Q})_{\mathrm{tors}})^2} \cdot \prod_{\ell \in \mathbb{N}} c_\ell(E/\mathbb{Q})\right)$$

(2) p-converse to Gross-Zagier & Kolyvagin

$$\mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1 \implies \mathrm{ord}_{s=1} L(E,s) = 1.$$

(3) p-part of BSD formula in rk 1

$$\mathrm{ord}_{s=1} L(E,s) = 1 \implies \mathrm{ord}_p\left(\frac{L'(E,1)}{\Omega_E \cdot \mathrm{Reg}_E}\right)$$

$$\| $$

$$\mathrm{ord}_p\left(\frac{\#Ш(E/\mathbb{Q})}{(\#E(\mathbb{Q})_{\mathrm{tors}})^2} \cdot \prod_{\ell \in \mathbb{N}} c_\ell(E/\mathbb{Q})\right).$$

Goal of this mini-course:

Explain the proof of (1) – (3) for $\boxed{\text{Eisenstein primes } p}$

i.e. $\exists$ rational p-isogeny $E \to E'$

$\Updownarrow$

$E[p]$ is reducible as $G_{\mathbb{Q}}$-module

Today: Explain how, for any ordinary prime p (Eisenstein or not),

(1) – (3) follow from certain "main conjectures" in Iwasawa theory.

§1. Mazur's Main Conjecture

$$\bigcup_{n\geq 0} \mathbb{Q}_n = \mathbb{Q}_\infty \subset \mathbb{Q}(\mu_{p^\infty})$$

$$\begin{array}{c} \mathbb{Q}_\infty \\ \Big| \\ \mathbb{Q} \end{array} \quad \Gamma \cong \mathbb{Z}_p \quad \text{cyclotomic } \mathbb{Z}_p\text{-ext'n}$$

Mazur's Control Theorem: The restriction maps

$$\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_n) \longrightarrow \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^{\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}_n)}$$

Pontryagin dual

have finite ker & coker, of order bounded as $n \to \infty$.

$\Rightarrow$ $\#\,\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) < \infty$ then $X(E/\mathbb{Q}_\infty) := \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\vee$

is a f.g. <u>torsion</u> Λ-module.

$$\Lambda = \mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}_p[\![T]\!], \quad \gamma \longleftrightarrow 1+T$$

Mazur–Swinnerton-Dyer: $\exists\ \mathcal{L}_p(E/\mathbb{Q}) \in \Lambda \otimes \mathbb{Q}_p$

s.t. $\forall\ \chi : \Gamma \longrightarrow \mu_{p^\infty}$:

$$\mathcal{L}_p(E/\mathbb{Q})(\chi) = \begin{cases} \left(1-\frac{1}{\alpha}\right)^2 \cdot \frac{L(E,1)}{\Omega_E} & \text{if } \chi = \mathbb{1} \\ \frac{p^n}{\tau(\overline{\chi})\alpha^n} \cdot \frac{L(E,\overline{\chi},1)}{\Omega_E} & \text{if } \operatorname{cond}(\chi) = p^n > 1 \end{cases}$$

$$\chi \longleftrightarrow \chi(\gamma)-1$$

where α = unit root of $x^2 - a_p(E)x + p$

$$\Omega_E = \int_{\substack{\mathbb{Z}\text{-basis} \\ \text{of } H_1(E(\mathbb{C}),\mathbb{Z})^+}} \omega_E$$

<u>Mazur's Main Conjecture</u>: $X(E/\mathbb{Q}_\infty)$ is Λ-torsion,

with

$$\operatorname{char}_\Lambda X(E/\mathbb{Q}_\infty) = (\mathcal{L}_p(E/\mathbb{Q})).$$

Proposition 1 Mazur MC $\Longrightarrow$ p-part of BSD formula in rk 0.

Proof. Suppose $L(E,1) \neq 0$, so $\mathcal{L}_p(E/\mathbb{Q})(0) \neq 0$

By Mazur's MC,

$$\# X(E/\mathbb{Q}_\infty)_\Gamma < \infty, \text{ so } \# \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) < \infty.$$

Let $\mathcal{F}(E/\mathbb{Q}_\infty) \in \Lambda \simeq \mathbb{Z}_p[\![T]\!]$ a char. power series for $X(E/\mathbb{Q}_\infty)$.

Then

$$\mathcal{F}(E/\mathbb{Q}_\infty)(0) \overset{\text{Schneider–Perrin-Riou}}{\sim_p} \left(1-\frac{1}{\alpha}\right)^2 \cdot \# \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) \cdot \frac{\prod_{\ell \in \mathbb{N}} c_\ell(E/\mathbb{Q})}{(\# E(\mathbb{Q})_{\mathrm{tors}})^2}$$

by Mazur's MC $\Big\|$ mod $\mathbb{Z}_p^\times$

$$\mathcal{L}_p(E/\mathbb{Q})(0) = \left(1-\frac{1}{\alpha}\right)^2 \cdot \frac{L(E,1)}{\Omega_E}$$

$\therefore$ p-part of BSD formula holds $\square$

§2. Perrin-Riou's Main Conjecture

$K/\mathbb{Q}$ imaginary quadratic field satisfying

<u>Heegner hypothesis</u>:

every prime $\ell \mid N$ splits in K.

$$\begin{tikzcd}
K_\infty^- = \bigcup_{n\geq 0} K_n^- \arrow[d, dash, "\Gamma^- \cong \mathbb{Z}_p \text{ anti-cycl. } \mathbb{Z}_p\text{-ext'n}"] \arrow[dd, dash, bend right, "\Gamma^- \rtimes \langle\tau\rangle"'] \\
K \arrow[d, dash, "\langle\tau\rangle"] \\
\mathbb{Q}
\end{tikzcd}$$

Via $\pi_E : X_0(N) \to E$ get Heegner points $x_n \in E(K_n^-)$, $n \geq 0$

$$\underset{p\text{-ord}}{\rightsquigarrow} \kappa_\infty^{Hg} \in \check{S}(E/K_\infty^-) := \varprojlim_n \mathrm{Sel}(K_n^-, T_pE).$$

Perrin-Riou's Main Conjecture

$$X(E/K_\infty^-) := \mathrm{Sel}_{p^\infty}(E/K_\infty^-)^\wedge \text{ has } \underset{\substack{\| \\ \mathbb{Z}_p[\![\Gamma^-]\!]}}{\Lambda^-}\text{-rk } 1,$$

and

$$\mathrm{char}_{\Lambda^-} X(E/K_\infty^-) = \mathrm{char}_{\Lambda^-}\left(\frac{\check{S}(E/K_\infty^-)}{(\kappa_\infty^{Hg})}\right)^2 \cdot \frac{1}{u_K^2\, c_E^2},$$

where $u_K = \frac{1}{2}\#\mathcal{O}_K^\times$

c_E = Manin constant : $\pi_E^* \omega_E = c_E \cdot 2\pi i \underset{\substack{\uparrow \\ \text{newform.}}}{f(z)}\, dz.$

Proposition 2. Perrin-Riou's MC $\Rightarrow$ p-converse to Gross-Zagier & Kolyvagin.

Proof. Supp. $\mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$.

Choose $K/\mathbb{Q}$ imag. quadratic s.t.

- Heegner hyp. holds.
- $L(E^K, 1) \neq 0$.

By Kato, $\mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/K) = \mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$

$$\begin{tikzcd}
\mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/K) \arrow[d, equal, "\text{control thm.}"'] \\
\mathrm{rk}_{\mathbb{Z}_p} X(E/K_\infty^-)_{\Gamma^-}
\end{tikzcd}$$

$$\overset{\text{Perrin-Riou's MC}}{\Longrightarrow} (\gamma - 1) \nmid \mathrm{char}_{\Lambda^-}\left(\frac{\check{S}(E/K_\infty^-)}{(\kappa_\infty^{Hg})} \right), \quad \gamma \in \Gamma^- \text{ top. gen.}$$

$\Longrightarrow$ κ_∞^{Hg} has non-torsion image κ_0 under

$$\begin{array}{ccccc}
\check{S}(E/K_\infty^-) & \twoheadrightarrow & \check{S}(E/K_\infty^-)_{\Gamma^-} & \longrightarrow & \mathrm{Sel}(K, T_pE) \\
\rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\
\kappa_\infty^{Hg} & & \longmapsto & & \kappa_0
\end{array}$$

But $\kappa_0 =$ Kummer image of classic Heegner point

$$\overset{\text{Gross-Zagier}}{\Longrightarrow} \mathrm{ord}_{s=1} L(E/K, s) = 1.$$

$$\overset{L(E^K,1) \neq 0}{\Longrightarrow} \mathrm{ord}_{s=1} L(E, s) = 1 \qquad \square$$

§3. BDP Main Conjecture

Suppose also that $p\mathcal{O}_K = v\bar{v}$ splits in K.

Bertolini–Darmon–Prasanna : $\exists\ \mathcal{L}_v^{BDP}(E/K) \in \tilde{\Lambda}^-$

$$\tilde{\Lambda}^- = \mathbb{Z}_p^{ur}[[\Gamma^-]]$$

s.t. $\forall\ \chi : \Gamma^- \longrightarrow \mathbb{C}_p^\times$ crystalline at v & $\bar{v}$ of wts $(n,-n)$, $n>0$

$$\mathcal{L}_v^{BDP}(E/K)^2(\chi) = \underset{\substack{\uparrow \\ \text{explicit, } \neq 0}}{(*)}\, L^{alg}(E/K,\chi,1)$$

Greenberg Selmer group

$$\mathrm{Sel}_v(E/K_\infty^-) := \ker\left(H^1(K_\infty^-, E[p^\infty]) \longrightarrow H^1(K^-_{\infty,\bar{v}}, E[p^\infty]) \times \prod_{w \nmid p} H^1(K^-_{\infty,w}, E[p^\infty]) \right)$$

BDP Main Conjecture

$$X_v(E/K_\infty^-) := \mathrm{Sel}_v(E/K_\infty^-)^\vee \text{ is } \Lambda^- \text{-torsion,}$$

with

$$\mathrm{char}_{\tilde{\Lambda}^-} X_v(E/K_\infty^-) = \left(\mathcal{L}_v^{BDP}(E/K)^2\right).$$

Proposition 3. Supp. p-part of BSD formula known in rk 0.

Then

$$\text{BDP MC} \implies p\text{-part of BSD formula in rk } 1.$$

Proof. Supp. $\mathrm{ord}_{s=1} L(E,s) = 1$.

Choose $K/\mathbb{Q}$ imag. quadratic s.t.

- Heegner hyp. holds.
- $p = v\bar{v}$ splits in K.
- $L(E^K, 1) \neq 0$.

By Kolyvagin, + GZ $\quad E(K)/_{\text{tors}} = \mathbb{Z}\, y_K$ & $\#Ш(E/K) < \infty$.

y_K: Heegner pt.

$\Longrightarrow \#\mathrm{Sel}_v(E/K) < \infty$.

Let $\mathcal{F}_v(E/K_\infty^-) \in \Lambda^- \cong \mathbb{Z}_p[\![T]\!]$ char. power series for $X_v(E/K_\infty^-)$.

Then

$$\mathcal{F}_v(E/K_\infty^-)(0) \overset{\text{Jetchev-Skinner-Wan}}{\sim_p} \left(\frac{1-a_p(E)+p}{p}\right)^2 \cdot \prod_{\ell \mid N} c_\ell(E/\mathbb{Q})^2 \times \#Ш(E/K) \cdot \frac{\log_{\omega_E}(y_K)^2}{[E(K):\mathbb{Z}y_K]^2}$$

by BDP Main Conj. $\quad \Big\| \quad (\bmod \mathbb{Z}_p^\times)$

$$\mathcal{L}_v^{BDP}(E/K)^2(0) \overset{\text{BDP formula}}{=} \frac{1}{u_K^2 c_E^2} \cdot \left(\frac{1-a_p(E)+p}{p}\right)^2 \times \log_{\omega_E}(y_K)^2$$

But by GZ

$$\begin{pmatrix} \text{p-part of BSD} \\ \text{for } r_{an}(E/K) = 1 \end{pmatrix} \Longleftrightarrow [E(K):\mathbb{Z}y_K]^2$$

$$\sim_p$$

$$\#Ш(E/K) \cdot \prod_{\ell \mid N} c_\ell(E/\mathbb{Q})^2 \cdot u_K^2 \cdot c_E^2$$

$\therefore$ Get p-part of BSD for $r_{an}(E/K) = 1$,

so done if p-part of BSD known in rk 0. $\boxtimes$