

On The Work Of Narasimhan and Seshadri

Edward Witten

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It was an honor to be invited to lecture at this event in memory of M. S. Narasimhan and C. S. Seshadri. At first I was not sure if I should accept, since I realized that my point of view would be somewhat limited. But I decided I could try to give a perspective on how parts of their work have influenced theoretical physics, including my own work. I am going to start with a short overview and then I will go over some of the topics in a little more detail.

In 1964, Narasimhan and Seshadri proved a fundamental theorem relating representations of the fundamental group of a Riemann surface Σ to holomorphic vector bundles over Σ . I doubt that they, or – anyone, at that time – anticipated a connection to physics. The first inkling of a connection to physics came in 1982 when Atiyah and Bott placed the Narasimhan-Seshadri theorem in the context of gauge theory. By this time, Seshadri had made several important extensions of the original theorem, which I will be telling you about, involving semi-stable bundles and “parabolic structure.” Some important background for the work of Atiyah and Bott came from a result of Narasimhan and Harder (in 1975) on the cohomology of moduli spaces of bundles on a curve, and I will say a word about this also.

In 1983, Simon Donaldson gave a new proof of the Narasimhan-Seshadri theorem in the framework of gauge theory, as suggested by Atiyah and Bott, and this was later generalized in many directions. Hitchin (1987) introduced Higgs bundles and Hitchin's fundamental result on the equivalence of complex flat connections and Higgs bundles was, in fact, the analog of the Narasimhan-Seshadri theorem for Higgs bundles. Of course this is very important for understanding geometric Langlands via gauge theory so it figures in various lectures in this school. Also in the 1980's, Hitchin and Kobayashi conjectured an analog of the Narasimhan-Seshadri theorem for holomorphic vector bundles over a complex manifold of dimension > 1 and versions of this were proved by Donaldson, and by Uhlenbeck and Yau, with many later generalizations.

Physicists became more familiar with these things in the mid-1980's when holomorphic vector bundles on Calabi-Yau manifolds were used in deriving models of particle physics from string theory. However, the time that the Narasimhan-Seshadri really became most important for me was in 1988 when I was trying to understand the quantization of Chern-Simons gauge theory in $2 + 1$ dimensions. I will explain that story later.

Finally even in this mini-overview, there are a couple of other developments that I should mention. As part of their work on holomorphic vector bundles on Riemann surfaces, Narasimhan and Ramanan introduced the geometric analogs of “Hecke transformations” of number theory. These became very fundamental in the geometric Langlands program, and their physical interpretation is actually something I described the other day. Finally in one of his later papers, Seshadri, with V. Balaji, made yet another important generalization of the notion of vector bundles on a Riemann surface by developing the idea of vector bundles with “parahoric” structure – as opposed to the “parabolic” structure that Seshadri had investigated decades earlier. I will say a few words about this, but as Balaji is speaking today he may say much more.

The Narasimhan-Seshadri theorem is a comparison between flat bundles and holomorphic bundles over a Riemann surface Σ . Any flat bundle has a natural holomorphic structure. Narasimhan and Seshadri would have explained this as follows. Let H be the universal cover of Σ and let $H \times \mathbb{C}^N$ be a trivial rank N complex bundle over H . As a trivial bundle, it is a flat bundle over H and also a holomorphic bundle. Any flat bundle $E \rightarrow \Sigma$ is a quotient $E = (H \times \mathbb{C}^N)/\pi_1(\Sigma)$, with $\pi_1(\Sigma)$ acting on \mathbb{C}^N via some homomorphism $\rho : \pi_1(\Sigma) \rightarrow GL(N)$. The quotient $E = (H \times \mathbb{C}^N)/\pi_1(\Sigma)$ comes with an obvious holomorphic structure: a local section s of E is holomorphic if it pulls back to a holomorphic section of the trivial holomorphic bundle $H \times \mathbb{C}^N$. In a gauge theory language that in this context was introduced later by Atiyah and Bott, one would say the following: a flat bundle E has a flat connection A ; the corresponding gauge-covariant exterior derivative d_A has a $(0, 1)$ part $\bar{\partial}_A$ which defines a holomorphic structure of E .

It was a classical result that if Σ is a Riemann surface and $\mathcal{L} \rightarrow \Sigma$ is a holomorphic line bundle, then \mathcal{L} admits a unitary flat connection if and only if \mathcal{L} has degree 0, or equivalently $c_1(\mathcal{L}) = 0$. Narasimhan and Seshadri became interested in the analogous question for the case of a holomorphic vector bundle $E \rightarrow \Sigma$ of rank $N > 1$. Does E admit a unitary flat connection, which in this case would have structure group $U(N)$? There is an obvious necessary condition $c_1(E) = 0$, but this is far from the whole story.

For example, suppose that

$$E = \mathcal{O}(p) \oplus \mathcal{O}(-p),$$

where p is a point in Σ . Then $c_1(E) = 0$, but E does not admit any unitary flat connection. (Proof: $\mathcal{O}(p)$, and therefore E , has a holomorphic section s that vanishes at p . But if E has a flat unitary connection A , and $\bar{\partial}_A s = 0$, then by integration by parts, after picking a Kahler metric on Σ , one can prove

$$\int_{\Sigma} d^2x \sqrt{g} |d_A s|^2 = \int_{\Sigma} d^2x \sqrt{g} |\bar{\partial}_A s|^2 = 0,$$

and hence $d_A s = 0$. Therefore, s (if not identically zero) cannot vanish anywhere. I do not know how close this explanation is to what Narasimhan and Seshadri would have said in 1964.

Mumford had defined the notion of a *stable* holomorphic vector bundle over a Riemann surface. A bundle E with $c_1(E) = 0$ is *unstable* if it has a holomorphic subbundle E' of positive degree:

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0.$$

For example, the bundle

$$E = \mathcal{O}(p) \oplus \mathcal{O}(-p)$$

is unstable because it has the subbundle $\mathcal{O}(p)$ of positive degree. Mumford also defined E to be semi-stable if it has a proper subbundle E' of degree 0 but none of positive degree. If every proper subbundle of E has negative degree, then in Mumford's terminology, E is stable.

The Narasimhan-Seshadri theorem, in its original version (1964), says that *irreducible* unitary flat bundles of rank N are in 1-1 correspondence with stable bundles. The easier direction is to show that for a holomorphic bundle E to have a flat unitary structure, it is necessary for it to be stable. This can be proved by generalizing what I said about the case $\mathcal{O}(p) \oplus \mathcal{O}(-p)$. (That is probably not too close to how Narasimhan and Seshadri expressed the argument.) The opposite direction is more difficult: stability is sufficient for existence of a flat unitary structure. I don't think I can usefully explain the proof of this.

I have stated this for bundles of degree 0, but they actually had a more general statement about bundles of rank N and degree d , for any d . In gauge theory language, for $d \neq 0$, instead of flat unitary bundles, one can talk about unitary bundles whose curvature is central. (Their formulation was slightly different.) The resulting moduli space is smooth and compact if and only if N and d are relatively prime, i.e. $(N, d) = 1$

Seshadri went on to extend the Narasimhan-Seshadri theorem in two directions. The moduli space of *irreducible* unitary flat bundles is not compact if d and N are not relatively prime (for example in the most basic case $d = 0$), because a reducible flat bundle $E_1 \oplus E_2$ can be slightly perturbed to make it irreducible (i.e. it can arise as a limit of irreducible flat bundles). Likewise in view of the N-S theorem, the moduli space of stable bundles is not compact. However, Seshadri extended the N-S theorem to an equivalence between compact moduli spaces: the moduli space of unitary flat bundles (not necessarily irreducible) and the moduli space of holomorphic bundles that are stable and/or semi-stable (which is usually called for short the moduli space of semi-stable bundles).

A primary subtlety here is that Seshadri had to understand and incorporate an equivalence relation for semistable bundles. There isn't a sensible (Hausdorff) moduli space that parametrizes all equivalence classes of semistable bundles. I will explain this in gauge theory language. Consider a rank 2 bundle E that is a direct sum

$$E = \mathcal{L} \oplus \mathcal{L}^{-1}$$

of line bundles \mathcal{L} of degree 0. Such an E generically is semi-stable (it would be unstable if \mathcal{L} has nonzero degree).

The direct sum $\mathcal{L} \oplus \mathcal{L}^{-1}$ has a \mathbb{C}^* group of automorphisms acting as $\text{diag}(\lambda, \lambda^{-1})$ on the two summands $\mathcal{L}, \mathcal{L}^{-1}$. Now consider a triangular deformation of E , by adding an upper triangular perturbation to its $\bar{\partial}$ operator

$$\bar{\partial} + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \rightarrow \bar{\partial} + \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix},$$

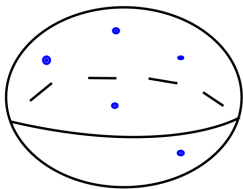
for some c . In general such a deformation exists. By the action of \mathbb{C}^* , we can replace this with

$$\bar{\partial} + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \rightarrow \bar{\partial} + \begin{pmatrix} a & \lambda c \\ 0 & -a \end{pmatrix}$$

for any $\lambda \neq 0$. So the bundle E'_λ defined by the $\bar{\partial}$ operator on the right hand side has a holomorphic type that is independent of λ as long as $\lambda \neq 0$; on the other hand, at $\lambda = 0$, E' reduces to E and its holomorphic type is different. Since E' is thus “infinitesimally close” to E , if we are going to define a sensible moduli space of semistable bundles, we have to treat E and E' as equivalent.

So Seshadri had to develop an equivalence relation, S -equivalence, on semistable bundles, basically considering any triangular deformation of $\mathcal{L} \oplus \mathcal{L}^{-1}$ (where \mathcal{L} has degree 0) to be equivalent to the direct sum. He was able to define a compact moduli space of equivalence classes of semi-stable (or stable) bundles, usually called in brief the moduli space of semistable bundles, and to prove that it is equivalent to the moduli space of unitary flat bundles, not necessarily irreducible. (This way to state the result is for the basic case of degree 0.). Apart from the importance of this particular problem, the ideas in Seshadri's construction were important in many later constructions of moduli spaces in algebraic geometry.

Seshadri's other generalization of the N-S theorem, in the 1970's, was to consider flat unitary bundles on a surface with punctures:



On one side of the correspondence, he considers flat bundles with prescribed conjugacy classes of the monodromy around the punctures. On the other side, he considers holomorphic bundles with “parabolic structure” at the punctures (meaning a reduction of the structure group of a bundle from $GL(N, \mathbb{C})$ to a parabolic subgroup).

Here it is necessary to generalize the ideas of stability, semi-stability, and S -equivalence to parabolic bundles. This is more complicated than the previous case because the stability condition depends on some parameters (which correspond to the conjugacy class of the monodromy in the other description). After finding the right definitions, Seshadri proved a relation between flat bundles with prescribed monodromy and semi-stable parabolic bundles. I will mention two applications later.

I will also mention two contributions by Narasimhan from the 1970's. First, with Ramanan, he introduced a geometric analog of the Hecke transformations of number theory. In number theory, a Hecke operator acts in a sense only at one prime. The geometric analog of a Hecke transformation by Narasimhan and Ramanan modifies the holomorphic structure of a holomorphic vector bundle $E \rightarrow \Sigma$ at just one point in Σ . Geometric Hecke transformations played a central role when the geometric Langlands program was developed, starting around 1990 by Beilinson and Drinfeld and then many others. In my lecture on Friday, I explained a gauge theory interpretation of geometric Hecke transformations in terms of 't Hooft operators.

Also in the 1970's, Narasimhan and Harder computed the Betti numbers of the moduli space of stable bundles (in the case $(N, d) = 1$ where this is smooth and compact). They did this in a very surprising way by comparing to characteristic p and using the Weil conjectures (which were proved by Deligne at just about this time). They introduced a stratification on the set of all bundles over a curve, basically according to how unstable a bundle is. Then, in the case of a curve over a finite field of characteristic p , they counted the stable bundles. The unstable bundles are relatively easy to count by induction on the rank. Knowing that the "Tamagawa number" of SL_N is 1 gave a "sum rule" which then gave a count of stable bundles. The form of the answer was such that, with the help of the Weil conjectures, the Betti numbers of the moduli space of stable bundles could be determined.

Atiyah and Bott in 1982 introduced a gauge theory perspective on the Narasimhan-Seshadri theorem and the Narasimhan-Harder results about the topology of the moduli space of stable bundles. They considered gauge theory of a compact group G on a Riemann surface Σ , with gauge field A and curvature $F = da = A \wedge A$. The Yang-Mills action is

$$I = \int_{\Sigma} \text{Tr} F \wedge \star F.$$

Their basic idea was to consider I as an equivariant Morse function on the space of all connections. Here one has to consider the critical points of I . The critical points with $I = 0$ are the unitary flat connections. According to the Narasimhan-Seshadri theorem, these correspond to stable holomorphic bundles. Atiyah and Bott showed that the other components of critical points of I are in 1-1 correspondence with the strata of the Narasimhan-Harder stratification.

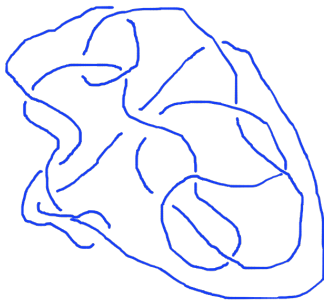
Because the Morse indices of the critical point sets are all even, Atiyah and Bott observed that *if* Morse theory applies to this situation, as they conjectured, then I is an equivariantly perfect Morse function on the space of all connections (equivariant with respect to the action of the gauge group). They showed that this would imply formulas for the Betti numbers of the moduli space of stable bundles that agreed with those of Narasimhan and Harder. Here they used the fact that the space of all connections (ignoring the action of the gauge group) is contractible, a fact that for them played the role of “Tamagawa number equals 1” for Narasimhan and Harder. Eventually it was proved (G. Daskopoulos, 1992; J. Rade, 1992) that Morse theory does apply in this problem. In 1982, however, Atiyah and Bott avoided the analytical difficulties of proving that by considering instead the action on the space of connections of the complexification of the gauge group. I won't go into this because it would take us away from our main subject today of the work of Narasimhan and Seshadri. I will just add that near the beginning of this lecture, in my way of explaining some facts relevant to the Narasimhan-Seshadri theorem, I used gauge theory language as introduced for this problem by Atiyah and Bott.

The Atiyah-Bott paper inspired a new gauge theory proof of the Narasimhan-Seshadri theorem by Donaldson, in 1983. This in turn inspired many generalizations. One that is of particular importance for geometric Langlands was due to Hitchin. Narasimhan and Seshadri had interpreted *unitary* representations of the fundamental group of a Riemann surface in terms of algebraic geometry. What about representations that might be *nonunitary*? Hitchin proved that (semistable and not necessarily unitary) representations of the fundamental group of a surface correspond to stable Higgs bundles (E, φ) , where E is a holomorphic bundle and φ is a holomorphic map $\varphi : E \rightarrow E \otimes K$. Extending this, he proved that the moduli space of either of those two types of object has a *hyper-Kähler* structure. This was the generalization of the Narasimhan-Seshadri theorem for non-unitary flat bundles.

Also in the 1980's, using the gauge theory formulation, the Narasimhan-Seshadri theorem was generalized to complex manifolds of dimension greater than 1 by Donaldson as well as by Uhlenbeck and Yau. In higher dimensions, the theorem says that a semistable holomorphic bundle admits a hermitian metric such that the corresponding connection satisfies the Yang-Mills equations; conversely, a holomorphic bundle with such a hermitian metric is semi-stable.

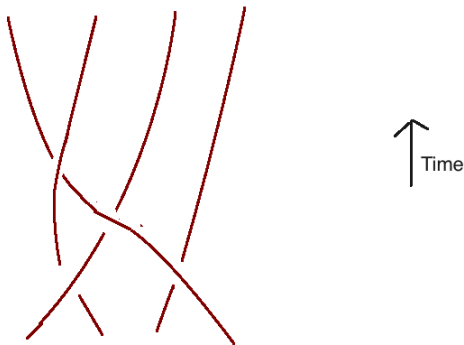
I well remember how obscure these things seemed to me when I heard about them from Atiyah and Bott in the early 1980's. However, physicists became familiar with such matters starting in the mid-1980's when Calabi-Yau threefolds and holomorphic vector bundles over them were used to make models of particle physics in the context of string theory. In this context, the existence of hermitian Yang-Mills metrics – in other words, the generalization by Donaldson and by Uhlenbeck and Yau of the Narasimhan-Seshadri theorem – is completely crucial. This has actually become a very important direction.

But instead of talking about that, I will explain the role of the Narasimhan-Seshadri theorem in my own work a few years later on rational conformal field theory and the Jones polynomial. Vaughn Jones had discovered the Jones polynomial of a knot in 1983



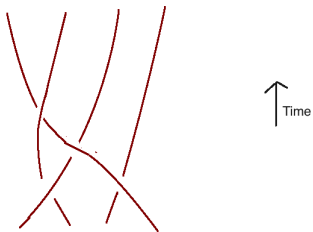
It is a subtle invariant that from the beginning was related to mathematical physics – in a bewildering variety of ways.

Jones's original definition involved the Jones representations of the braid group. A braid in the mathematical sense is a picture like this:



Points move around in the plane, then return to their starting positions. In this example, I've drawn a braid with 4 strands. Braids form a group because they can be composed by gluing one braid on top of another.

Jones's original definition of the Jones polynomial of a knot was as follows. We can build a knot by gluing together the top and bottom ends of a braid:



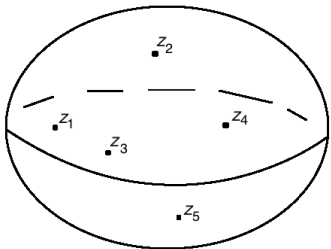
This gluing is a little bit like taking a trace. Let B be a braid. Let R be a representation of the braid group and let us write $\mathcal{R}(B)$ for the matrix that represents this braid in the representation $R_i(q)$. In his original work, Jones constructed representations $\mathcal{R}_i(q)$ of the braid group and defined the Jones polynomial as a linear combination of the corresponding traces:

$$J(q) = \sum_i c_i(q) \text{Tr}_{R_i(q)} \mathcal{R}_{i,q}(B).$$

It worked, but it was unclear why, since there was no manifest three-dimensional symmetry in the construction. Michael Atiyah recommended this as a problem for physicists. A very important step was taken by A. Tsuchiya and Y. Kanie. They showed that the Jones representations of the braid group are the ones that arise when one decomposes the correlation functions of the two-dimensional WZW model in conformal blocks, as originally analyzed by Knizhnik and Zamolodchikov.

Consider genus 0 correlation functions of a primary field Φ in this theory:

$$G(z_1, \bar{z}_1; z_2, \bar{z}_2; \cdots ; z_n, \bar{z}_n) = \langle \Phi(z_1, \bar{z}_1) \Phi(z_2, \bar{z}_2) \Phi(z_3, \bar{z}_3) \cdots \Phi(z_n, \bar{z}_n) \rangle.$$



These functions are neither holomorphic nor antiholomorphic, and they cannot be factored as the product of a holomorphic and an antiholomorphic function. But Knizhnik and Zamolodchikov had shown that they are *finite* sums of products of holomorphic and antiholomorphic functions:

$$G(z_1, \bar{z}_1; z_2, \bar{z}_2; \cdots; z_n, \bar{z}_n) = \sum_{\alpha} f_{\alpha}(z_1, z_2, \cdots, z_n) \bar{f}_{\alpha}(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n).$$

Here the functions $f_{\alpha}(z_1, z_2, \cdots, z_n)$ are multivalued holomorphic functions. For each n , we can define a vector bundle V_n over the configuration space of n distinct points z_1, z_2, \cdots, z_n with a basis given by the f_{α} . These are automatically *flat* vector bundles (with the f_{α} understood as covariantly constant sections), and their *monodromies* when the points move around give representations of the braid group. (Single-valuedness of the correlation functions of the WZW model implies that these representations are unitary.) The observation of Tsuchiya and Kanie was essentially that, for symmetry group $G = SU(2)$ and a primary field in the 2-dimensional representation, these are the Jones representations.

The holomorphic functions f_α are called conformal blocks. Eric Verlinde had defined operators that act on the space of conformal blocks, and intuitively it seemed that he was treating the space of conformal blocks of the WZW model as the Hilbert space of some quantum system. But what was that system? At some point, I had the idea that the appropriate quantum system was simply Chern-Simons gauge theory, with the action

$$I = \frac{k}{4\pi} \int_W d^3x \epsilon^{ijk} \text{Tr} \left(A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k \right)$$

where A is a connection on a G -bundle over a three-manifold W and k is the level of the WZW model. Because the only structure of W used in defining this action is an orientation, one can hope that a quantum theory derived from this action will be topologically invariant and will give three-manifold invariants.

We can include a knot or embedded circle $K \subset W$ by including a Wilson loop operator

$$\mathcal{W}_R(K) = \text{Tr}_R P \exp \oint_K A,$$

where R is a representation of G and the symbol P represents “holonomy.” Now the path integral

$$\int DA \exp(iI) \mathcal{W}_R(K)$$

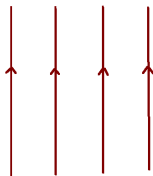
depends on K (and R) as well as on W , but nothing else. So (taking $W = \mathbb{R}^3$) this will potentially give an invariant of a knot. (It turns out that these statements need slight modification: because of an anomaly, both W and K must be “framed.”)

But to get anywhere, we need to know that the quantum Hilbert space \mathcal{H}_Σ of Chern-Simons theory, on a Riemann surface Σ , is the same as the space of conformal blocks of the WZW model on Σ (with the same symmetry group G and the same “level” k). That is where the Narasimhan-Seshadri theorem came in. First of all, to construct \mathcal{H} we are supposed to “quantize” the classical phase space. The Euler-Lagrange equations of the Chern-Simons action just say that $F = dA + A \wedge A$, the curvature of a connection A , should vanish. The phase space that we have to quantize to construct \mathcal{H}_Σ is thus the moduli space M of flat connections on $\mathbb{R} \times \Sigma$ (here \mathbb{R} parametrizes the “time”) which is the same as the moduli space of homomorphisms $\rho : \pi_1(\Sigma) \rightarrow G$. Somehow we have to quantize M and show that the result is the same as the space of conformal blocks of the WZW model. M should be quantized with symplectic structure $k\omega_0$ where $\omega_0/2\pi$ generates $H^2(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.

There is not any known way to quantize M without picking some additional structure. Suppose we choose the additional structure to be a complex structure on Σ . Then we can invoke the Narasimhan-Seshadri theorem, which says that M is the same as the moduli space \mathcal{M} of semistable holomorphic G -bundles over Σ . Knowing this, we can quantize \mathcal{M} using geometric quantization. There is a fundamental holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{M}$, whose first Chern class is represented in de Rham cohomology by ω_0 (for $G = SU(N)$, \mathcal{L} is the determinant line bundle of the Dirac operator on Σ , coupled to a rank N holomorphic vector bundle $E \rightarrow \Sigma$). Then \mathcal{L}^k is a prequantum line bundle for this problem in the language of geometric quantization, and geometric quantization tells us that we should define $\mathcal{H} = H^0(\mathcal{M}, \mathcal{L}^k)$ as a quantization of \mathcal{M} .

At this point we get “lucky.” The answer $H^0(\mathcal{M}_\Sigma, \mathcal{L}^k)$ coincides with a known and in a sense standard – though rather abstract – description of the space of WZW model conformal blocks on a genus g surface. (I probably learned this description from Graeme Segal and I was lucky here as this description was certainly not well-known among physicists at the time and possibly also not today. It is widely used – in a more general form – in research on the geometric Langlands program.) This is the basic link between 2 and 3 dimensions, and as I explained it depends on the Narasimhan-Seshadri theorem.

Actually, I explained this in the absence of knots. To incorporate knots, we need to use the later refinement of the Narasimhan-Seshadri theorem by Seshadri to include parabolic structure. To get the Jones representations of the braid group, we pick some points $p_i \in \Sigma$ and place knots, in some representations R_i , on $p_i \times \mathbb{R} \subset \Sigma \times \mathbb{R}$. Here is a local picture:



Now one can show that the phase space that has to be quantized is a moduli space M' of flat bundles on $\Sigma \setminus \{p_1, \dots, p_s\}$ with a prescribed conjugacy class of the monodromy around the p_i . So now we have to quantize M' . To do this, we again pick a complex structure on Σ , and now, invoking Seshadri's extension of the Narasimhan-Seshadri theorem, we can identify M' with the moduli space \mathcal{M}' of semistable holomorphic $G_{\mathbb{C}}$ bundles over Σ with parabolic structure at the points p_i .

A key ingredient is missing in what I've said so far. We picked a complex structure J on Σ in order to quantize the moduli space M of flat bundles or its counterpart M' of flat bundles with point singularities. Of course, the choice of complex structure violated topological invariance. So now we have to show that the Hilbert space \mathcal{H}_J that we construct in complex structure J really does not depend on J . This is proved by constructing a (projectively) flat connection that lets us compare the \mathcal{H}_J 's for different (nearby) J 's. This is a very important part of the story, which involves adapting to infinite dimensions some simple facts about quantization in finite dimensions. It is a rare case in which the issues about quantization that we discussed Thursday are important in quantum field theory. But as this would take us rather far afield from the Narasimhan-Seshadri theorem, I will not explain this point today.

However, I do want to make contact with something I said Thursday: we do not have a good understanding of Chern-Simons gauge theory with a noncompact real semi-simple gauge group G such as $SL(N, \mathbb{R})$

$$I = \frac{k}{4\pi} \int_W d^3x \epsilon^{ijk} \text{Tr} \left(A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k \right)$$

because the Narasimhan-Seshadri theorem does not apply to flat bundles with non-compact gauge group. The closest analog is Hitchin's theorem about Higgs bundles, and after picking a complex structure J on Σ , we can indeed use that theorem, as was shown by Hitchin, to construct a Kahler polarization of the appropriate phase space M . But we do not understand in what sense the resulting Hilbert space \mathcal{H}_J is independent of J , so we do not know how to exhibit topological invariance. (This is a broad brush statement; there are some important special cases in which the problem has been solved or circumvented.)

There is one much more recent contribution by Seshadri that I wish to mention. Seshadri and V. Balaji (2010) developed a theory of holomorphic vector bundles over a Riemann surface with “parahoric” structure, which is more general than parabolic structure. (There is also a gauge theory approach by P. Boalch, and there have been other more recent papers.) I only learned about this paper in the last few weeks and don’t understand enough to describe it in any detail. Perhaps in Balaji’s talk today he will describe this work. I will just make the following remark. In 2006-8, Sergei Gukov and I wrote two papers on “surface operators” in the context of the gauge theory approach to geometric Langlands. In the first paper, we used Seshadri’s theory of parabolic structure, or more precisely the extension of this to Higgs bundles by Simpson, to describe in gauge theory the “ramified” case of geometric Langlands. But then we realized that there were more general singularities to consider; we tried to analyze these in the second paper but were not able to get a systematic picture. Since learning about “parahoric structure,” I’ve been thinking that this might have been a better framework for at least a large part of what we were trying to do.

In conclusion, I've described some of the celebrated contributions of Narasimhan and Seshadri and I hope I have given at least a few hints of the role that they have played in theoretical physics.