Introduction to Quantum Information Theory

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Quantum theory was born out of some revolutionary observations:

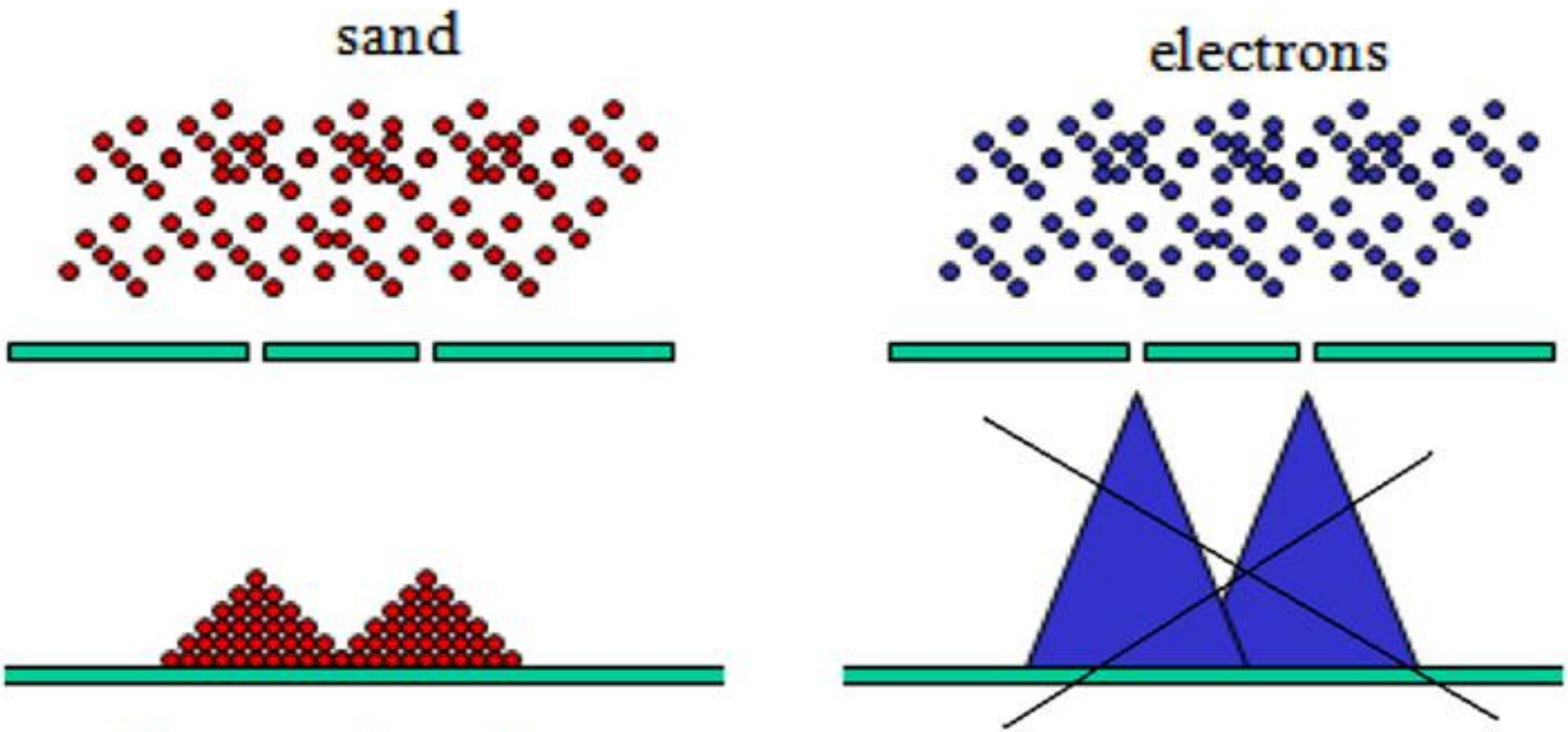
- **Light has also particle nature:**

$$E = h\nu$$

- **Micro particle has also wave nature:**

$$\lambda = \frac{\hbar}{p}$$

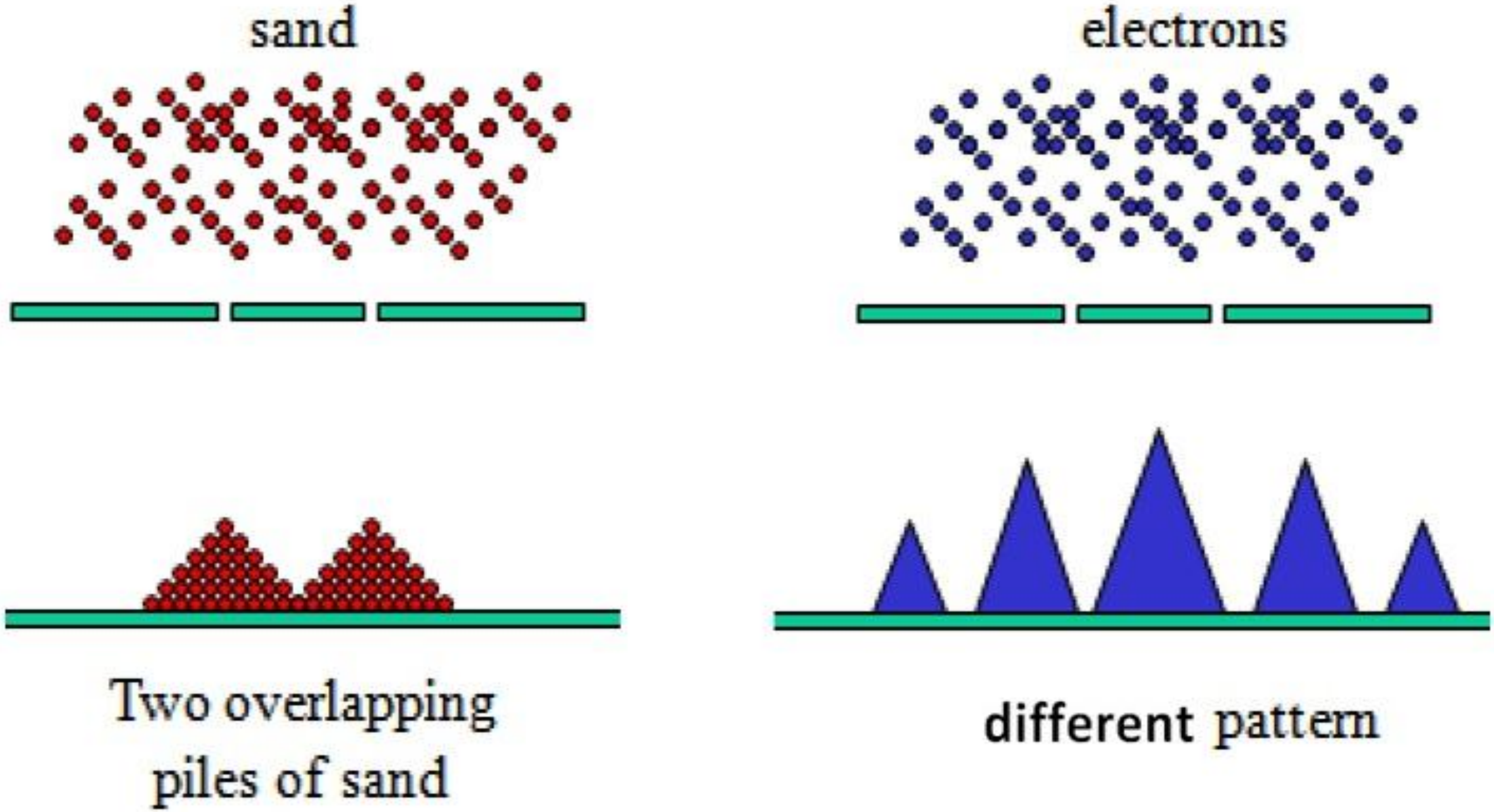
Two slits experiment with particles



Two overlapping piles of sand

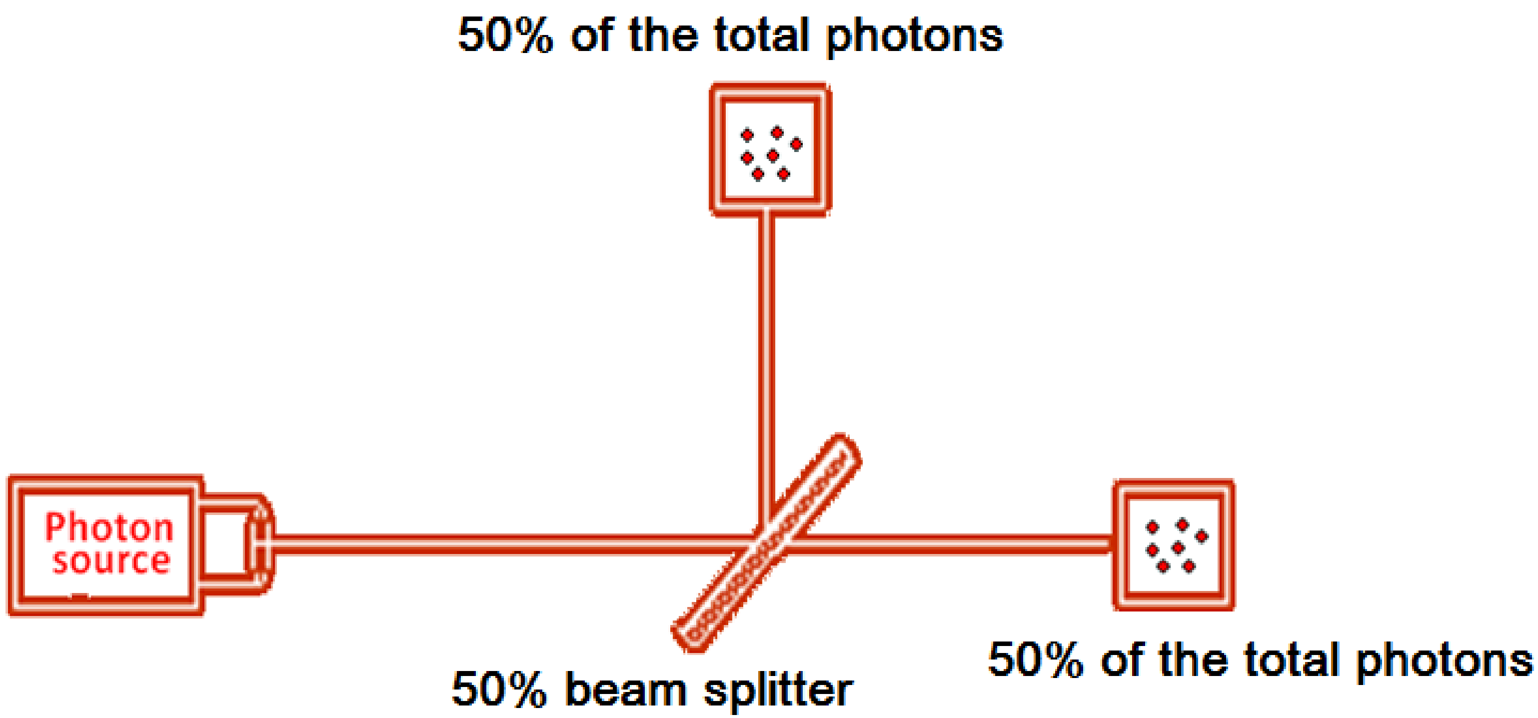
Not just two overlapping "piles" of electrons

A different pattern for micro particles



Interference pattern!

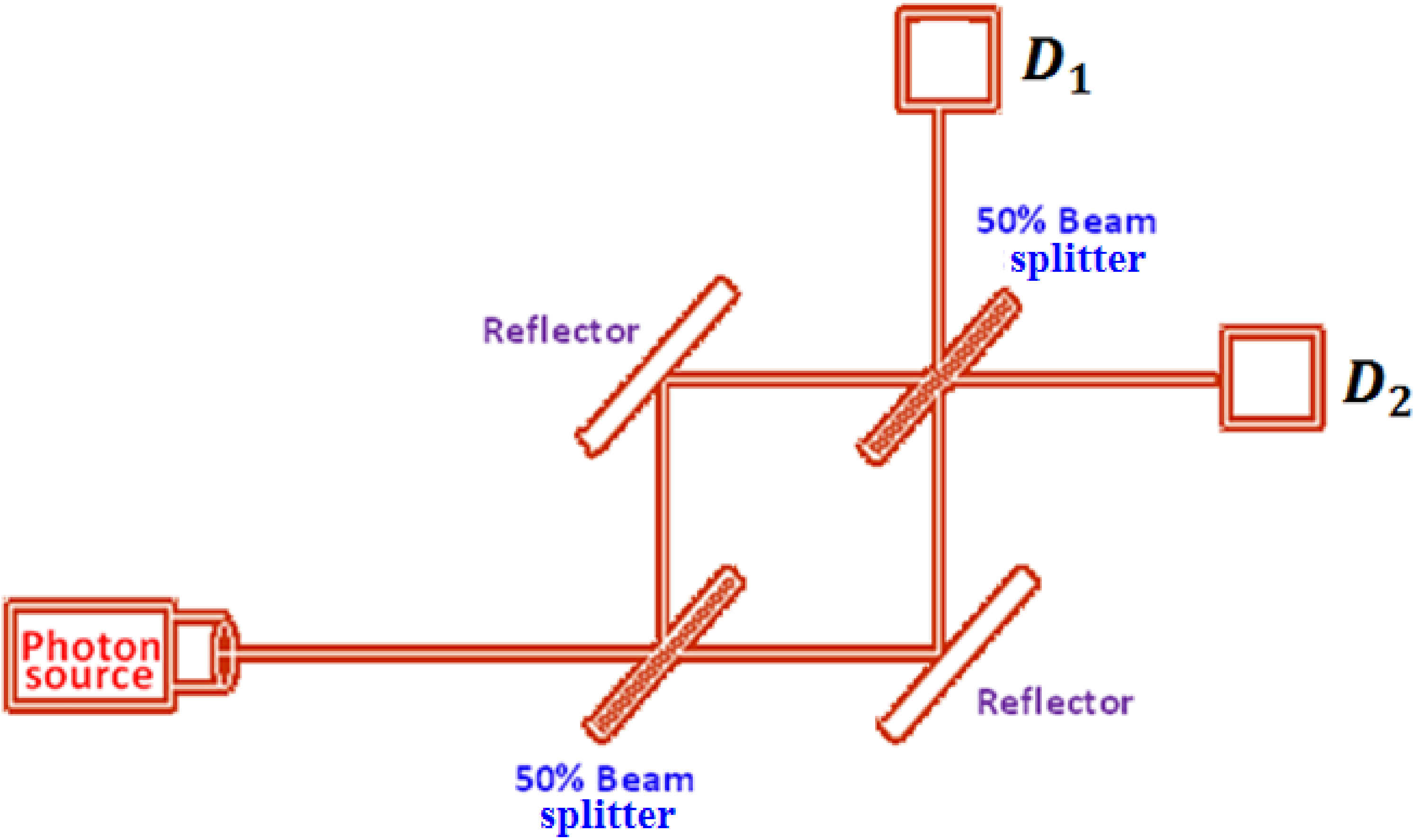
Interferometer Experiment



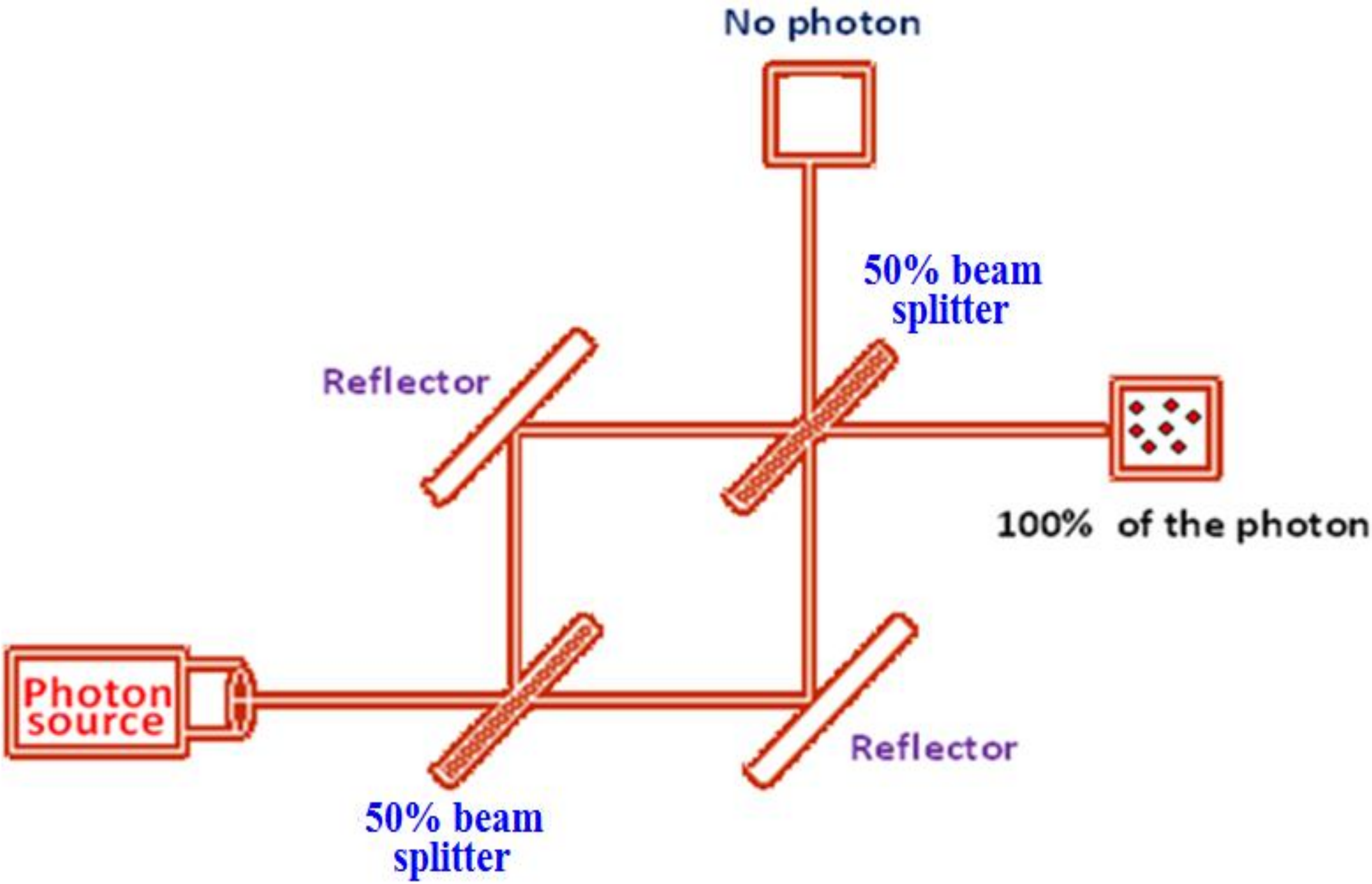
Photon source sends photons one by one to the beam splitter.

Photon is indivisible, only one detector clicks at a time.

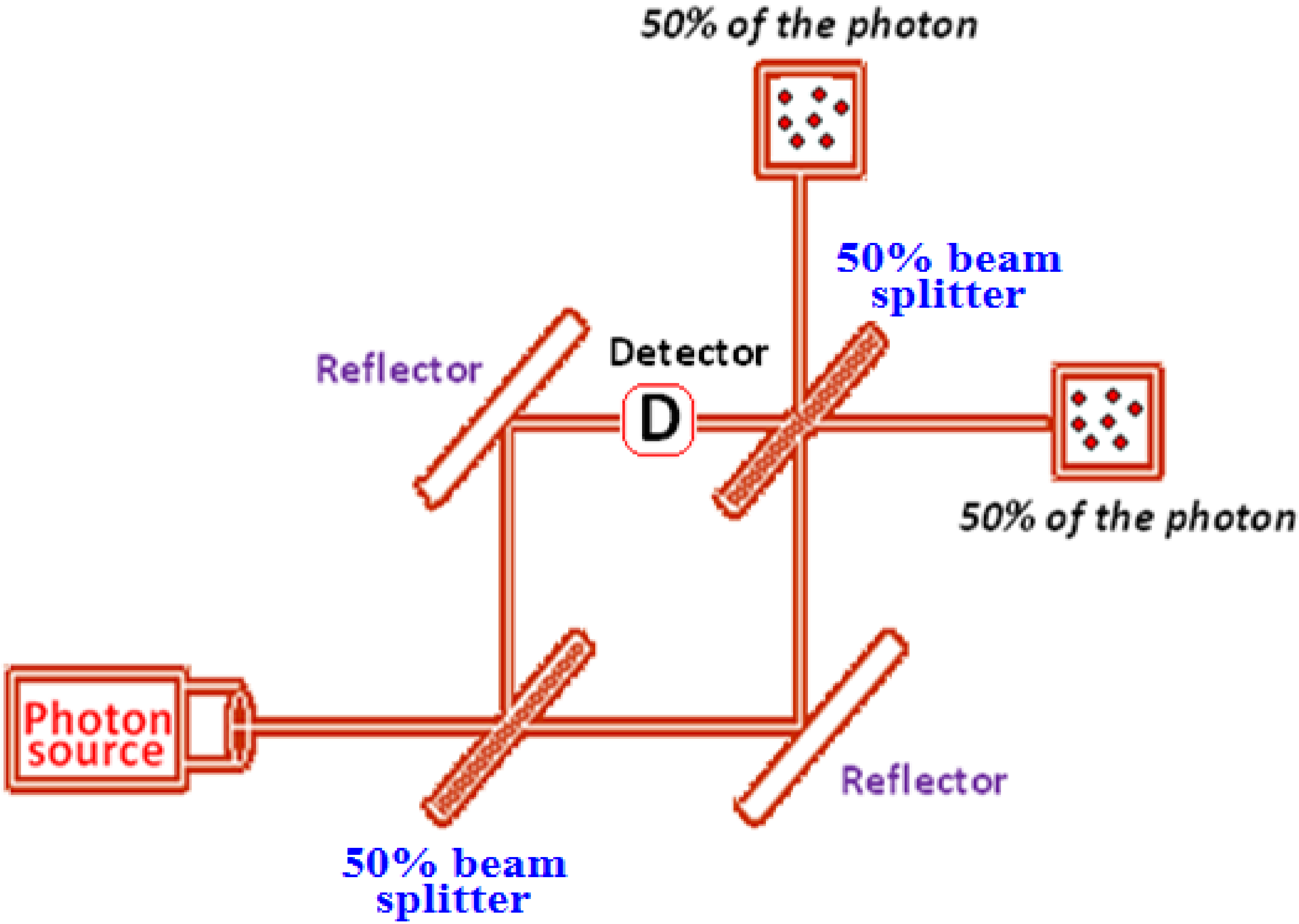
Our common sense based on classical physics tells;



A single photon will be detected in one of the detector with 50% probability.

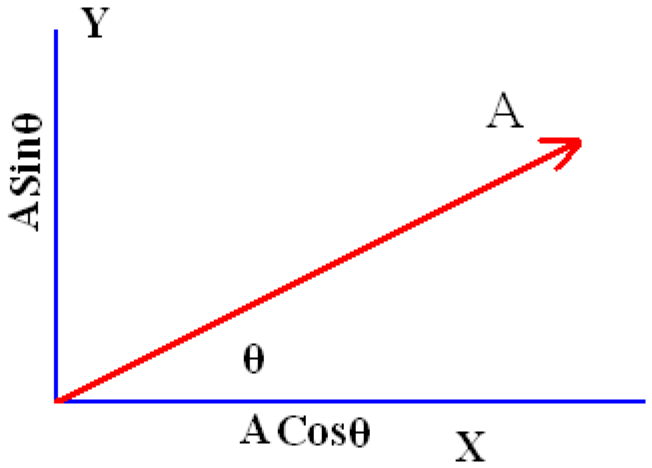


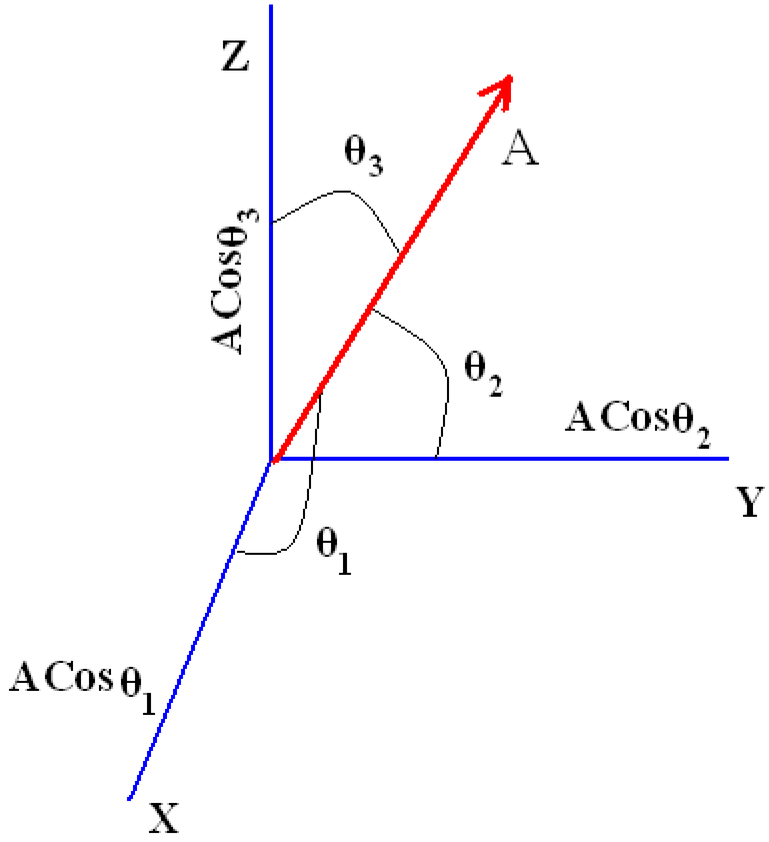
Classical physics has no explanation.



Knowledge of path taken by the photon erases interferences.

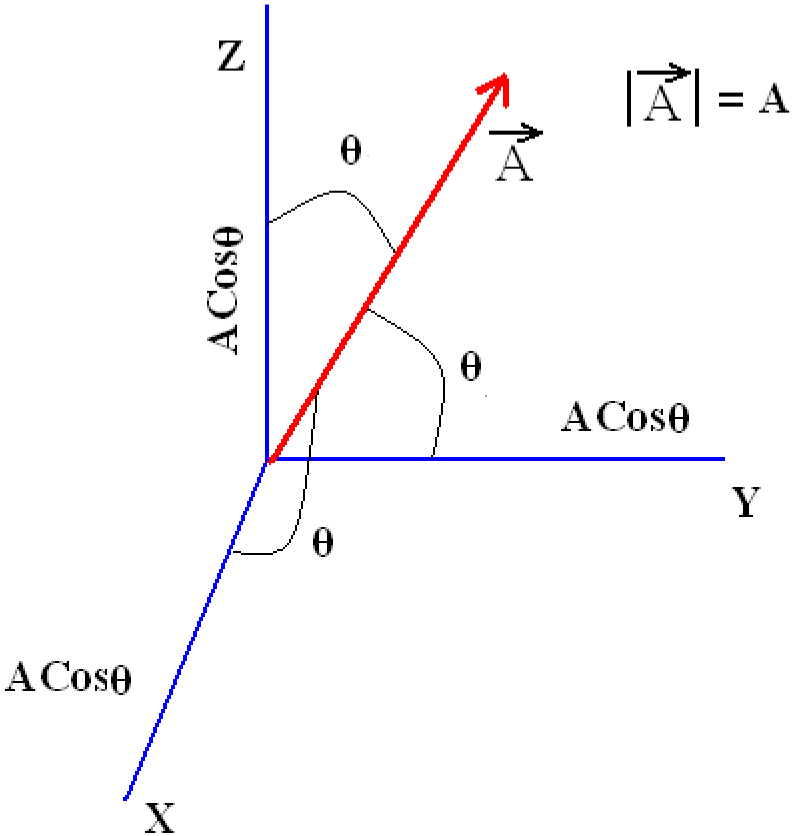
Vector and its components





The Vector makes equal angle with the axes

$$\theta_1 = \theta_2 = \theta_3 = \theta$$

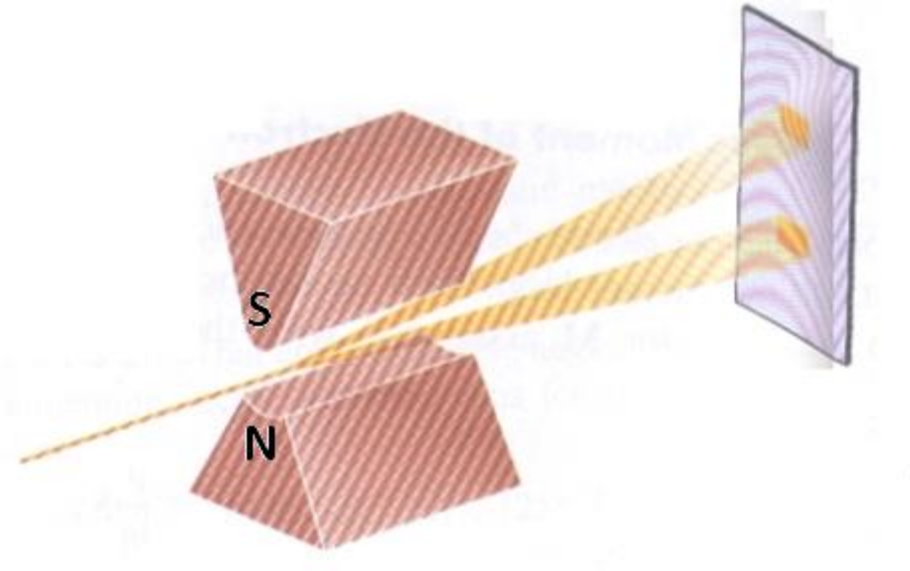


$$A_X = \frac{A}{\sqrt{3}}$$

$$A_Y = \frac{A}{\sqrt{3}}$$

$$A_Z = \frac{A}{\sqrt{3}}$$

The Stern-Gerlach experiment



Stern-Gerlach experiment measures spin angular momentum

Spin angular momentum is a vector!

Result of spin angular momentum measurement

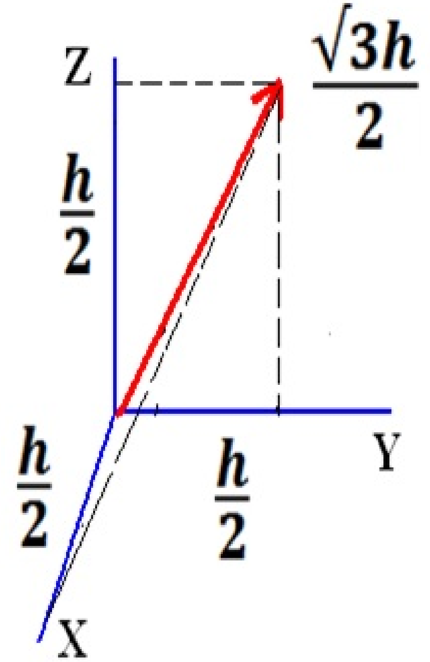
Direction	Value
X	$\frac{h}{2}$
Y	$\frac{h}{2}$
Z	$\frac{h}{2}$

Expected value of spin angular momentum :

$$\sqrt{\frac{h^2}{4}+\frac{h^2}{4}+\frac{h^2}{4}} = \frac{\sqrt{3}h}{2}$$

The idea of vector in classical world does not work

Expected direction of the spin angular momentum makes equal angle with all the axes

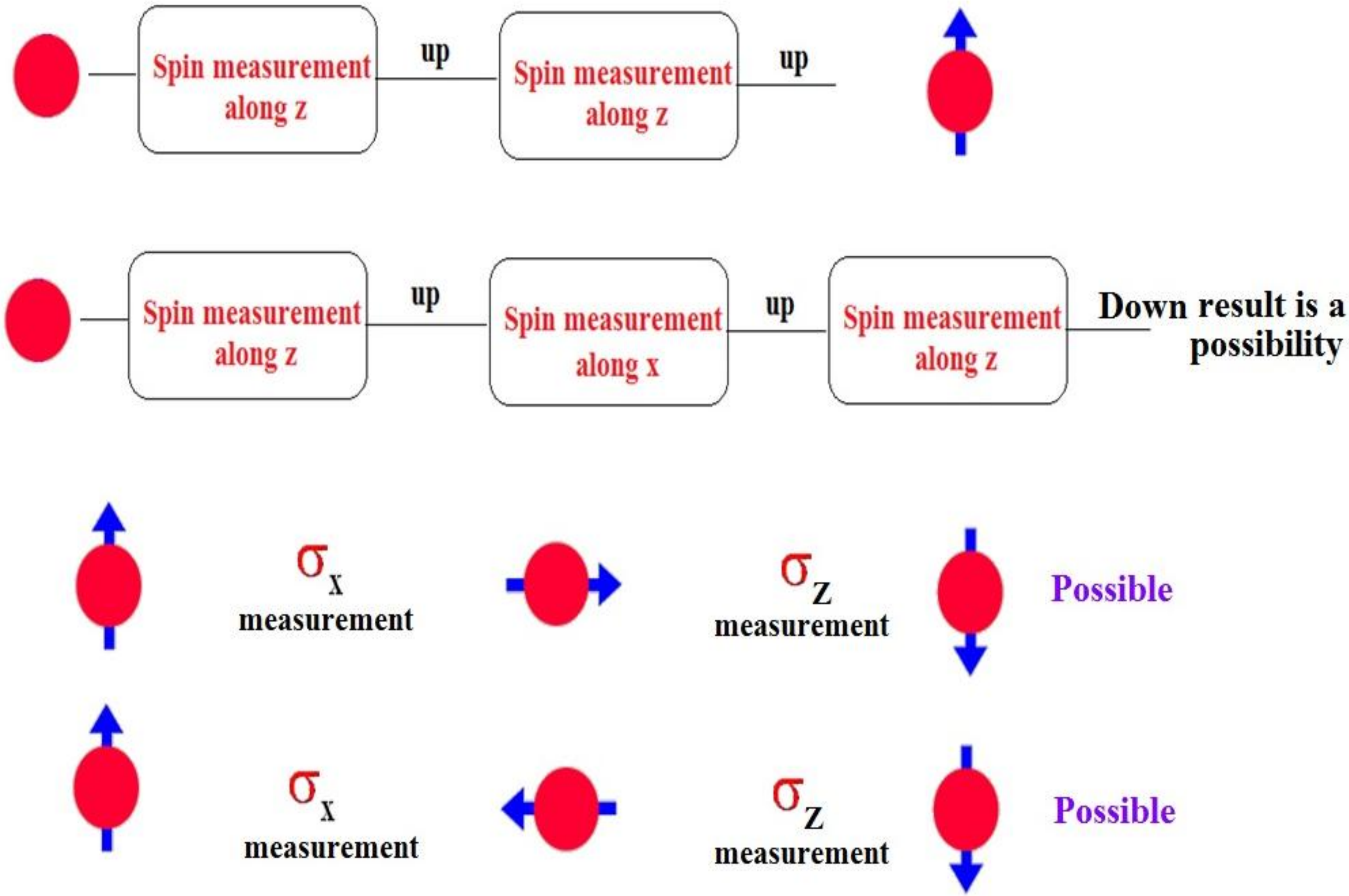


Now if spin is measured along that direction the result is $\frac{h}{2}$ instead of $\frac{\sqrt{3}h}{2}$

In whichever direction one measures spin, the result is

either $+\frac{h}{2}$

or $-\frac{h}{2}$



Spin measurement along z
up
Spin measurement along z
up
Spin measurement along z
up
Spin measurement along x
up
Spin measurement along z
Down result is a possibility
σ_x measurement
σ_z measurement
Possible
σ_x measurement
σ_z measurement
Possible

- Classical physics were unable to explain this and some other peculiarities.

- Through trial and error process a new physical theory arose which is

Quantum Mechanics

A Mathematical formalism for description of physical theories.

Hilbert space:

Hilbert space is a normed linear space enriched with scalar product.

- **Hilber space of dimension d:**

$$|\alpha\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix},$$ **where $\alpha_1, \alpha_2 \ldots\ldots \alpha_d$, are in general complex.**

- **Conjugate transpose of $|\alpha\rangle$:**

$$\langle\alpha| = [\alpha_1^* \quad \alpha_2^* \quad \ldots \alpha_d^*]$$

- **Scalar product of $|\alpha\rangle$ and $|\beta\rangle$:**

$$\langle\alpha|\beta\rangle = [\alpha_1^* \quad \alpha_2^* \quad \ldots] \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix}$$

$$= \alpha_1^*\beta_1 + \alpha_2^*\beta_2 + \ldots\ldots \alpha_d^*\beta_d$$

- **Two vectors $|\alpha\rangle$ and $|\beta\rangle$ are orthogonal when**

$$\langle\alpha|\beta\rangle = = \alpha_1^*\beta_1 + \alpha_2^*\beta_2 + \ldots . \alpha_d^*\beta_d = 0$$

- **$|\alpha\rangle$ is unit vector when the following condition is satisfied:**

$$\langle\alpha|\alpha\rangle = [\alpha_1^* \quad \alpha_2^* \quad \ldots] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix}$$

$$= |\alpha_1|^2 + |\alpha_2|^2 + \ldots . = 1$$

Linear operator:

A is an linear operator when it acts in the following way:

$$A\,(a|\alpha\rangle + b|\beta\rangle) = a\,A|\alpha\rangle + bA|\beta\rangle$$

where a and b are scalars.

An operator acting on a d dimensional Hilbert space can be represented by a $d \times d$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

where $a_{11}, a_{12}, \ldots.$ are in general complex.

Conjugate transpose of A:

$$A^{\dagger} = \begin{bmatrix} a_{11}^{*} & a_{21}^{*} & \cdots \\ a_{12}^{*} & a_{22}^{*} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

A is called hermitian or self edjoint when

$$A = A^{\dagger}$$

If for some linear operator A the following equation holds:

$$A\,|a_i\rangle = \lambda_i|a_i\rangle, \quad \lambda_i \text{ being scalars.}$$

It is called Eigen value equation where $|a_i\rangle$'s are called Eigen vectors and λ_i's are eigen values.

One can find the Eigen values and Eigen vectors by solving the following characteristics equation:

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots \\ a_{21} & a_{22} - \lambda & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix} = 0$$

Some properties of self adjoint operator

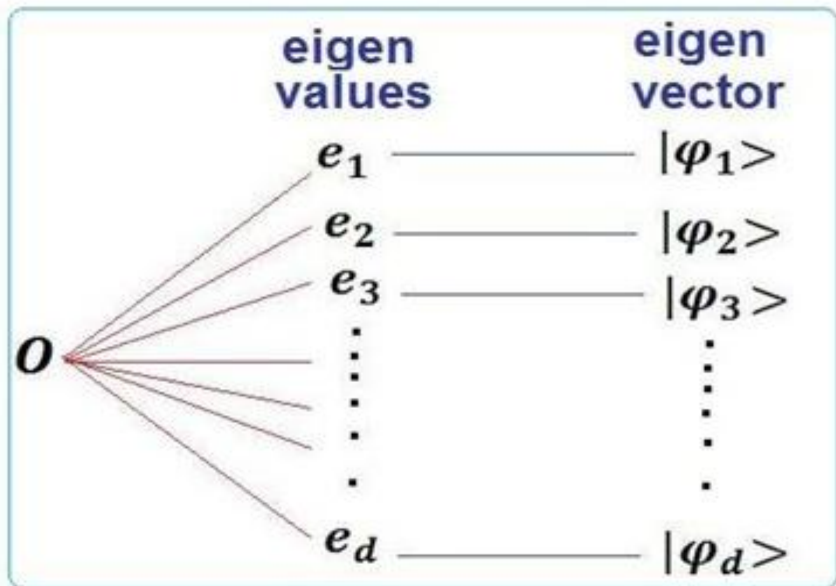
Operators are $d \times d$ matrix. $O = \begin{bmatrix} o_{11} & o_{12} & \cdots & o_{1d} \\ o_{21} & o_{21} & \cdots & o_{2d} \\ \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ o_{d1} & o_{d2} & \cdots & o_{dd} \end{bmatrix}$

Self adjoint operator:

$$O = O^{\dagger}$$

Eigen value equations:

$$O|\varphi_r> = e_r|\varphi_r>$$



(1) Eigen values are real.

(2) Eigen vector are orthogonal for non-degenerate eigen values.

$$<\varphi_i|\varphi_j> = \delta_{ij} \text{ for } a_i \neq a_j$$

The set of orthonormal vectors $\{|\phi_i\rangle\}_{i=1}^{d}$ forms an orthonormal basis for the d dimensional Hilbert.

Which means any arbitrary vectors $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_i c_i \, |\phi_i\rangle$$

where $c_i = \langle\phi_i|\psi\rangle$

Rule of quantum mechanics

System $\longrightarrow$ Hilbert space

State $\longrightarrow$ Unit vector in the Hilbert space

Observable $\longrightarrow$ Self adjoint operator

– Measurement rules –

(1) measurement results are one of the eigen values.

(2) Qunatum state, in general, does not specify definite values for measurement but specifies probabilities for various results.

(3) After measurement, intial state of the system collapses to the eigen state corresponding to the result.

$$\text{Initial state} = |\psi>$$

Measurement of $\boldsymbol{O}$

Possible results	Probabilities	Final State
e_1	$\lvert<\psi\lvert\varphi_1>\rvert^2$	$\lvert\varphi_1>$
e_2	$\lvert<\psi\lvert\varphi_2>\rvert^2$	$\lvert\varphi_2>$
•		
•		
e_d	$\lvert<\psi\lvert\varphi_d>\rvert^2$	$\lvert\varphi_d>$

Prescription for finding Born probability:

Initial State is $|\xi\rangle$

Observable to be measured is B

Eigen values of B: $\{b_k\}_{k=1}^{d}$

Eigen vectors of B: $\{|\tau_k\rangle\}_{i=1}^{d}$

Express $|\xi\rangle$ in the orthonormal basis $\{|\tau_k\rangle\}_{i=1}^{d}$:

$$|\xi\rangle = \sum_k d_k \, |\tau_k\rangle$$

Upon measurement of B, the probability of getting the ith Eigen value as result:

$$p(b_i|B,\xi) = |\langle\tau_i|\xi\rangle|^2 = |d_i|^2$$

Quantum mechanical description of spin

- **Spin is associated with a two dimensional Hilbert space**

State $\longrightarrow$ Normalised vector $\begin{bmatrix} a \\ b \end{bmatrix}$

a, b complex and

$$|a|^2 + |b|^2 = 1$$

Observable $\longrightarrow$ 2×2 self adjoint matrix

$$\begin{bmatrix} m & p \\ n & q \end{bmatrix} = \begin{bmatrix} m^* & n^* \\ p^* & q^* \end{bmatrix}$$

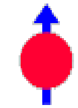
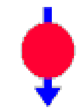
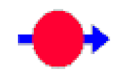
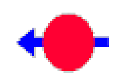
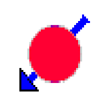
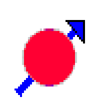
S.A. operator

	Eigen values	Eigen vector
A →	a_1	$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$
A →	a_2	$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

- **Eigen values are real.**
- **Eigen vectors are orthogonal, When** $a_1 \neq a_2$

$$\begin{bmatrix} x_1^* & y_1^* \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0$$

Some examples

Observable	Eigen values	Eigen vector		Symbol
$\sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	+1	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$		$\vert 0\rangle$
	−1	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$		$\vert 1\rangle$
$\sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	+1	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$		$\vert 0_x\rangle = \frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle)$
	−1	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$		$\vert 1_x\rangle = \frac{1}{\sqrt{2}}(\vert 0\rangle - \vert 1\rangle)$
$\sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	+1	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$		
	−1	$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ i \end{bmatrix}$		

Spin measurement along arbitrary direction:

Let $\hat{n}$ be a unit vector. Then spin measurement along this direction is given by the following operator:

$$\hat{n}\,.\sigma = n_x\sigma_x + n_y\sigma_x + n_z\sigma_x$$
$$= Sin\theta Cos\phi\,\sigma_x + Sin\theta Sin\phi\,\sigma_y + Cos\theta\,\sigma_z$$

$$= \begin{bmatrix} Cos\theta & Sin\theta Cos\phi - iSin\theta Sin\phi \\ Sin\theta Cos\phi + iSin\theta Sin\phi & -Cos\theta \end{bmatrix}$$

Problem:

Show that the Eigen values of $\hat{n}\,.\sigma$ are $+1$ and -1 and corresponding Eigen vectors are

$$\begin{bmatrix} Cos\frac{\theta}{2} \\ e^{i\phi}\,Sin\frac{\theta}{2} \end{bmatrix} = Cos\frac{\theta}{2}|0\rangle + e^{i\phi}\,Sin\frac{\theta}{2}|1\rangle$$

$$\begin{bmatrix} e^{-i\phi}\,Sin\frac{\theta}{2} \\ Cos\frac{\theta}{2} \end{bmatrix} = Cos\frac{\theta}{2}|1\rangle - e^{-i\phi}\,Sin\frac{\theta}{2}|0\rangle$$

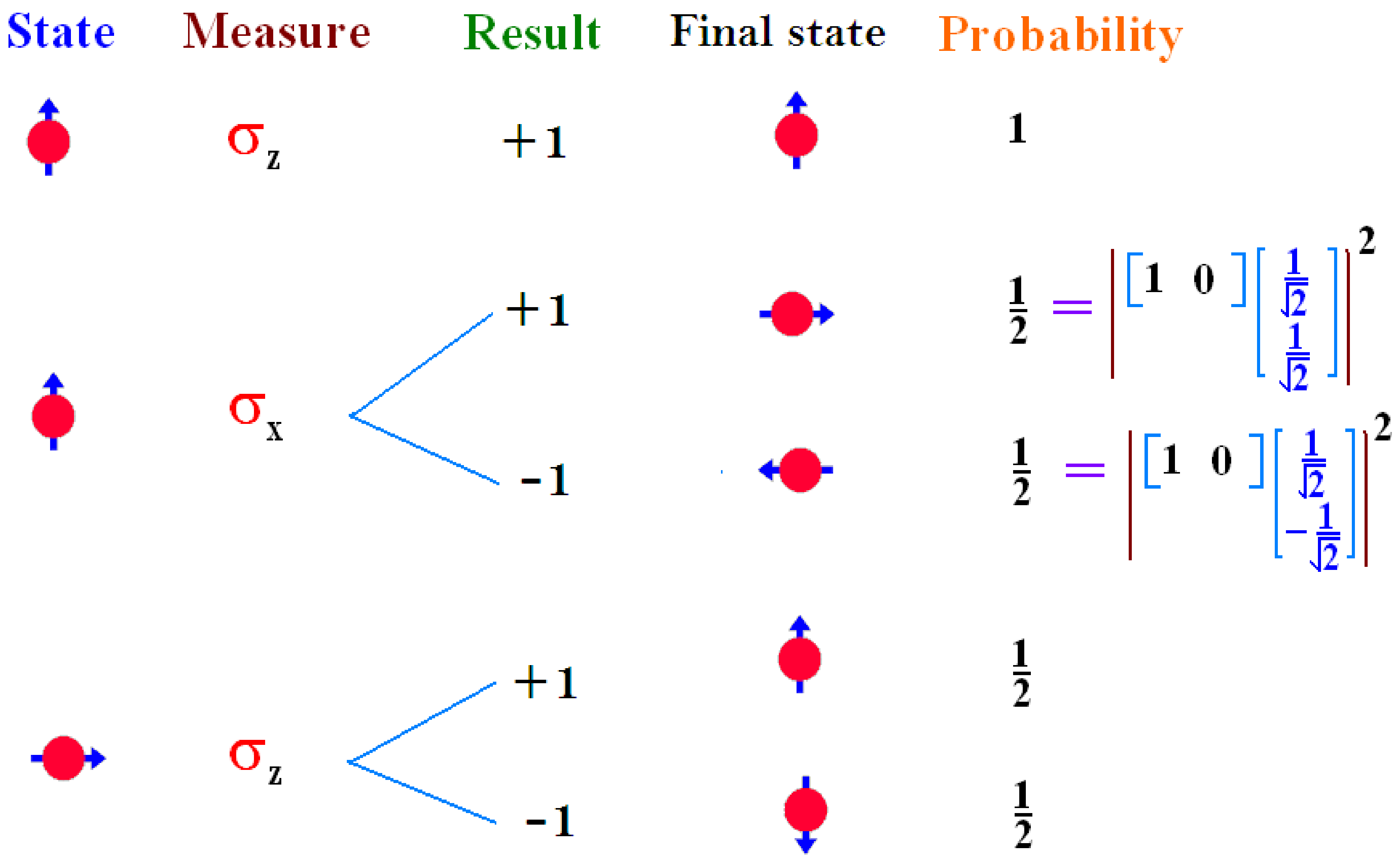
Measurement rules

$$\text{Initial state} = \begin{bmatrix} x \\ y \end{bmatrix}$$

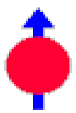
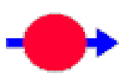
Measurement of A

Possible results	Probabilities	Final State
a_1	$\left\| \begin{bmatrix} x^* & y^* \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right\|^2$	$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$
a_2	$\left\| \begin{bmatrix} x^* & y^* \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\|^2$	$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

Example of spin measurement



Uncertainty relation

State	Measure	Result	Probability
	σ_z	$+1$	1 (certain)
	σ_x	$+1$	$\frac{1}{2}$ (uncertain)
		-1	$\frac{1}{2}$
	σ_z	$+1$	$\frac{1}{2}$ (uncertain)
		-1	$\frac{1}{2}$
	σ_x	$+1$	1 (certain)

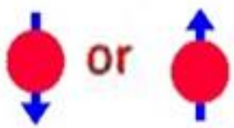
There is no state for which both σ_z and σ_x are certain.

Physical Consequences of non-orthogonal states

Alice prepares a qubit

Can Bob determine the state?

either in $|0\rangle$ or $|1\rangle$



Yes Measure σ_z

 : **Probability of getting down is zero.**

either in $|0_x\rangle$ or $|1_x\rangle$

Yes Measure σ_x

Orthogonal states can be determined.

Non-orthogonal states

Can Bob determine the state?

Alice prepares a qubit

either in $|0\rangle$ or $|0_x\rangle$

	result	conclusion
Measure σ_z	+1	fail
	-1	$\|0_x\rangle$

Probability for up is non-zero for both and

	result	conclusion
Measure σ_x	+1	fail
	-1	$\|0\rangle$

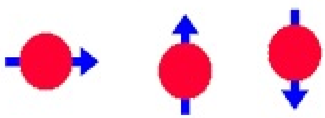
Non-orthogonal states can neither be reliably determined nor cloned.

Linearly Dependent set of states

Can Bob determine the state?

Alice prepares a qubit in one of the following states

$|0_X\rangle$, $|0\rangle$, $|1\rangle$



	result	conclusion
Measure σ_Z	+1	$\lvert 0\rangle$ or $\lvert 0_X\rangle$
	-1	$\lvert 1\rangle$ or $\lvert 0_X\rangle$
Measure σ_X	+1	$\lvert 0_X\rangle$ or $\lvert 0\rangle$ or $\lvert 1\rangle$
	-1	$\lvert 0\rangle$ or $\lvert 1\rangle$

Linearly dependent states can not even probabilistically be determined.

More about linear operators:

What kind of mathematical object $|\alpha\rangle\langle\beta|$ is?

$$|\alpha\rangle\langle\beta| = \begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\end{bmatrix}[\alpha_1^* \quad \alpha_2^* \quad \ldots]$$

This is a linear operator i.e a $d \times d$ matrix.

One can show that collection of linear operators acting on a d dimensional Hilbert space forms a linear space of dimension d^2.

Problem: Consider an orthonormal basis $\{|\phi_i\rangle\}_{i=1}^{d}$ of Hilbert space of dimension d.

Show that the d^2 operators $\{|\phi_i\rangle\langle\phi_j|\}_{i,j=1}^{d}$ form a linearly independent set.

Hence any operator A can be expressed as

$$A = \sum_{i,j} c_{ij} \, |\phi_i\rangle\langle\phi_j|$$

Projection operator:

(1) It is positive i.e $\langle\alpha|P|\alpha\rangle \geq 0$ *for all* $|\alpha\rangle$.

(2) It is idempotent i.e $P^2 = P$.

Let A is a self adjoint operator with eigen value equations:

$$A|\xi_j\rangle = a_j|\xi_j\rangle, j = 1, 2 \quad d$$

Then A can be expressed as

$$A = \sum_{j=1}^{d} a_j \, |\xi_j\rangle\langle\xi_j| = \sum_{j=1}^{d} a_j \, P_j$$

Where $P_j's (= |\xi_j\rangle\langle\xi_j|)$ are one dimensional projection operators.

This is called spectral representation.

Born rule of quantum probability

$$p(a_i|\psi, A) = \langle\psi|P_i|\psi\rangle$$

$$= \langle\psi|P_i|\psi\rangle = \langle\psi|\xi_i\rangle\langle\xi_i|\psi\rangle = |\langle\psi|\xi_i\rangle|^2$$

Expectation value of observable:

$$\langle A\rangle_\psi = \sum_{i=1}^{d} a_i\, p(a_i|\psi, A) = \sum_{i=1}^{d} a_i \langle\psi|P_i|\psi\rangle$$

$$= \left\langle\psi\left|\left(\sum_{i=1}^{d} a_i P_i\right)\right|\psi\right\rangle = \langle\psi|A|\psi\rangle$$

Description for Composite System:

$$S_1 \Rightarrow H_1 \; and \; S_2 \Rightarrow H_2$$

$$S_1 + S_2 \Rightarrow H_1 \otimes H_2$$

Tensor product Hilbert space:

$$|\psi\rangle \in H_1 \; and \; |\phi\rangle \in H_2$$

$$|\psi\rangle \otimes |\phi\rangle \in H_1 \otimes H_2$$

- **The dimension of $H_1 \otimes H_2$ is $d_1 \otimes d_2$.**
- $(a|\psi_1\rangle + b|\psi_2\rangle) \otimes |\phi\rangle = a\,|\psi_1\rangle \otimes |\phi\rangle + b\,|\psi_2\rangle \otimes |\phi\rangle$
- **Scalar Product between $|\psi_1\rangle \otimes |\phi_1\rangle$ and $|\psi_2\rangle \otimes |\phi_2\rangle$:**

$$(\langle\psi_1| \otimes \langle\phi_1|)(|\psi_2\rangle \otimes |\phi_2\rangle) = \langle\psi_1|\psi_2\rangle\langle\phi_1|\phi_2\rangle$$

- **Action of linear operator:**

$$A \otimes B\,(|\psi\rangle \otimes |\phi\rangle) = A|\psi\rangle \otimes B|\phi\rangle$$

Understanding the dimension of $H_1 \otimes H_2$

Example:

$$\text{Dim}(H_1) = 3 \text{ and } \text{Dim}(H_2) = 2$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \otimes \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a\alpha \\ a\beta \\ b\alpha \\ b\beta \\ c\alpha \\ c\beta \end{bmatrix}$$

$$\text{Dim}(H_1 \otimes H_2) = 6$$

Composite System

Product state:

The state $|\psi\rangle_{12}$ for a joint system $(S_1 + S_2)$ is called product state when

$$|\psi\rangle_{12} = |\xi\rangle_1 \otimes |\tau\rangle_2$$

Example:

$$|\psi\rangle_{12} = a|0\rangle_1|1\rangle_2 + b|1\rangle_1|1\rangle_2 = (a|0\rangle_1 + b|1\rangle_1)|1\rangle_2$$

Entangled states:

States that can not be reduced in the product form.

Example: $|\phi\rangle_{12} = a|0\rangle_1|0\rangle_2 + b|1\rangle_1|1\rangle_2, \quad a, b \neq 0$

Hint:

$$|\phi\rangle_{12} = |\alpha\rangle_1|\beta\rangle_2$$

$$= (c|0\rangle_1 + d|1\rangle_1)(e|0\rangle_1 + f|1\rangle_1)$$

$$= ce|0\rangle_1|0\rangle_2 + cf|0\rangle_1|1\rangle_2 + de|1\rangle_1|0\rangle_2 + df|1\rangle_1|1\rangle_2$$

Unitary operator:

An operator U is unitary when it satisfies the following condition;

$$UU^\dagger = U^\dagger U = I$$

✱ What is the nature of the Eigen values of U ?

$$U|\psi\rangle = \lambda|\psi\rangle$$

Conjugate transpose of the equation:

$$\langle\psi|U^\dagger = \lambda^*\langle\psi|$$

Taking scalar product: $\langle\psi|U^\dagger U|\psi\rangle = |\lambda|^2$

$$|\lambda|^2 = 1 \Rightarrow \quad \lambda = e^{i\theta}$$

✱ **Unitary operator preserves scalar product:**

Scalar product between $U|\psi\rangle$ and $U|\phi\rangle$:

$$\langle\psi|U^\dagger U|\phi\rangle = \langle\psi|\phi\rangle$$

Mathematical Theorem:

U is unitary if and only if $U = e^{iB}$

Where B is some self adjoint operator.

- Dynamics -

The evolution of closed quantum system is described by a unitary transformation:

$$|\psi>_{t=T} \;=\; U|\psi>_{t=0}$$

where U is a unitary operator satisfying the following relation:

The unitary operator is determined by the Hamiltonian operator H acting on the system in the following way:

$$U = e^{\frac{-i}{\hbar}\int_0^T H dt}$$

Example of Unitary operator in C^2 :

$\sigma_x, \sigma_y, \sigma_z$ are self adjoint operator.

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$$

Hence $\sigma_x, \sigma_y, \sigma_z$ are unitary operator.

$$\sigma_z|0\rangle = |0\rangle \quad \sigma_z|1\rangle = -|1\rangle$$

$$\sigma_x|0\rangle = |1\rangle \quad \sigma_x|1\rangle = |0\rangle$$

Prove that $\hat{n}.\sigma$ ($|\hat{n}| = 1$) is unitary operator

Hints:

(1) It is self adjoint

(2) $(\hat{n}.\sigma)(\hat{n}.\sigma)^\dagger = (\hat{n}.\sigma)(\hat{n}.\sigma) = I$

(3) $\sigma_x\sigma_y = i\sigma_z, \sigma_y\sigma_z = i\sigma_x, \sigma_z\sigma_x = i\sigma_y$ etc.

Hadamard gate

$$H = \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H^{\otimes 2}|0\rangle^{\otimes 2} = H|0\rangle H|0\rangle = \frac{1}{2}[|00\rangle + |01\rangle + [|10\rangle + |11\rangle)$$

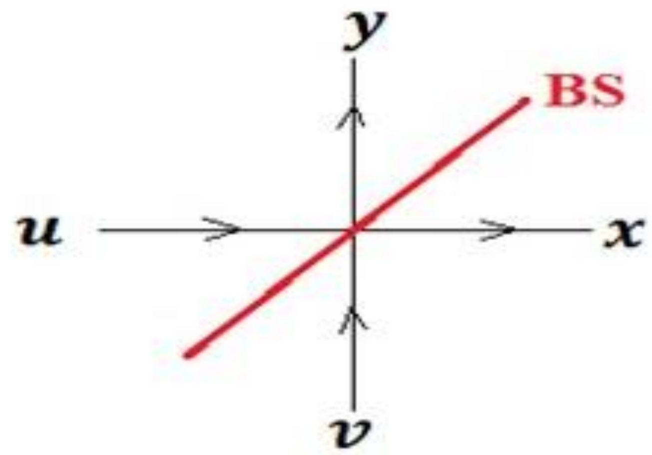
where $|ab\rangle \equiv |a\rangle|b\rangle$

$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$$

Where $|x\rangle$ is all possible binary string.

Understanding the peculier feature in interferometric experiment

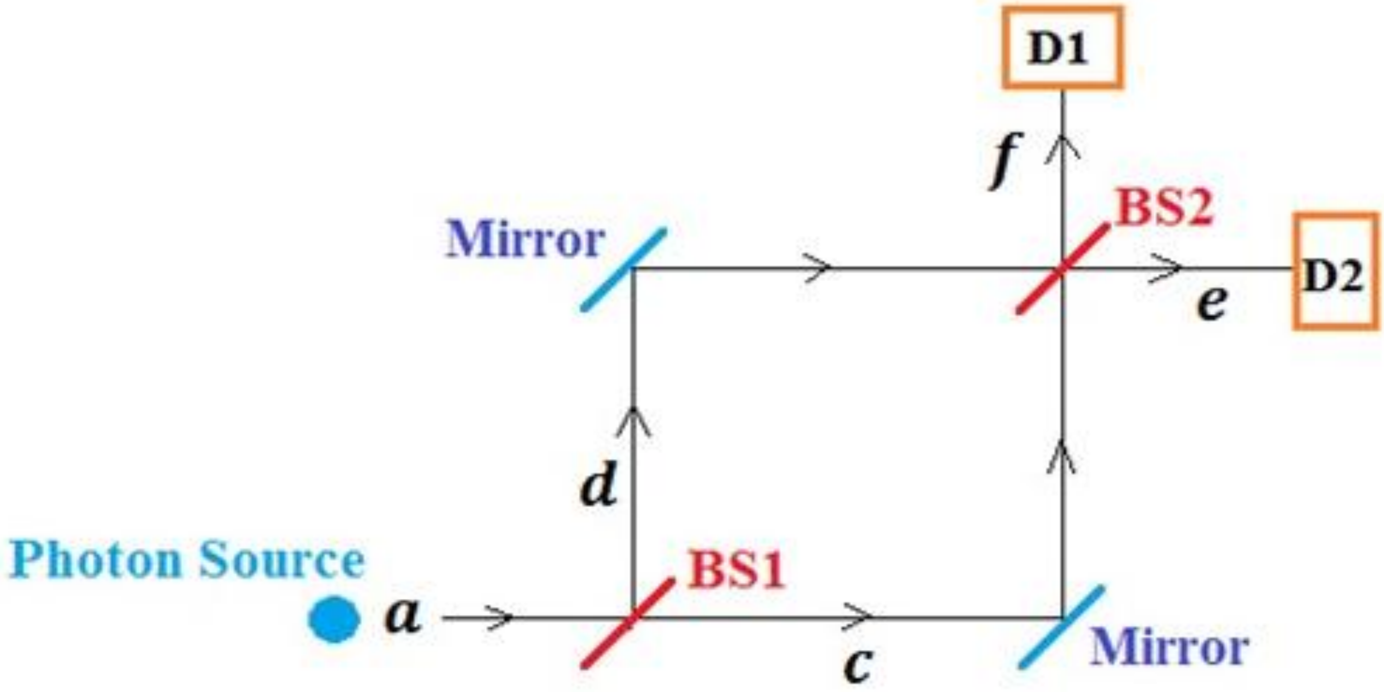
How beam Splitter functions



BS acts as unitary operator in the following way:

$$|u> \xrightarrow{BS} \frac{1}{\sqrt{2}}(|x> +i|y>)$$

$$|v> \xrightarrow{BS} \frac{1}{\sqrt{2}}(i|x> +|y>)$$



$$|a> \xrightarrow{\text{BS1}} \frac{1}{\sqrt{2}}(|c> + i|d>)$$

$$\xrightarrow{\text{BS2}} \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(i|e> + |f>\right) + \frac{i}{\sqrt{2}}(|e> + i|f>)\right] = i|e>$$

So all the particles will be collected at D2.

For given two vectors $|\alpha\rangle$ and $|\beta\rangle$, does there exist an unitary operator U such that

$$U|\alpha\rangle = |\beta\rangle$$

$$\begin{bmatrix} a_{11}^* & a_{21}^* & \cdots \\ a_{12}^* & a_{22}^* & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix}$$

$$UU^\dagger = I$$

- **Such unitary operator of course exists.**
- **There are infinitely many unitary operators.**

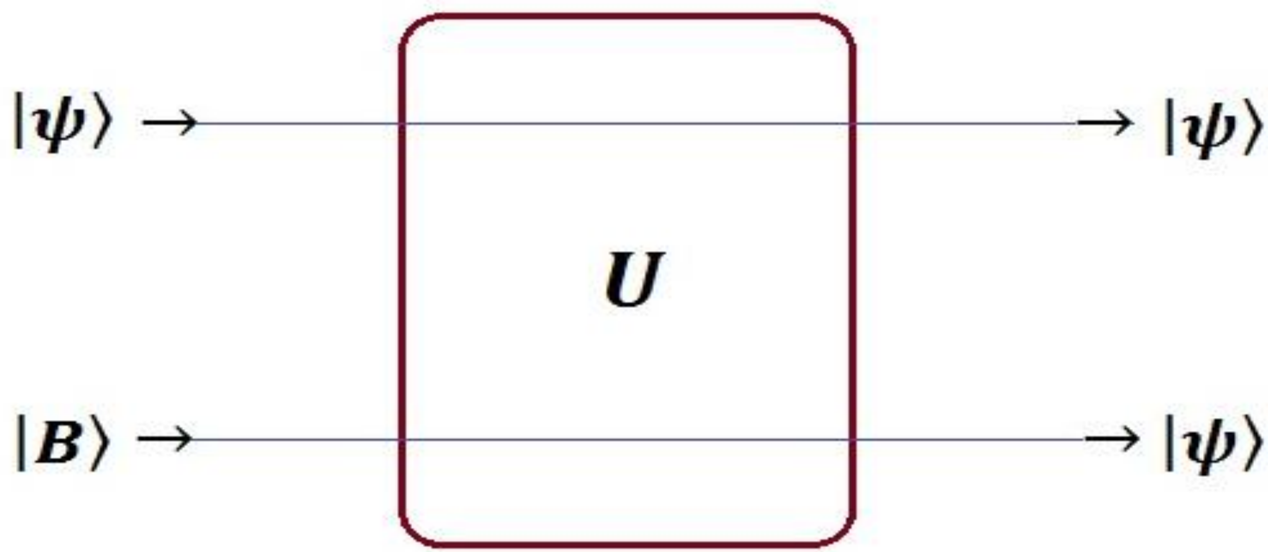
No cloning theorem

Quantum cloning machine

$$|\psi\rangle|B\rangle \rightarrow |\psi\rangle|\psi\rangle \quad \textit{for all } |\psi\rangle$$

(a) Deterministic evolution

(b) Universal



$$U|\psi\rangle|B\rangle \rightarrow |\psi\rangle|\psi\rangle \quad \textit{for all } |\psi\rangle$$

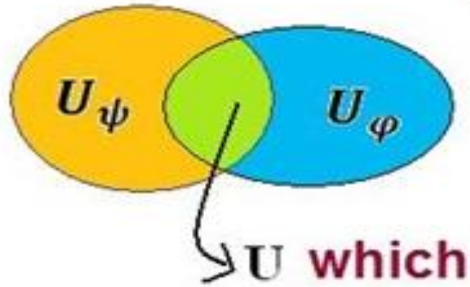
Consider two states $|\psi\rangle$ and $|\phi\rangle$.

$$U_\psi|\psi\rangle|b\rangle = |\psi\rangle|\psi\rangle$$

Infinitely many U_ψ

$$U_\phi|\phi\rangle|b\rangle = |\phi\rangle|\phi\rangle$$

Infinitely many U_ϕ



$$U|\psi\rangle|B\rangle \to |\psi\rangle|\psi\rangle$$

$$U|\phi\rangle|B\rangle \to |\phi\rangle|\phi\rangle$$

Taking scalar product:

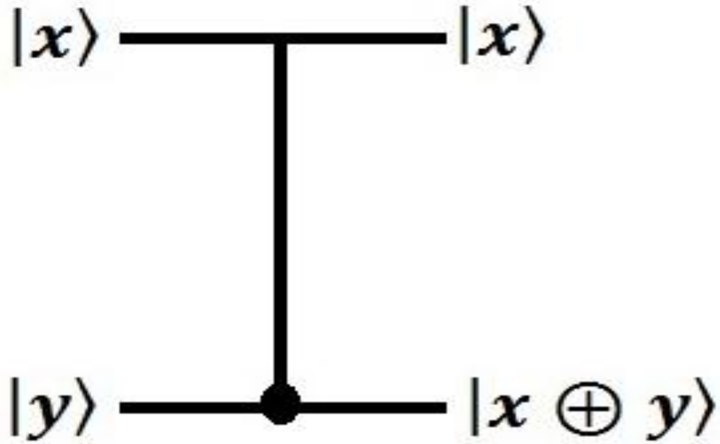
$$\langle\psi|\langle B\left|U^\dagger U\right||\phi\rangle|B\rangle = \langle\psi|\langle\psi|\phi\rangle|\phi\rangle$$

$$\langle\psi|\phi\rangle = \langle\psi|\phi\rangle^2$$

$$\langle\psi|\phi\rangle = 0 \; or \; 1$$

Just two non-orthogonal states cannot be cloned.

Two qubits Unitary gate:



$$U_{CNOT}|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle, \quad x, y \in \{0, 1\}$$

Flips the second qubit when the first qubit is in $|1\rangle$.

$$U_{CNOT}|0\rangle|0\rangle = |0\rangle|0\rangle$$

$$U_{CNOT}|0\rangle|1\rangle = |0\rangle|1\rangle$$

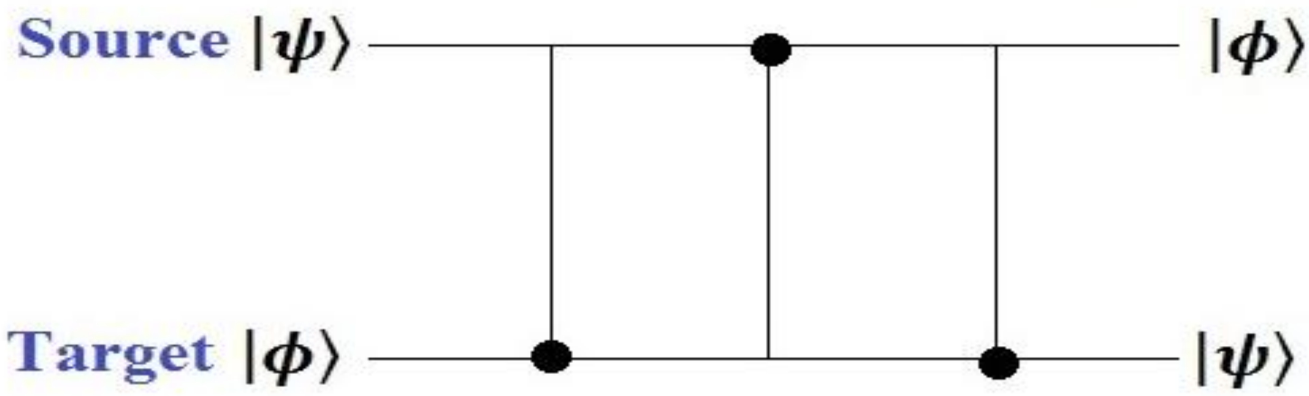
$$U_{CNOT}|1\rangle|0\rangle = |1\rangle|1\rangle$$

$$U_{CNOT}|1\rangle|1\rangle = |1\rangle|0\rangle$$

Swap gate

$$U_{SWAP}|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle \quad \textit{for all } |\psi\rangle \textit{ and } |\phi\rangle$$

Realization of U_{SWAP} by U_{CNOT}:



$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$|\phi\rangle = c|0\rangle + d|1\rangle$$

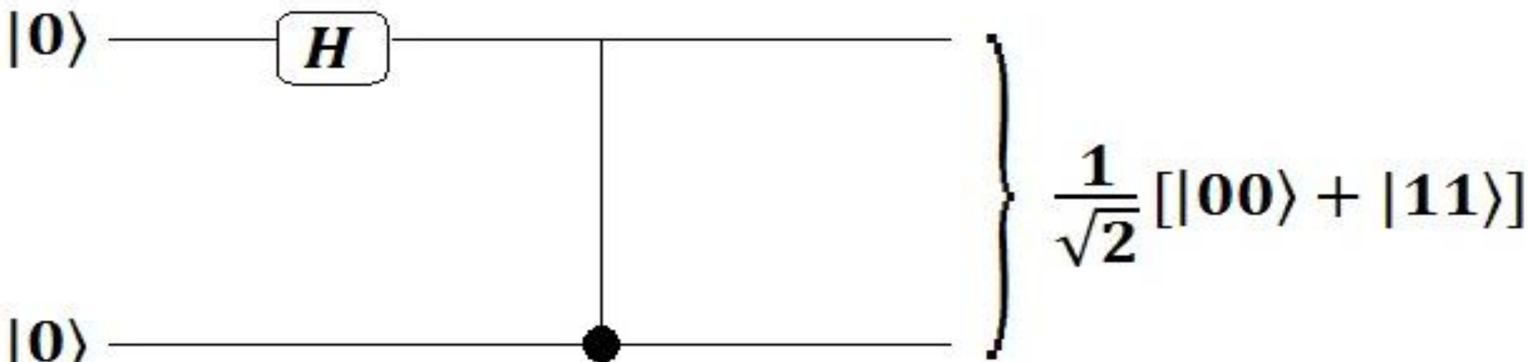
$$|\psi\rangle|\phi\rangle = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

$$\xrightarrow{U_{CNOT}} ac|00\rangle + ad|01\rangle + bc|11\rangle + bd|10\rangle$$

$$\to \; ac|00\rangle + ad|10\rangle + bc|01\rangle + bd|11\rangle$$

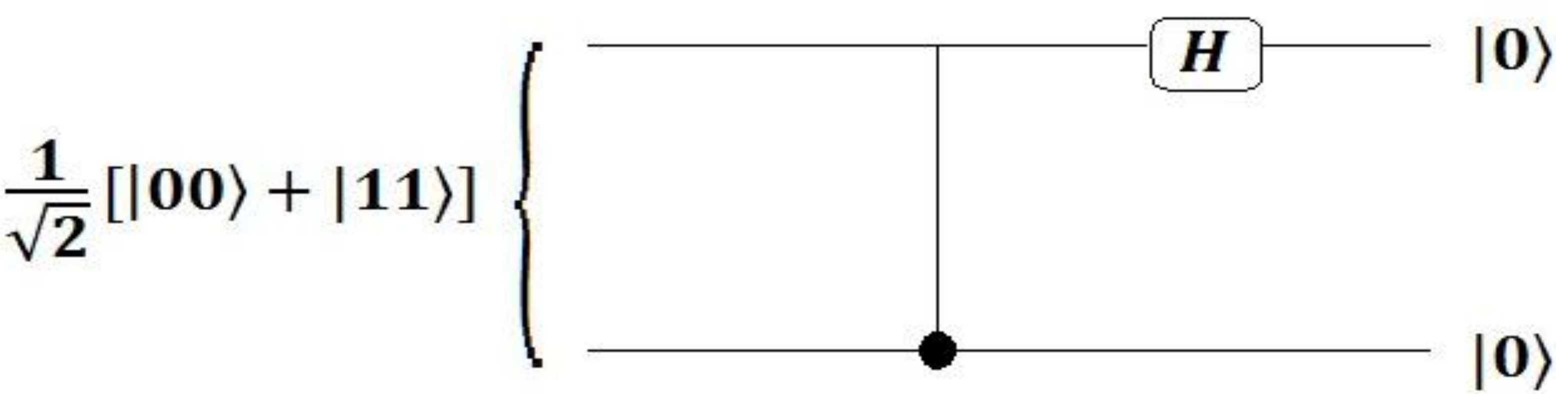
$$= (c|0\rangle + d|1\rangle)(a|0\rangle + b|1\rangle) = |\phi\rangle|\psi\rangle$$

Creating entanglement by U_{CNOT}



$$|00\rangle \rightarrow \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]|0\rangle \rightarrow \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

The reverse operation:



Two qubit Bell states

$$|\phi^+\rangle \rightarrow \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle] \qquad |\phi^-\rangle \rightarrow \frac{1}{\sqrt{2}}[|00\rangle - |11\rangle]$$

$$|\psi^+\rangle \rightarrow \frac{1}{\sqrt{2}}[|01\rangle + |10\rangle] \qquad |\psi^-\rangle \rightarrow \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$$

They form an orthonormal basis.

How to distinguish among Bell states:

$$(H \otimes I)\; U_{CNOT}\, |\phi^+\rangle \rightarrow |00\rangle$$

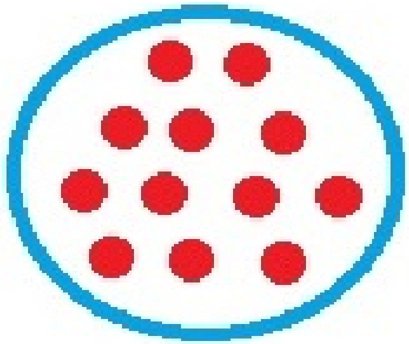
$$(H \otimes I)\; U_{CNOT}\, |\phi^-\rangle \rightarrow |10\rangle$$

$$(H \otimes I)\; U_{CNOT}\, |\psi^+\rangle \rightarrow |01\rangle$$

$$(H \otimes I)\; U_{CNOT}\, |\psi^-\rangle \rightarrow |11\rangle$$

Measuring in the $\{|0\rangle, |1\rangle\}$ basis for both the qubits, one can determine the Bell state.

General description of quantum state



A source contains quantum particles whose states are $|\psi_i\rangle$ with probability p_i.

If one randomly picks one particle from this source what is the quantumstate of the particle?

Obviously a single unit vector cannot do.

Trace operation:

How to find the trace of an operator A?

Let $\{|i\rangle\}$ ***is an orthonormal basis.***

$$Tr[A] = \sum_i \langle i|A|i\rangle$$

$$A = \sum_{jk} c_{jk}|j\rangle\langle k|$$

$$Tr[A] = \sum_i \langle i|\left(\sum_{jk} c_{jk}|j\rangle\langle k|\right)|i\rangle = \sum_i c_{ii}$$

(1)*Linear operation*: $Tr[A+B] = Tr[A] + Tr[B]$

(2) $Tr[AB] = Tr[BA]$

(a) $Tr[|\psi\rangle\langle\phi|] = \langle\phi|\psi\rangle$

(b) $Tr[A|\psi\rangle\langle\phi|] = \langle\phi|A|\psi\rangle$

States are density operator with following properties:

(1) It is self adjoint; $D = D^\dagger$

(2) It is positive; $\langle\psi|D|\psi\rangle \geq 0 \; for \; all \; \psi.$

(3) Its trace is one; $Tr[D] = 1$

Let D is density operator with the following extra property:

$$D^2 = D$$

Then $D = |\phi\rangle\langle\phi| \;\; for \; some \; unit \; vector \; |\phi\rangle.$

Show that the set of density operators forms a convex set.

Generalized Born rule:

State ⇛ Density operator

Observable ⇛ Self adjoint operator:

$$A = \sum_i a_i P_i$$

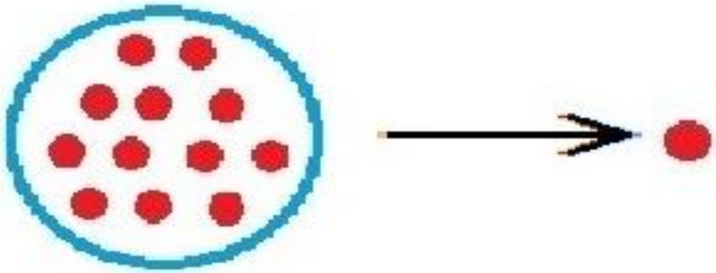
Probability for i-th result:

$$p(i|D, A) = Tr[DP_i]$$

When D *is pure i.e* $D = |\psi\rangle\langle\psi|$

$$p(i|D, A) = Tr[P_i|\psi\rangle\langle\psi|] = \langle\psi|P_i|\psi\rangle$$

Non-unique decomposition of Mixed state



$$D = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$D = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_j q_j |\phi_j\rangle\langle\phi_j| = \sum_k r_k |\tau_k\rangle\langle\tau_k| \dots..$$

Examples:

$$\frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|0_x\rangle\langle 0_x| + \frac{1}{2}|1_x\rangle\langle 1_x|$$

$$\frac{I}{2} = \frac{1}{4}\sum_{i=0,x,y,z} \sigma_i |\phi\rangle\langle\phi|\sigma_i$$ **where** $|\phi\rangle = a|0\rangle + b|1\rangle,\ |a|^2 + |b|^2 = 1$

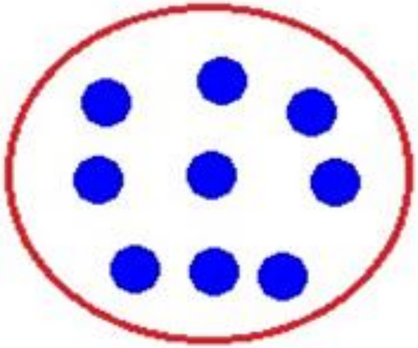
$$\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| = \frac{1}{2}|R\rangle\langle R| + \frac{1}{2}|L\rangle\langle L|$$

where

$$|R\rangle = \sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle, |L\rangle = \sqrt{p}|0\rangle - \sqrt{1-p}|1\rangle$$

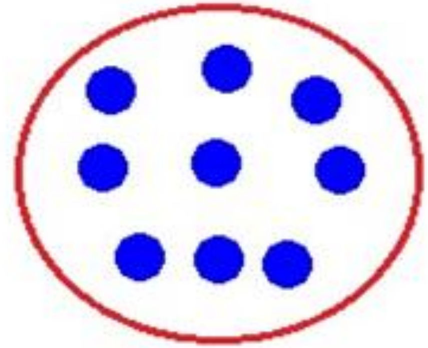
Which Box?

Box-1



$50\%\ |0\rangle \,\&\, 50\%\ |1\rangle$

Box-2



$50\%\ |+\rangle \,\&\, 50\%\ |-\rangle$

Consider any self adjoint operator A

$$\frac{1}{2}\langle 0|A|0\rangle + \frac{1}{2}\langle 1|A|1\rangle = \frac{1}{2}\langle +|A|+\rangle + \frac{1}{2}\langle -|A|-\rangle$$

These two ensembles can not be discriminated.

Density matrix of a particle taken from either of the boxes is same.

Symmetry of Maximally entangled state

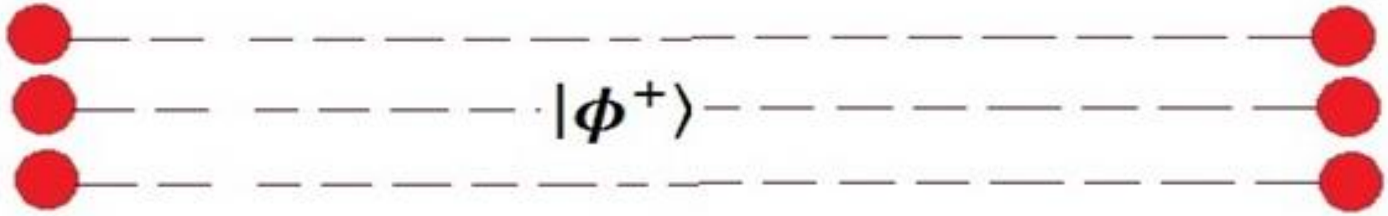
$$|\phi^+\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

The same state can also be written as

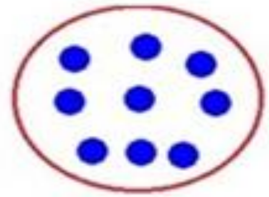
$$|\phi^+\rangle = \frac{1}{\sqrt{2}}[|+\,+\rangle + |-\,-\rangle]$$

where

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \textit{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

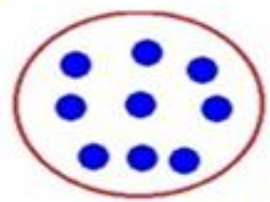


Measurement of $\boldsymbol{\sigma_z}$ on each left qubit:



50% $|\mathbf{0}\rangle$& **50%** $|\mathbf{1}\rangle$

Measurement of $\boldsymbol{\sigma_x}$ on each left qubit:



50% $|+\rangle$& **50%** $|-\rangle$

Possibility of distinguishing between these two ensembles would imply signaling.

General form of Density matrix for qubit:

$$D = a_0 I + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$$

(a) *Hermitian* $\Rightarrow a_0, a_x, a_y, a_z$ are real.

(b) Trace condion $\Rightarrow a_0 = \frac{1}{2}$

$$D = \frac{1}{2}\left[I + 2a_x\sigma_x + 2a_y\sigma_y + 2a_z\sigma_z\right]$$

$$= \frac{1}{2}[I + 2a.\sigma] = \frac{1}{2}[I + n.\sigma]\ (n = 2a)$$

$$n.\sigma = |n|\hat{n}.\sigma$$

Eigen values of D are $\frac{1}{2}(1 + |n|)$ and $\frac{1}{2}(1 - |n|)$.

(c) Positivity implies $\Rightarrow |n| \leq 1$

General form of density matrix for qubit

$$D = \frac{1}{2}[I + n.\sigma], |n| \leq 1$$

Pure state:

$$D^2 = D \Rightarrow D = \frac{1}{2}[I + n.\sigma], |n| = 1$$

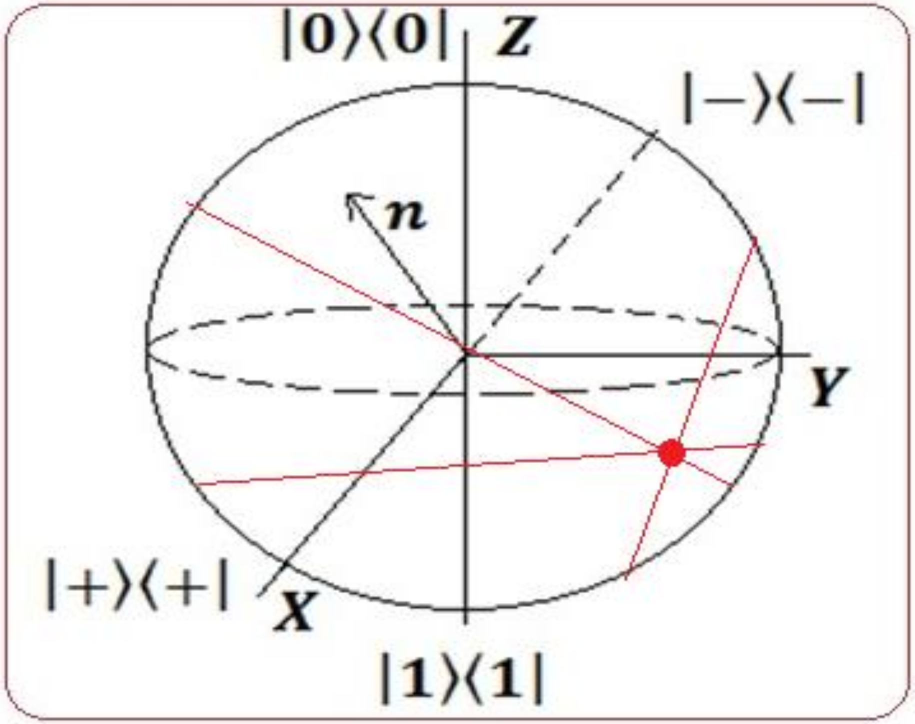
Example:

$$|0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}[I + \sigma_z]$$

$$|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}[I - \sigma_z]$$

$$|+\rangle\langle +| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} [\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}[I + \sigma_x]$$

Qubit states are represented by points in unit sphere of unit radious.



(a) Points on the surface represents pure state.

(b) Every non-pure or mixed state can be expressed as mixture of pure states in infinitely many ways.

Spin measurement along the vector $\boldsymbol{m}$:

$$\boldsymbol{m}.\boldsymbol{\sigma} = (+)\frac{1}{2}[I + \boldsymbol{m}.\boldsymbol{\sigma}] + (-1)\frac{1}{2}[I - \boldsymbol{m}.\boldsymbol{\sigma}], \text{ with } |\boldsymbol{m}| = 1$$

Born rule:

$$p(+|\boldsymbol{m}.\boldsymbol{\sigma}) = Tr\left[D\frac{1}{2}[I + \boldsymbol{m}.\boldsymbol{\sigma}]\right]$$

$$= Tr\left[\frac{1}{2}[I + \boldsymbol{n}.\boldsymbol{\sigma}]\frac{1}{2}[I + \boldsymbol{m}.\boldsymbol{\sigma}]\right]$$

$$= \frac{1}{2}(1 + \boldsymbol{n}.\boldsymbol{m})$$

Expectation value:

$$\langle \boldsymbol{m}.\boldsymbol{\sigma} \rangle = (+1)\frac{1}{2}[I + \boldsymbol{m}.\boldsymbol{n}] + (-1)\frac{1}{2}[I - \boldsymbol{m}.\boldsymbol{n}]$$

$$= \boldsymbol{m}.\boldsymbol{n}$$

Partial Trace

Consider a tensor product Hilbert space: $\boldsymbol{H_A \otimes H_B}$

$$\{|\phi_i\rangle\} \in H_A \ \textit{and} \ \{|\psi_j\rangle\} \in H_B$$

$$\{|\phi_i\rangle \otimes |\psi_j\rangle\} \in H_A \otimes H_B$$

$$\{|\phi_i\rangle \otimes |\psi_j\rangle \langle\phi_k| \otimes \langle\psi_l|\} \ \textit{form basis for operator space.}$$

$$O = \sum_{ijkl} c_{ijkl} \ |\phi_i\rangle_A\langle\phi_k| \otimes |\psi_j\rangle_B\langle\psi_l|$$

$$Tr_B[O] = \sum_m \langle\psi_m|O|\psi_m\rangle$$

$$= \sum_{ijklm} c_{ijkl} \ |\phi_i\rangle_A\langle\phi_k|\langle\psi_m|\psi_j\rangle_B\langle\psi_l|\psi_m\rangle$$

$$= \sum_{ik} d_{ik} \, |\phi_i\rangle_A\langle\phi_k|; \ \textit{an operator on } H_A.$$

How to find the density matrix of a subsystem?

Density matrix of the composite system: D_{AB}

$$\rho_A = Tr_B[D_{AB}] = \sum_k \langle\phi_k|D_{AB}|\phi_k\rangle$$

$$\rho_B = Tr_A[D_{AB}] = \sum_j \langle\xi_j|D_{AB}|\xi_j\rangle$$

$\{|\phi_k\rangle\}$ $\{|\xi_j\rangle\}$ ***are orthonormal bases of*** H_B ***and*** H_A ***respectively.***

Example:

$$|\psi\rangle_{AB} = a|0\rangle_A \otimes |0\rangle_B + b|1\rangle_A \otimes |1\rangle_B$$

$$\langle F\rangle_B = \langle\psi|I \otimes F|\psi\rangle = Tr_{AB}[|\psi\rangle\langle\psi|(I \otimes F)]$$

$$= |a|^2\langle 0|F|0\rangle_B + |b|^2\langle 1|F|1\rangle_B = Tr_B[\rho_B F]$$

where

$$\rho_B = |a|^2|0\rangle\langle 0|_B + |b|^2|1\rangle\langle 1|_B = Tr_A[\,|\psi\rangle\langle\psi|]$$

Maximally entangled state

$$|\psi_{max}\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

$$\rho_A = Tr_B[\,|\psi_{max}\rangle\langle\psi_{max}|]$$

$$= \frac{1}{2}|0\rangle\langle 0|_A + \frac{1}{2}|1\rangle\langle 1|_A = \frac{I}{2}$$

Similarly; $\rho_B = \frac{I}{2}$

Bipartite Maximally entangled Sates:

(i) Density matrix are pure

(ii) Marginal density matrix are completely random.

Schmidt Decomposition

$\{|k\rangle, k = 1,2..d_A\}$ and $\{|j\rangle, j = 1,2..d_B\}$ are orthonormal bases of H_A and H_B respectively.

For any pure state of composite system:

$$|\psi\rangle_{AB} = \sum_{kj} \beta_{kj} |k\rangle_A \otimes |j\rangle_B$$

There exist another set of orthonormal base

$$\{|\phi_l\rangle, l = 1,2..d_A\} \; and \; \{|\xi_m\rangle, m = 1,2..d_B\},$$

such that

$$|\psi\rangle_{AB} = \sum_{i=1}^{min\{d_A,d_B\}} c_i \, |\phi_i\rangle_A \otimes |\xi_i\rangle_B$$

How to find Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_{kj} \beta_{kj} |k\rangle_A \otimes |j\rangle_B$$

Find:

$\rho_A = Tr_B[|\psi\rangle\langle\psi|]$ **and** $\rho_B = Tr_A[|\psi\rangle\langle\psi|]$

Eigen values of ρ_A and ρ_B :

$$\rho_A = \sum_l \lambda_l |\phi_l\rangle\langle\phi_l| \qquad \rho_B = \sum_l \lambda_l |\xi_l\rangle\langle\xi_l|$$

Then

$$|\psi\rangle_{AB} = \sum_{i=1}^{min\{d_A, d_B\}} c_i |\phi_i\rangle_A \otimes |\xi_i\rangle_B$$

where $c_i = \sqrt{\lambda_i}$

Example:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$$

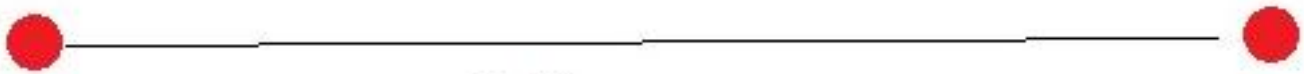
Show that the Schimdt form of the state will be the following:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}|0+\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1-\rangle$$

Characterization of entanglement:

- *A bipartite pure state is separable if the no. of Schimdt term is one.*
- *A bipartite pure state is entangled if and only if the no. of Schimdt term is more than one.*
- *For an entangled state, the density matrix of the subsystems are necessarily mixed.*

HJW Theorem



$$|\psi\rangle_{AB}$$

$$\rho_B = Tr_A[|\psi\rangle_{AB}\langle\psi|]$$

$$\rho_B = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_j q_j |\phi_j\rangle\langle\phi_j| = \sum_k r_k |\tau_k\rangle\langle\tau_k| \ldots\ldots$$

Alice can always choose appropriate measurement that prepares the desired ensemble.

Examples for qubits

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}[|00\rangle_{AB} + |11\rangle_{AB}]$$

Measure $\boldsymbol{\sigma_z}$

$$\frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

Measure $\boldsymbol{\sigma_x}$

$$\frac{I}{2} = \frac{1}{2}|0_x\rangle\langle 0_x| + \frac{1}{2}|1_x\rangle\langle 1_x|$$

$|\phi\rangle$

Measure in Bell basis

$$\frac{I}{2} = \frac{1}{4}\sum_{i=0,x,y,z} \sigma_i|\phi\rangle\langle\phi|\sigma_i$$

Exploiting No-go theorems to perform useful information processing tasks:

* No cloning
* No information without disturbance

} Secret key generation (BB84)

* Monogamous nature of entanglement
* No information without disturbance

} Secret key generation (Ekart, 1991)

* Entanglement
* Quantum oprations

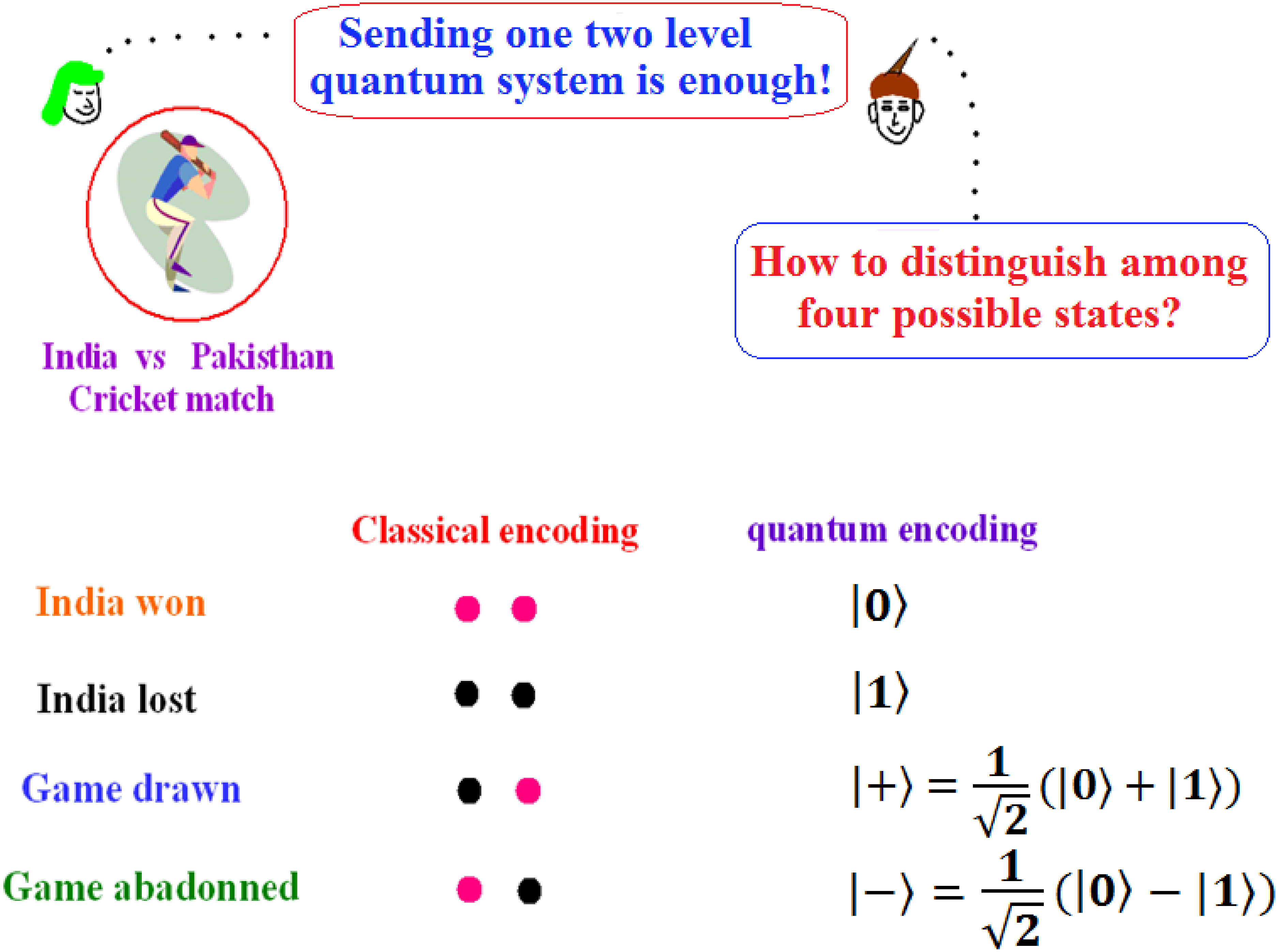
} Quantum dense coding (1992)
Quantum teleportation (1993)

Quantum dense coding

Two level system	Allowed state	
Classical	• •	Two different states
Quantum (Two dimensional Hilber space)	$|0\rangle$ $|1\rangle$ $a||0\rangle + b|1\rangle$	Infinitely many different states

$$|a|^2 + |b|^2 = 1$$

So one two level quantum system can be used to encode enormous classical information.


Sending one two level quantum system is enough!
How to distinguish among four possible states?
India vs Pakisthan Cricket match
Classical encoding
quantum encoding
India won
$|0\rangle$
India lost
$|1\rangle$
Game drawn
$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
Game abadonned
$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Successful quantum encoding

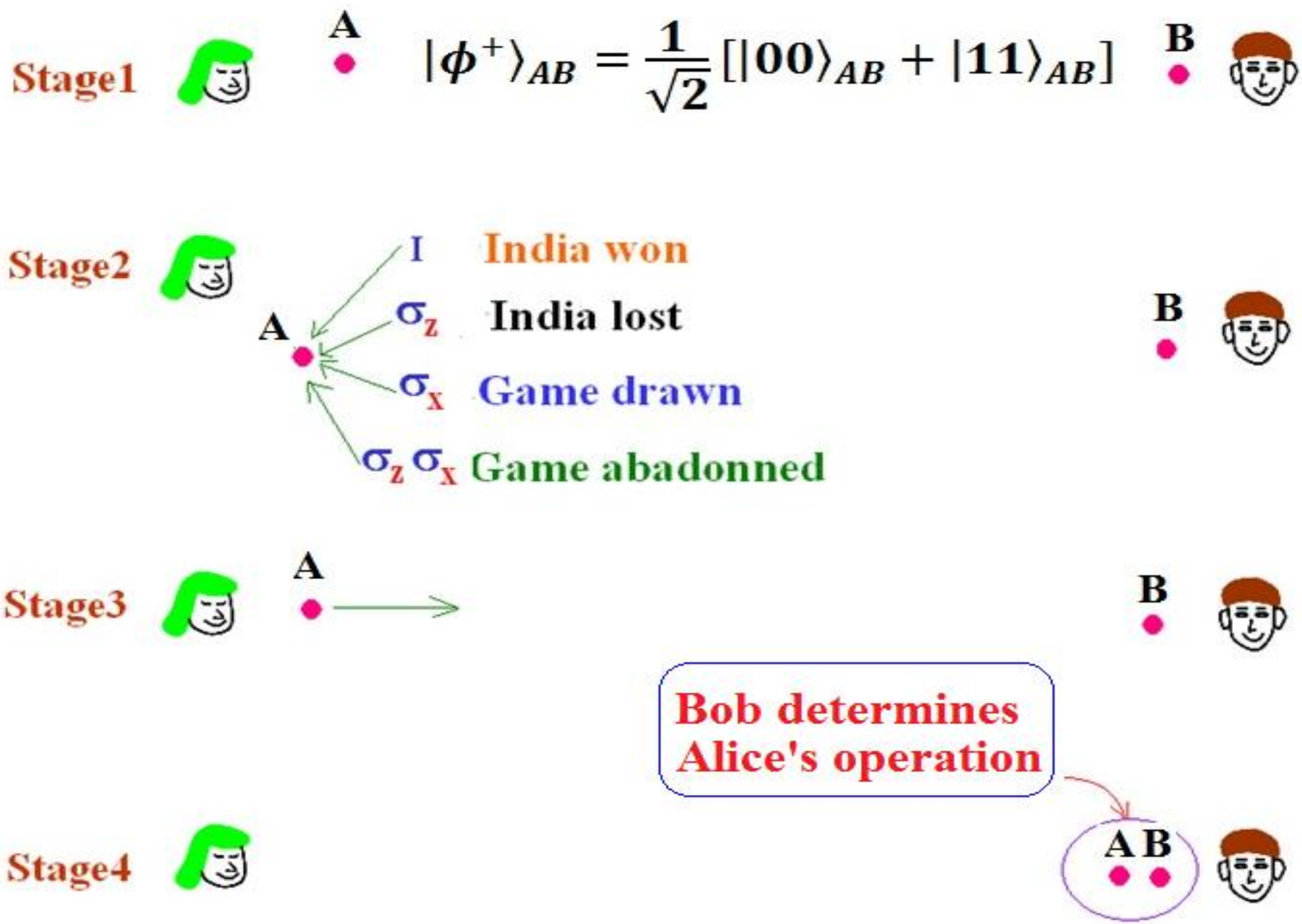
	Product states	entangled states
India won	$\|0\rangle_1\|0\rangle_2$	$\|\phi^+\rangle = \frac{1}{\sqrt{2}}[\|00\rangle + \|11\rangle]$
India lost	$\|1\rangle_1\|1\rangle_2$	$\|\phi^-\rangle = \frac{1}{\sqrt{2}}[\|00\rangle - \|11\rangle]$
Game drawn	$\|0\rangle_1\|1\rangle_2$	$\|\psi^+\rangle = \frac{1}{\sqrt{2}}[\|01\rangle + \|10\rangle]$
Game abadonned	$\|1\rangle_1\|0\rangle_2$	$\|\psi^-\rangle = \frac{1}{\sqrt{2}}[\|01\rangle - \|10\rangle]$

So quantum theory provide no advantage.

Restriction:

One is allowed to send only one two level (classical/quantum) system after the information is generated.

A successful Quantum protocol:



How it works:

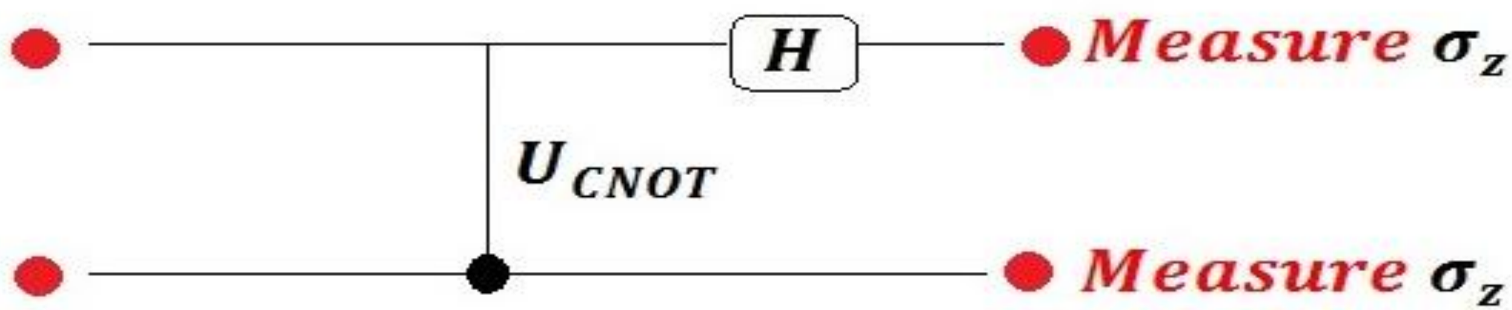
$$I_A \otimes I_B|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}[|00\rangle_{AB} + |11\rangle_{AB}] \;\; = |\phi^+\rangle_{AB}$$

$$\sigma_{z_A} \otimes I_B|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}[|00\rangle_{AB} - |11\rangle_{AB}] \;\; = |\phi^-\rangle_{AB}$$

$$\sigma_{x_A} \otimes I_B|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}[|01\rangle_{AB} + |10\rangle_{AB}] \;\; = |\psi^+\rangle_{AB}$$

$$\sigma_x\sigma_{z_A} \otimes I_B|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}[|01\rangle_{AB} - |10\rangle_{AB}] = |\psi^-\rangle_{AB}$$

Bob applies the following operations



to learn the information.

Quantum teleportation

The state $|\psi\rangle$ has to be prepared in Bob's lab.

$|\psi\rangle$

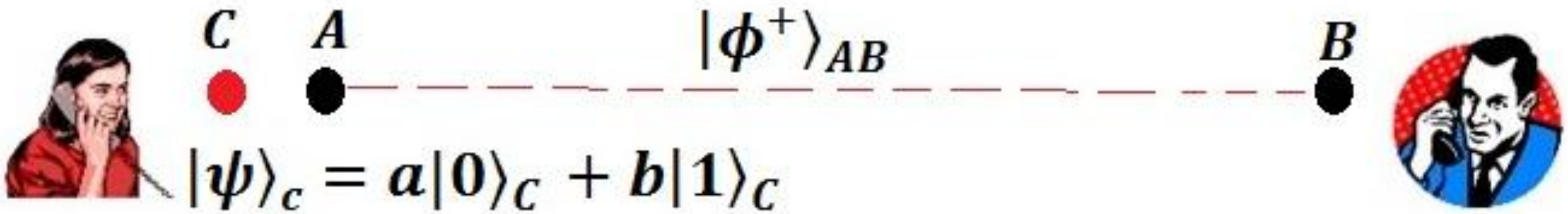
Physical transfer of particle is not allowed

$|\psi\rangle$

Preparing quantum state at distant location

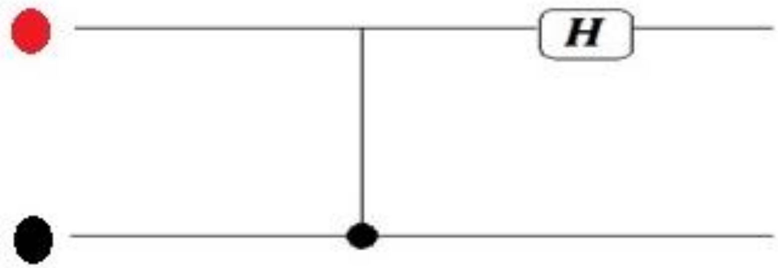
- **State unknown** — Quantum teleportation
- **State known** — Remote state preparation

Teleportation protocol



$|\psi\rangle_c = a|0\rangle_C + b|1\rangle_C$

- **Alice performs:**



- **Alice measures σ_z on both and convey the result to Bob by using 2 bits.**
- **Bob performs unitary operations in the following way:**

$$\begin{aligned} 00 &\rightarrow I \\ 10 &\rightarrow \sigma_z \\ 01 &\rightarrow \sigma_x \\ 11 &\rightarrow \sigma_z\sigma_x \end{aligned}$$

How it works:

$$|\psi\rangle_C\,|\phi^+\rangle_{AB} = (a|0\rangle_C + b|1\rangle_C)\frac{1}{\sqrt{2}}[|00\rangle_{AB} + |11\rangle_{AB}]$$

$$= \frac{a}{\sqrt{2}}|00\rangle_{AC}|0\rangle_B + \frac{a}{\sqrt{2}}|01\rangle_{AC}|1\rangle_B + \frac{b}{\sqrt{2}}|10\rangle_{AC}|0\rangle_B + \frac{b}{\sqrt{2}}|11\rangle_{AC}|1\rangle_B$$

Express product states in terms of Bell basis:

$$|00\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\phi^-\rangle)$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle)$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle)$$

$$|11\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle - |\phi^-\rangle)$$

The three qubits state can be expressed as

$$\frac{1}{2}[|\phi^+\rangle_{AC}(a|0\rangle_B + b|1\rangle_B) + |\phi^-\rangle_{AC}(a|0\rangle_B - b|1\rangle_B)$$

$$+ \ |\psi^+\rangle_{AC}(a|1\rangle_B + b|0\rangle_B) + |\psi^-\rangle_{AC}(a|1\rangle_B - b|0\rangle_B)]$$

$$= \frac{1}{2}[|\phi^+\rangle_{AC}\,|\psi\rangle_B + |\phi^-\rangle_{AC}\,\sigma_z|\psi\rangle_B$$

$$+ |\psi^+\rangle_{AC}\,\sigma_x|\psi\rangle_B + |\psi^-\rangle_{AC}\,\sigma_z\sigma_x|\psi\rangle_B]$$

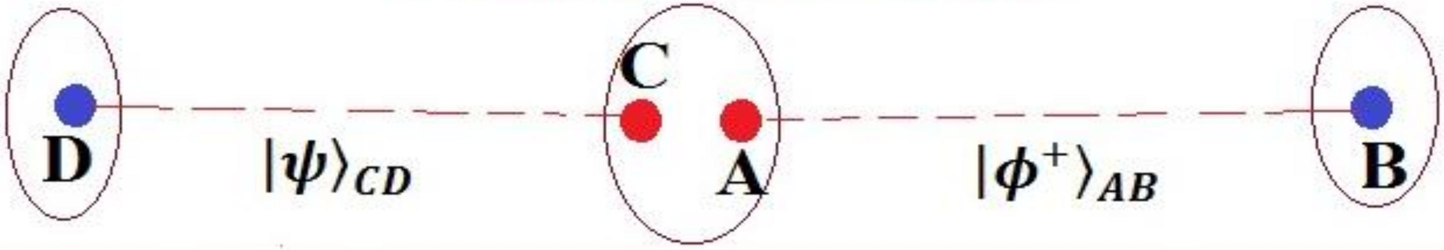
After Alice's operation $U_{CNOT}\,(H \otimes I)$:

$$\rightarrow \ \frac{1}{2}[|00\rangle_{AC}\,|\psi\rangle_B + |10\rangle_{AC}\,\sigma_z|\psi\rangle_B$$

$$+ \ |01\rangle_{AC}\,\sigma_x|\psi\rangle_B + |11\rangle_{AC}\,\sigma_z\sigma_x|\psi\rangle_B]$$

Hence the protocol works.

Entanglement swaping:



Alice and Bob performs the teleportation protocol and finally the state shared between Charlie and Alice is shared Charlie and Bob.

Hint:

$$|\psi\rangle_{CD}\,|\phi^+\rangle_{AB} = (a|00\rangle_{CD} + b|11\rangle_{Cd})\frac{1}{\sqrt{2}}[|00\rangle_{AB} + |11\rangle_{AB}]$$

$$= \frac{a}{\sqrt{2}}|0\rangle_D|00\rangle_{AC}|0\rangle_B + \frac{a}{\sqrt{2}}|0\rangle_D|01\rangle_{AC}|1\rangle_B + \frac{b}{\sqrt{2}}|1\rangle_D|10\rangle_{AC}|0\rangle_B + |1\rangle_D\frac{b}{\sqrt{2}}|11\rangle_{AC}|1\rangle_B$$

$$= \frac{1}{2}[|\phi^+\rangle_{AC}\,|\psi\rangle_{DB} + |\phi^-\rangle_{AC}\,I\otimes\sigma_z|\psi\rangle_{DB} + |\psi^+\rangle_{AC}\,I\otimes\sigma_x|\psi\rangle_{DB} + |\psi^-\rangle_{AC}\,I\otimes\sigma_z\sigma_x|\psi\rangle_{DB}]$$

Remote state preparation:

$|\phi\rangle_c = a|0\rangle_C + b|1\rangle_C$

- **In this scenario, the state is known and has to be preapared in Bob's lab.**

- **By using infinite bits of communication Alice can provide all the information of the parameter of the state and henceBob can prepare the state.**

- **Is there any protocol where by using entanglement one can reduce the cost of classical communication needed in teleportation protocol?**

$$|\psi^{-}\rangle \rightarrow \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$$

A ● - - - - - - - - - - - - - - - - ● B

$|\psi^{-}\rangle$ **has a beautiful symmetry property:**

$$|\psi^{-}\rangle \rightarrow \frac{1}{\sqrt{2}}[|\phi\rangle|\bar{\phi}\rangle - |\bar{\phi}\rangle|\phi\rangle]$$

for any qubit state $|\phi\rangle$.

Let us design the following protocol:

★ **Alice measures in the basis** $\{|\phi\rangle, |\bar{\phi}\rangle\}$ **and convey the result to Bob by using 1 bit.**

★ **Bob performs the universal Not or flip operation:**

$$U_f|\phi\rangle = |\bar{\phi}\rangle \textit{ for all } |\phi\rangle.$$

When Alice collapses on $|\phi\rangle$.

But unfortunately universal Not operation does not exist.

$$|\phi\rangle = a|0\rangle + b|1\rangle$$
$$|\overline{\phi}\rangle = b^*|0\rangle - a^*|1\rangle$$

Then there is no universal gate U such that

$$U|\phi\rangle = |\overline{\phi}\rangle \; \textit{for all } a \textit{ and } b.$$

Consider restricted set of state:

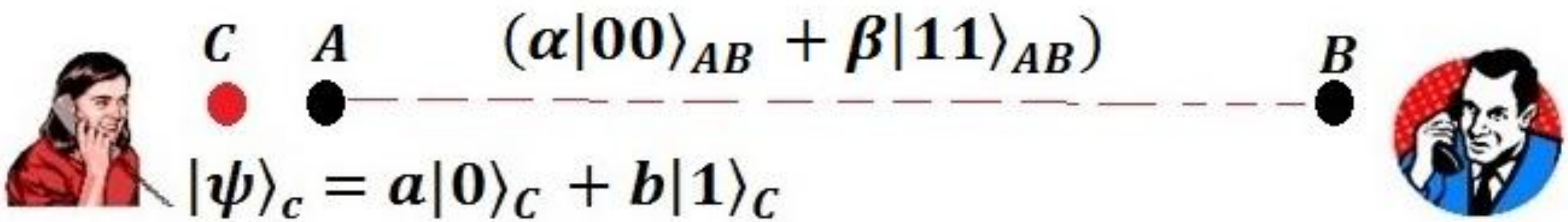
$$|\phi\rangle = a|0\rangle + b|1\rangle, \; a, b \textit{ are real}$$

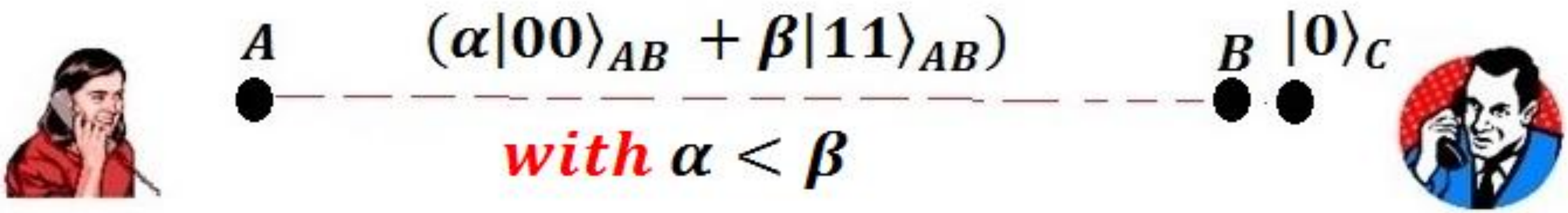
Then $|\overline{\phi}\rangle = b|0\rangle - a|1\rangle$

Then universal gate exists such that

$$\sigma_x \sigma_z (b|0\rangle - a|1\rangle) = a|0\rangle + b|1\rangle$$

Teleportation using non-maximally entangled state





Convert the state to a Bell state:

- Bob applies some unitary operatio on two qubits in his lab.
- Bob measures in $\{|0\rangle, |1\rangle\}$ basis on the extra qubit.
- Bob inform the result to Alice.

A $\quad (\alpha|00\rangle_{AB} + \beta|11\rangle_{AB}) \quad$ B $\quad |0\rangle_{B_1}$ $\quad B_1$

with $\alpha < \beta$

$$(\alpha|00\rangle_{AB} + \beta|11\rangle_{AB})\ |0\rangle_{B_1}$$

$$U|00\rangle_{BB_1} = |00\rangle_{BB_1}$$

$$U|10\rangle_{BB_1} = |1\rangle_B(c|0\rangle_{B_1} + d|1\rangle_{B_1})$$

Such that $\beta c = \alpha$

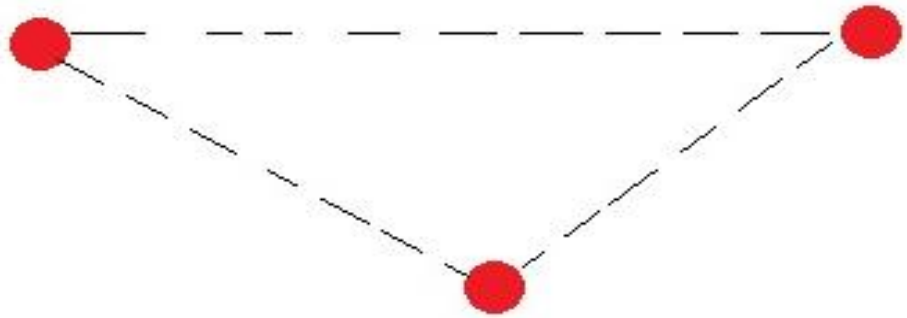
$$(\alpha|000\rangle_{ABB_1} + \beta c|110\rangle_{ABB_1}) + \beta d\ |111\rangle_{ABB_1}$$

$$= (\alpha|00\rangle_{AB} + \beta c|11\rangle_{AB})|0\rangle_{B_1} + \beta d\ |111\rangle_{ABB_1}$$

$$= \sqrt{2}\alpha \frac{1}{\sqrt{2}}[|00\rangle_{AB} + |11\rangle_{AB}]|0\rangle_{B_1} + \beta d\ |11\rangle_{AB}|1\rangle_{B_1}$$

With probability $2\alpha^2$ ***Bell state will be created.***

Three Qubits GHZ state:



$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}}[|000\rangle_{ABC} + |111\rangle_{ABC}]$$

Charlie can help Alice and Bob to share a Bell state.

- Charlie applies Hadamard H.
- Charlie measures in the computational basis and inform the result to one of them (say Alice).
- Alice applies some unitary operation accordingly.

How it works:

$$|GHZ\rangle_{ABC} \xrightarrow{H} \frac{1}{\sqrt{2}}|00\rangle_{AB}\frac{1}{\sqrt{2}}(|0\rangle_C + |1\rangle_C)$$

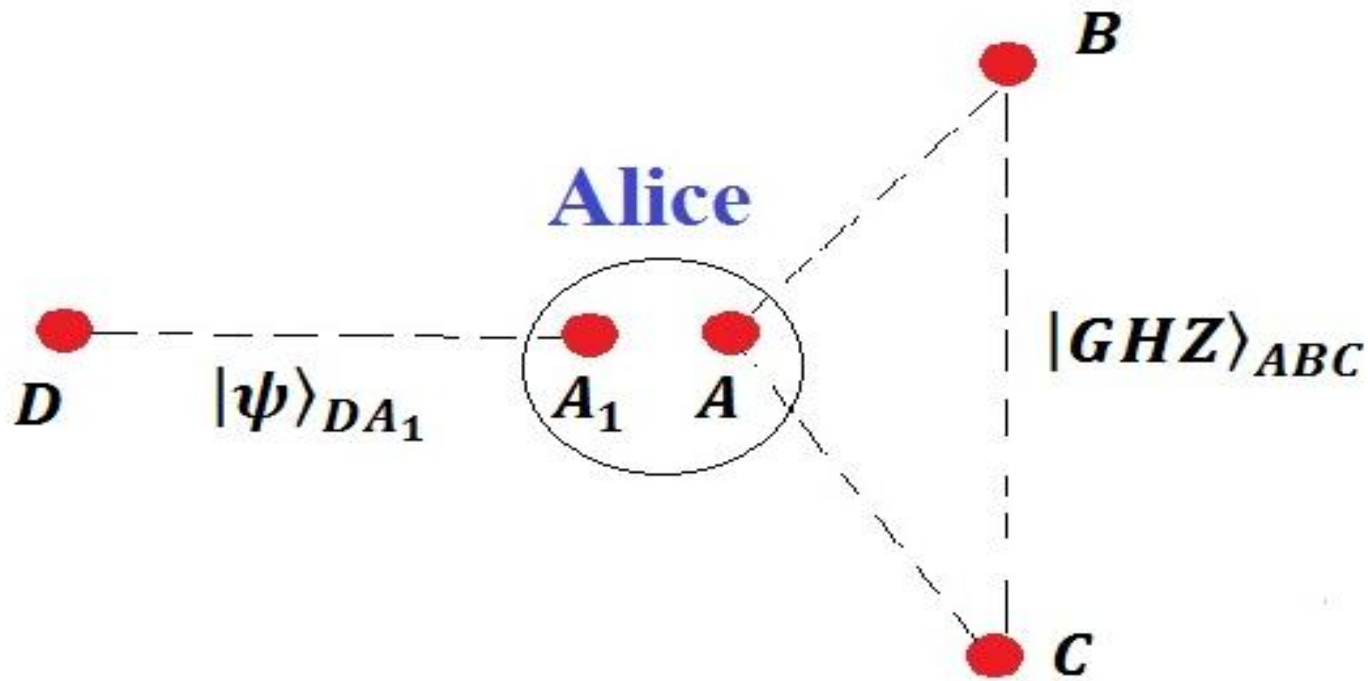
$$+\frac{1}{\sqrt{2}}|11\rangle_{AB}\frac{1}{\sqrt{2}}(|0\rangle_C - |1\rangle_C)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}[|00\rangle_{AB} + |11\rangle_{AB}]|0\rangle_C + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}[|00\rangle_{AB} - |11\rangle_{AB}]|1\rangle_C$$

$$= \frac{1}{\sqrt{2}}|\phi^+\rangle_{AB}|0\rangle_C + \frac{1}{\sqrt{2}}|\phi^-\rangle_{AB}|1\rangle_C$$

If Charlie collapses on $|0\rangle_C$ Alice does nothing otherwise she applies σ_Z on her qubit.

Transferring entanglement by GHZ state



where

$$|\psi\rangle_{DA_1} = |00\rangle_{DA_1} + b|11\rangle_{DA_1}$$

The task is to transfer the state between Alice and Dick to Bob and Charlie.

Protocol:

- Alice performs $(H \otimes I)C_{not}$ operation on her two qubits.
- She measures in the computational basis on both qubits and send the result to Bob and Charlie.
- Bob and Charlie perform following unitary operations based on the classical information from Alice.

00	$\rightarrow$	I^B
10	$\rightarrow$	σ_z^B
01	$\rightarrow$	σ_x^B
11	$\rightarrow$	$\sigma_z^B \sigma_x^B$

00	$\rightarrow$	I^C
10	$\rightarrow$	I^C
01	$\rightarrow$	σ_x^C
11	$\rightarrow$	σ_x^C

- Dick applies Hadamard H.
- Dick measures in the computational basis and inform the result to one of them (say Bob).
- Bob applies some unitary operation accordingly.

How the protocol works:

$$|\psi\rangle_{DA_1}\,|GHZ\rangle_{ABC}$$

$$= (a|00\rangle_{DA_1} + b|11\rangle_{DA_1})\,\frac{1}{\sqrt{2}}[|000\rangle_{ABC} + |111\rangle_{ABC}]$$

$$= \frac{a}{\sqrt{2}}|00\rangle_{A_1A}\,|000\rangle_{DBC} + \frac{a}{\sqrt{2}}|01\rangle_{A_1A}\,|011\rangle_{DBC}$$

$$+ \frac{b}{\sqrt{2}}|10\rangle_{A_1A}\,|100\rangle_{DBC} + \frac{b}{\sqrt{2}}|11\rangle_{A_1A}\,|111\rangle_{DBC}$$

$$\left.\begin{aligned} = {} & \frac{1}{\sqrt{2}}|\phi^+\rangle_{AA_1}\,(a|000\rangle_{DBC} + b|111\rangle_{DBC}) \\ + {} & \frac{1}{\sqrt{2}}|\phi^-\rangle_{AA_1}(a|000\rangle_{DBC} - b|111\rangle_{DBC}) \\ + {} & \frac{1}{\sqrt{2}}|\psi^+\rangle_{AA_1}\,(a|011\rangle_{DBC} + b|100\rangle_{DBC}) \\ + {} & \frac{1}{\sqrt{2}}|\psi^-\rangle_{AA_1}\,(a|011\rangle_{DBC} - b|100\rangle_{DBC}) \end{aligned}\right\} \rightarrow (a|000\rangle_{DBC} + b|111\rangle_{DBC})$$

$$(a|000\rangle_{DBC} + b|111\rangle_{DBC}$$

$$\xrightarrow{H} \frac{a}{\sqrt{2}}(|0\rangle_D + |1\rangle_D)\,|00\rangle_{BC} + \frac{b}{\sqrt{2}}(|0\rangle_D - |1\rangle_D)|11\rangle_{BC}$$

$$= \frac{1}{\sqrt{2}}\,|0\rangle_D[a|00\rangle_{BC} + b|11\rangle_{BC}] + \frac{1}{\sqrt{2}}\,|1\rangle_D[a|00\rangle_{BC} - b|11\rangle_{BC}]$$

Obviously the protocol works.

Impossibility of Bit Committment in Quantum Mechanics

Before the game starts, Alice has to commit one of the result.

India will win

Or

India will lose

If final results comes true, Bob has to pay

Otherwise Alice has to pay

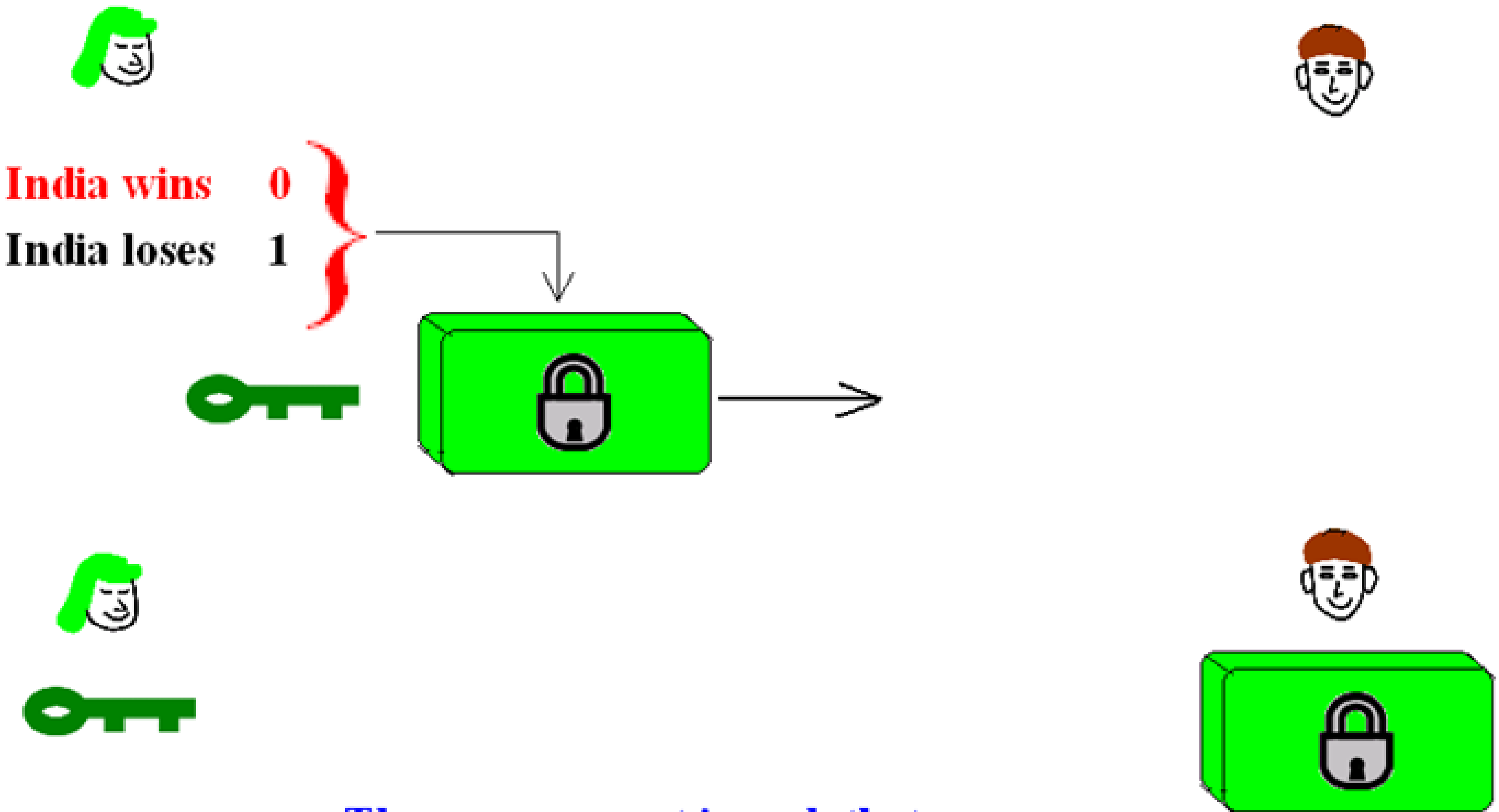
Condition :

Alice would not be able to change her commitment.

Bob would not be able to learn the Alice's commitment before she reveals it after the match ends.

Can it be made possible in QM?

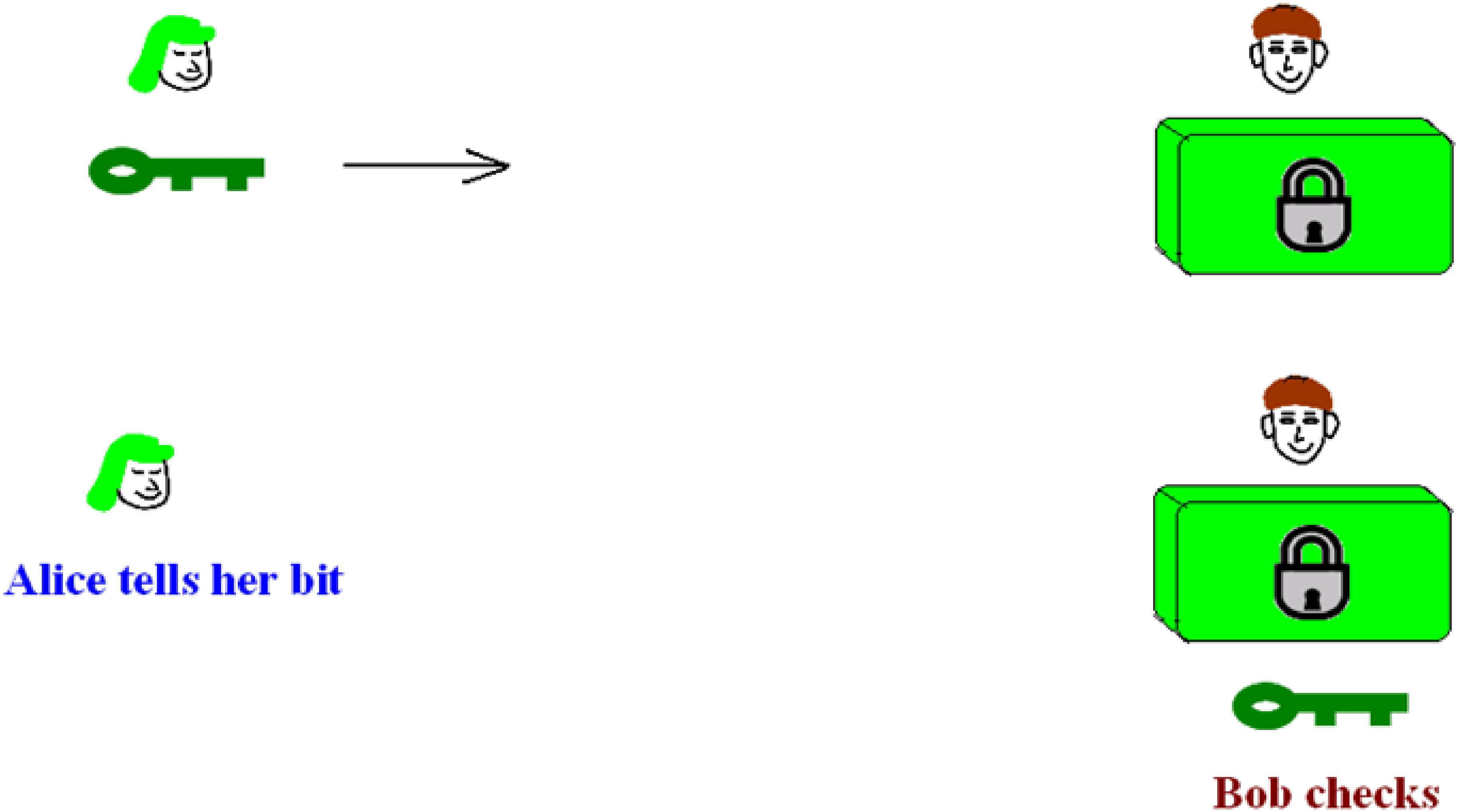
Commit phase



The arrangement is such that :

- **Alice can not change her commitment.**
- **Bob can not learn the commitment.**

Opening phase



In classical physics there is no law to make it successful.

Why not try with quantum laws?

Non-unique decomposition of mixed states provide an opportunity.

Prepare n no. of spin states

$$|0_y\rangle \otimes |0_y\rangle \otimes |0_y\rangle \ldots\ldots \otimes |0_y\rangle$$

Bit – 0 $\quad$ *Measure σ_z on all of them*

Bit - 1 $\quad$ *Measure σ_x on all of them*

Note down the result and send particles to Bob

$$\frac{I}{2} \otimes \frac{I}{2} \otimes \frac{I}{2} \ldots\ldots \otimes \frac{I}{2}$$

Bon cannot recover any information about the bit as each qubit is in completely random state.

Openning phase

Alice announces
her committment

and

spin polarization of the n particles in the basis according to the announced bit

Bit	measures
0	σ_z
1	σ_x

and

verifies Alice's answers

If Alice wants to cheat,
her probability of success

$$= \left(\frac{1}{2}\right)^n$$

But entanglement makes the protocol insecure

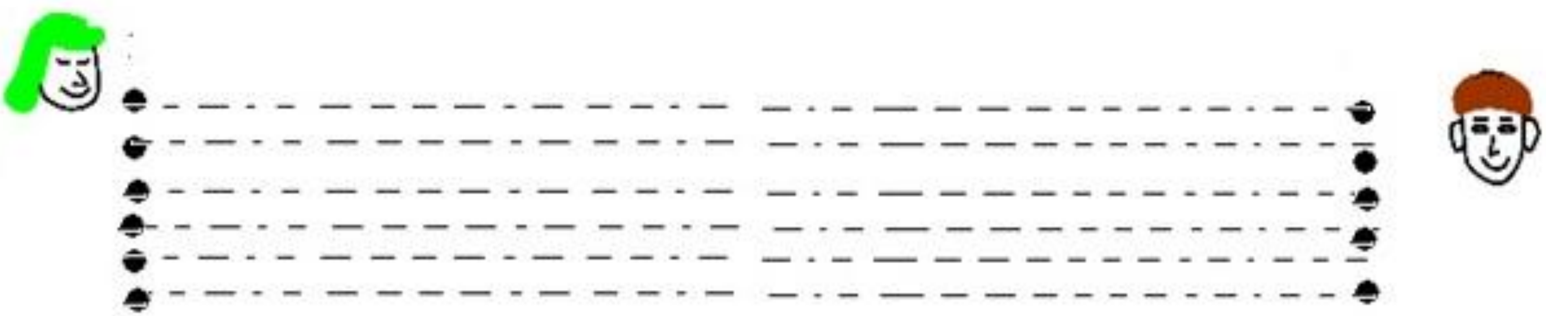
Commit phase

Alice prepares n pair of qubits in the following state

$$|\phi^{+}\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

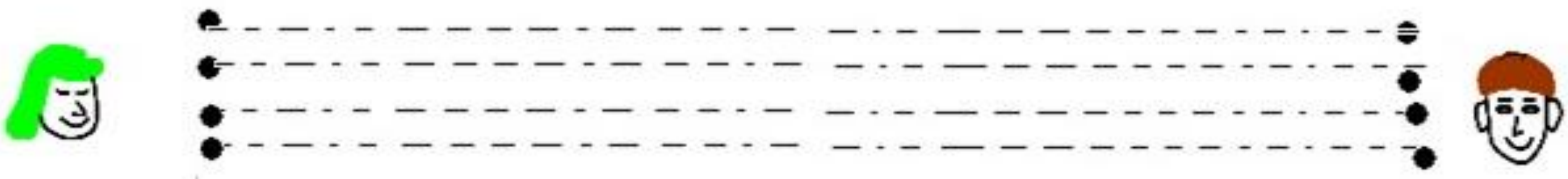
and send one qubit from each pair.

Commit no bit



Bob has no way to learn this cheating as density matrix are same for each qubit with Bob.

Openning phase



Alice decides her bit

and

choses the measurement accordingly.

Bit	measures
0	σ_z
1	σ_x

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}[|++\rangle + |--\rangle]$$

Due to perfect correlation in both the basis Alice will be able to tell the results of all measurements to be performed by Bob.

For some Scientists,

Quantum mechanics is characterized by:

(*a*)***Possibility of secret key generation.***

(*b*)***Impossibility of Bit committment.***