Collective description of trapped fermions: Exact results

Manas Kulkarni

M.K, S. N. Majumdar, G. Schehr
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Collaboration

Satya Majumdar (Université Paris-Sud)

Gregory Schehr (Sorbonne Université)

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Outline

- Review of some results on trapped fermions in 1D and 2D (Background and Motivation)
- Our results in Fermions in 2D rotating trap with additional repulsive central potential
- Deep connections to orthogonal polynomials, quantum spin chains and Random Matrix Theory

Background and Motivation

Great experimental progress in cold atoms

- Atoms confined in traps (such as harmonic, box-like)
- Different spacial dimensions
- Tunable interactions
- Variable temperature and number of atoms
- Collective density measurements by absorption imaging technique
- Techniques that can resolve at the level of a single atom (Quantum Gas Microscope)
- Time dynamics

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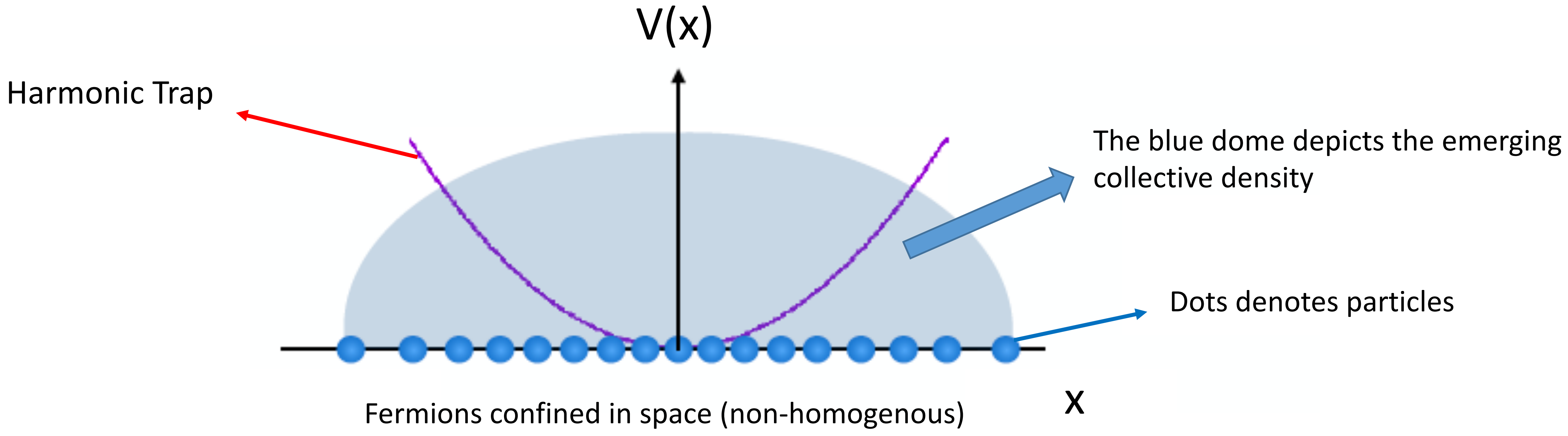
- Excellent platform to understand quantum and statistical behavior in many body systems

- Interesting quantum many body effects even in the absence of interactions.

Bosons - Bose-Einstein Condensation

Fermions – Pauli Exclusion leads to highly non-trivial spacial profiles/correlation

Cold fermions in confining potential (schematic)



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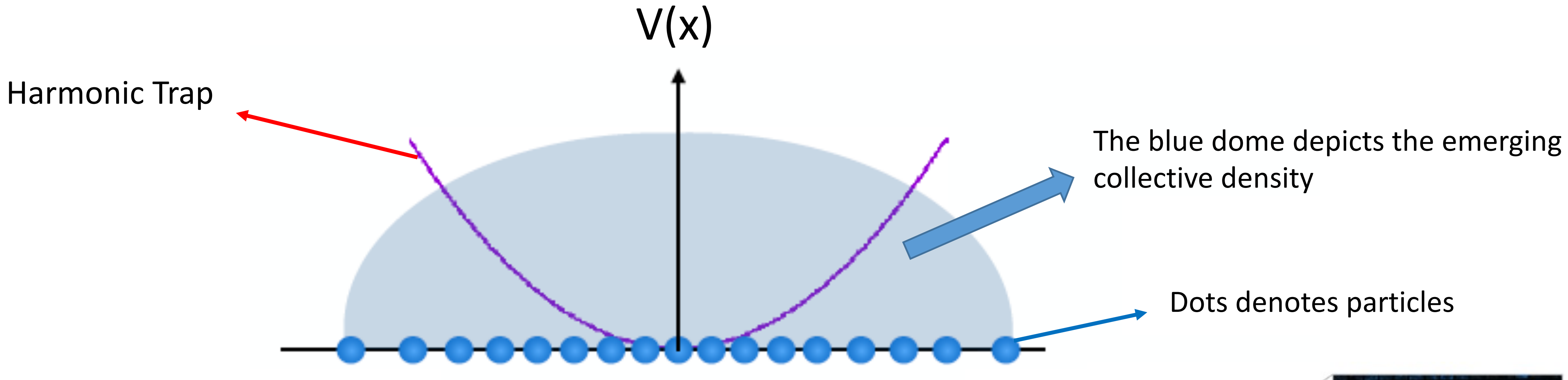


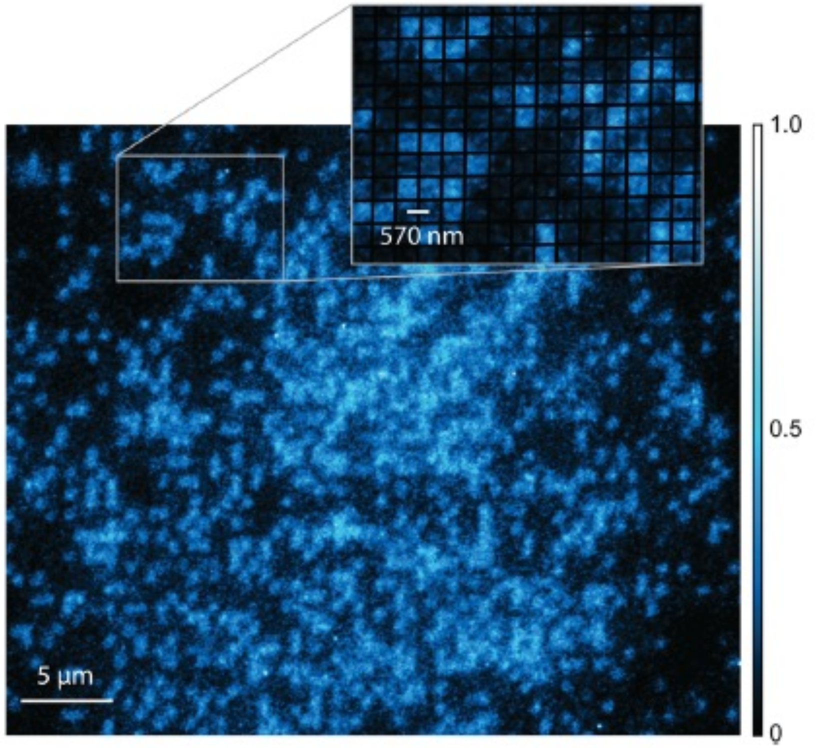
Fermions confined in space (non-homogenous)

Quantum Gas Microscope

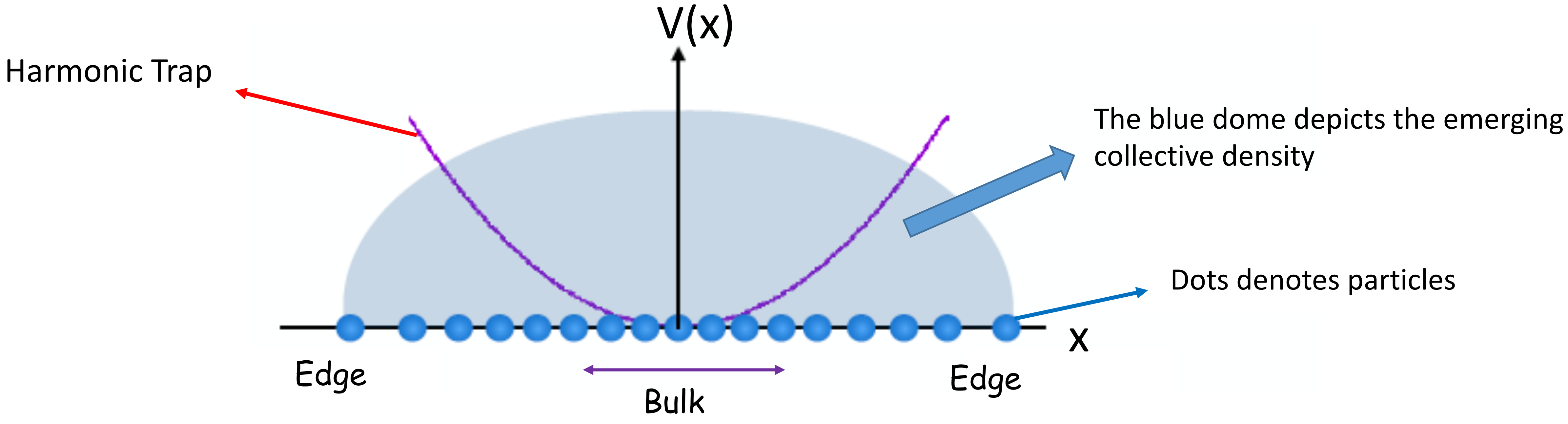
Greiner Group (Harvard)

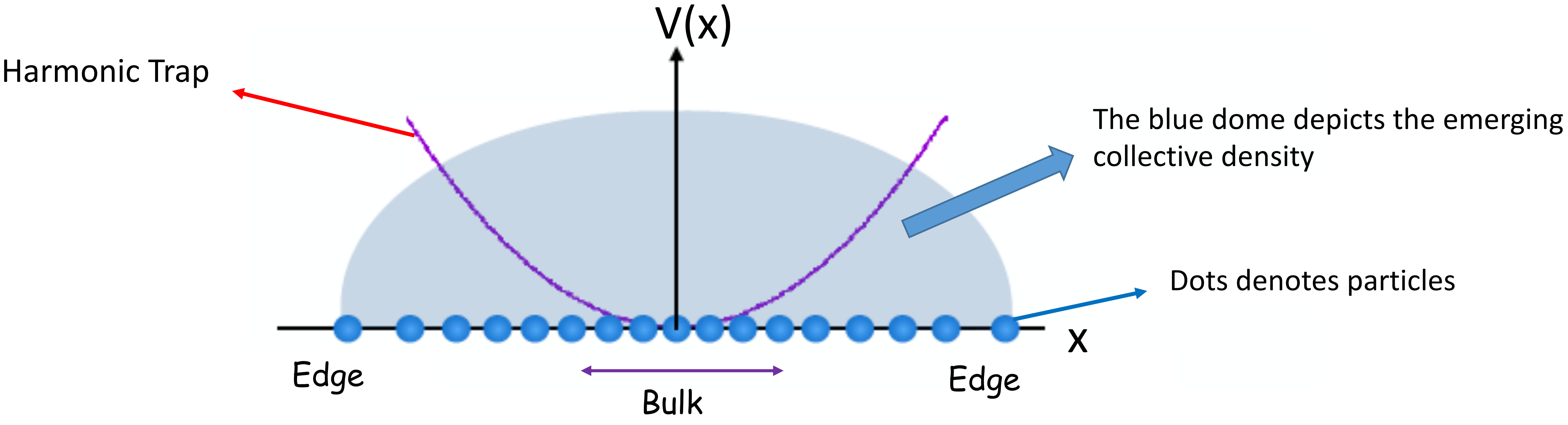
^{6}Li Fermions in trap

"demonstrate site-resolved imaging of individual fermionic ^{6}Li atoms in a single layer of a 3D optical lattice." - PRL 2015 (Parson's et al)



Many fascinating experiments


V(x)
Harmonic Trap
The blue dome depicts the emerging collective density
Dots denotes particles
x
Edge
Bulk
Edge



Bulk - Local Density/Thomas-Fermi approximation is mostly reasonable

You will get a relation between external potential V(x) and ground state density $\rho(x)$ without solving Schrodinger equation

Edge – As you move to the edge such approximations break down.

More generally, when translational symmetry is broken and when we have regions of extremely low densities, such approximations do not work

"The uniform electron gas, the traditional starting point for density-based many-body theories of inhomogeneous systems, is inappropriate near electronic edges" – Kohn and Mattison, PRL 1998

- Therefore, instead of density based approaches and approximation, let us solve the Schrödinger equation

Review: Dean, Le Doussal, Majumdar, Schehr (J. Phys. A: Math. Theor, 2019)

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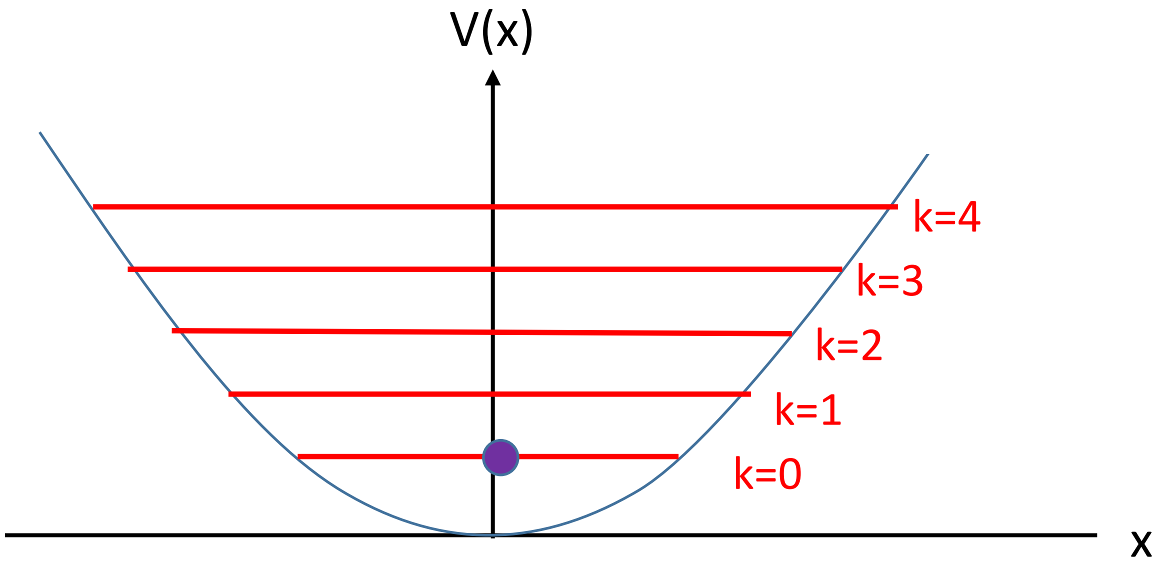
Free Fermions at Zero Temperature in a 1D Harmonic Trap

- Let us start with a single Fermion in a trap



Hamiltonian:

$$\hat{h} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Schrodinger Equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} + \frac{1}{2} m\omega^2 x^2 \varphi_k(x) = \epsilon_k \varphi_k(x)$$

$$\varphi_k(x \to \pm\infty) = 0$$

Solution to the eigenvalue problem

Single particle eigenfunctions: $\varphi_k(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}2^k k!}} e^{-\alpha^2 x^2/2} H_k(\alpha x)$

Energy levels: $\epsilon_k = (k + 1/2)\hbar\omega$

$\alpha = \sqrt{m\omega/\hbar}$ Inverse width of the ground state wave-packet (particle fluctuations because of quantum mechanics)

Hermite polynomials: $H_k(x) = (-1)^n e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$

$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2, \cdots, H_7(x) = x^7 - 21x^5 + 105x^3 - 105x$ … etc;

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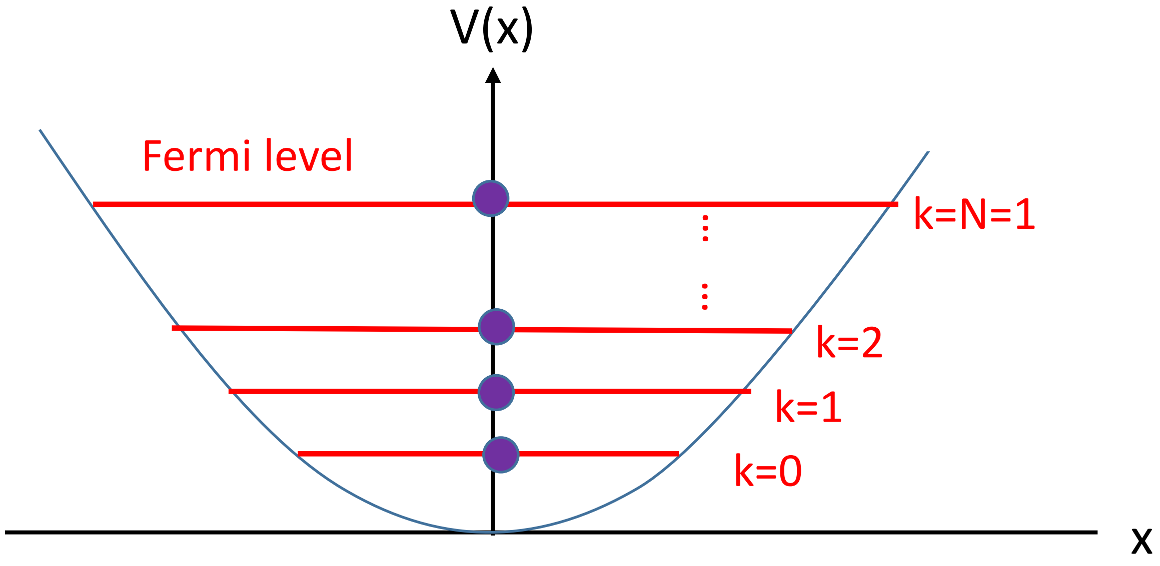
Some properties of Hermite polynomials for later:

- Zeros of Nth Hermite polynomial are real
- Zeros of this polynomial are solutions to the algebraic Stieltjes problem $$x_j = \sum_{\substack{k=1 \\ (k\neq j)}}^{N} \frac{1}{x_j - x_k}, \quad j = 1, 2, 3 \cdots N$$

N free fermions in a 1D Harmonic Trap



Many-body Hamiltonian (non-interacting)

$$\hat{H}_N = \sum_{i=1}^{N} \hat{h}_i = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_i^2 \right]$$

Eigenvalues and eigenfunctions of this Many Body Hamiltonian

Many Body Ground State wavefunction $\hat{H}_N \Psi_0 = E_0 \Psi_0$

$\Psi_0(x_1, x_2 \ldots x_N)$ Antisymmetric (Pauli Exclusion Principle)

$$\Psi_0(x_1, x_2, \ldots x_N) = \frac{1}{\sqrt{N!}} \det \left[\varphi_i(x_j) \right], \quad 0 \le i \le (N-1), 1 \le j \le N$$ (Slater determinant)

Ground state energy $E_0 = \sum_{k=0}^{N-1} (k + 1/2)\hbar\omega = \frac{N^2}{2} \hbar\omega$

Slater determinant can be evaluated

$$\Psi_0(x_1, x_2, \dots x_N) = \frac{1}{\sqrt{N!}} \det\left[\varphi_i(x_j)\right], \quad 0 \le i \le (N-1), 1 \le j \le N$$

$$\Psi_0(x_1, x_2, \dots x_N) \propto e^{-\frac{\alpha^2}{2}\sum_{i=1}^{N} x_i^2} \det_{1\le(i,j)\le N}\left[H_i(\alpha x_j)\right]$$

$$\propto e^{-\frac{\alpha^2}{2}\sum_{i=1}^{N} x_i^2} \prod_{j<k}(x_j - x_k)$$

Determinant of matrix of Hermite polynomials is Vandermonde

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Determinant of matrix of Hermite polynomials is Vandermonde

Quantum probability density that encodes quantum fluctuations

$$|\Psi_0(x_1, x_2, \dots x_N)|^2 = \frac{1}{Z_N} e^{-\alpha^2 \sum_{i=1}^N x_i^2} \prod_{j<k}(x_j - x_k)^2$$

Joint distribution of the positions of Fermions in the ground state

---

$$P(\lambda_1, \lambda_2, \dots \lambda_N) = \frac{1}{Z_N} e^{-\sum_{i=1}^N \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^2$$

Joint probability distribution of the eigenvalues of a GUE random matrix

- The positions of free fermions in a trap behave statistically as the eigenvalues of a GUE Random Matrix

$$(\alpha x_1, \alpha x_2, \alpha x_3 \dots \alpha x_N) \equiv (\lambda_1, \lambda_2, \lambda_3 \dots \lambda_N)$$

- Good because GUE is determinantal process

- m-point correlation functions are determinants of two-point correlation functions

$$R_m(x_1, x_2 \dots x_m) = \frac{N!}{(N-m)!} \int dx_{m+1} dx_{m+2} \dots dx_N |\Psi_0(x_1, x_2, \dots x_m, x_{m+1} \dots x_N)|^2$$

$$= \det_{1 \le (i,j) \le m} \left[K_N(x_i, x_j) \right] \quad \text{where} \quad K_N(x,y) = \sum_{k=0}^{N-1} \varphi_k(x) \varphi_k(y)$$

$$\rho_N(x) = \frac{1}{N} K_N(x,x) = \frac{1}{N} \sum_{k=0}^{N-1} |\varphi_k(x)|^2 \quad \text{(average density)}$$

- This average density is given by Wigner semicircle law
- Scaled distribution of rightmost trapped fermion is Tracy-Widom (GUE)

Classical Dyson’s Log Gas

Classical Calogero-Moser System

- Wigner Semi-circle $\sqrt{1-x^2}$
- Zeros of N^th Hermite Polynomial $H_N(x) = 0$
- Stieltjes problem $x_j = \sum_{\substack{k=1 \\ (k \neq j)}}^{N} \frac{1}{x_j - x_k}, \quad j = 1, 2, 3 \cdots N$

Free Fermions in 1D trap

GUE Random Matrix

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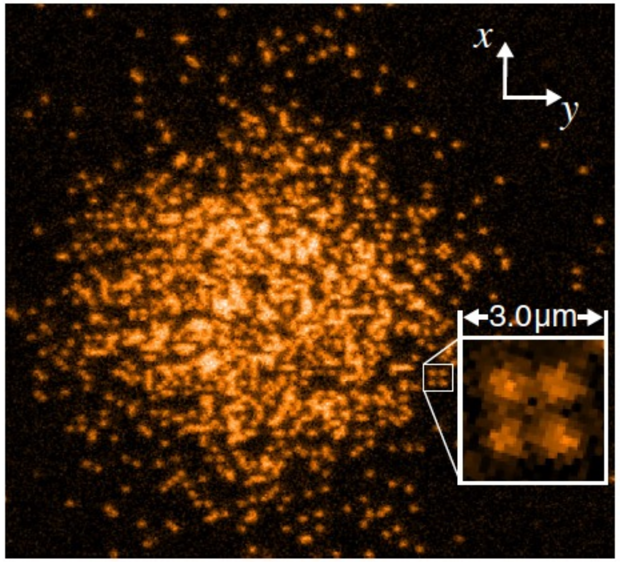
Free Fermions in 1D trap

GUE Random Matrix

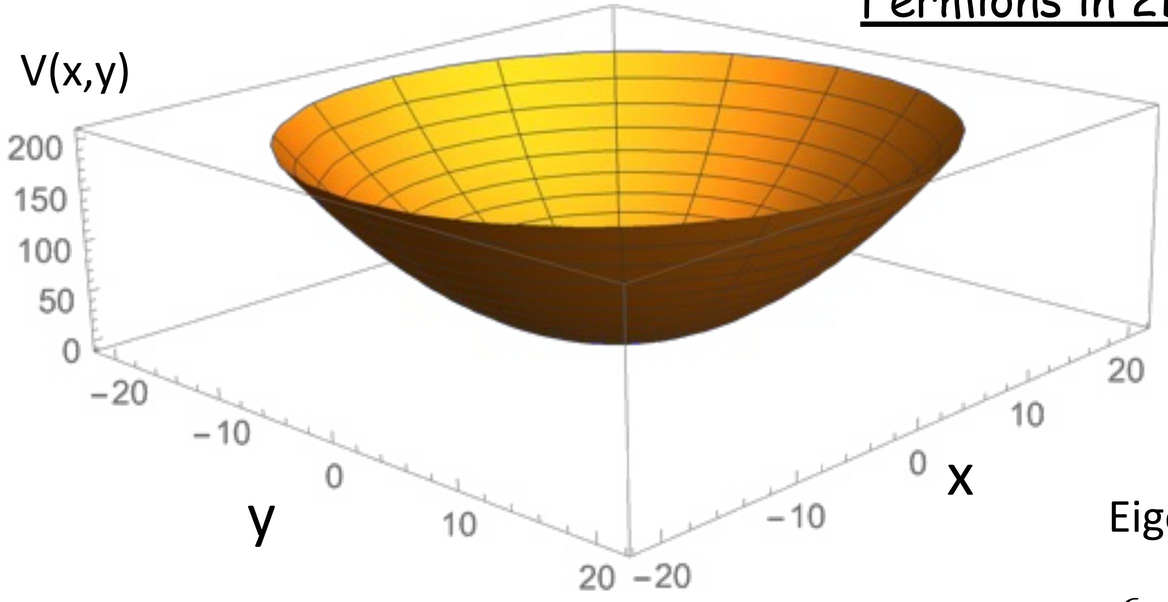
Some natural questions

- What happens to Fermions in higher dimensions ?
- Is there a RMT connection in higher dimensions ?
- Do orthogonal polynomials emerge in higher dimensions ?
- Is there a generalization of the Stieltjes problem ?
- Can we get exotic density profiles ?

Quantum Gas Microscope
Cheuk et al, PRL 2015
Fermionic ^{40}K atoms



Fermions in 2D Trap



- Separable Hamiltonian

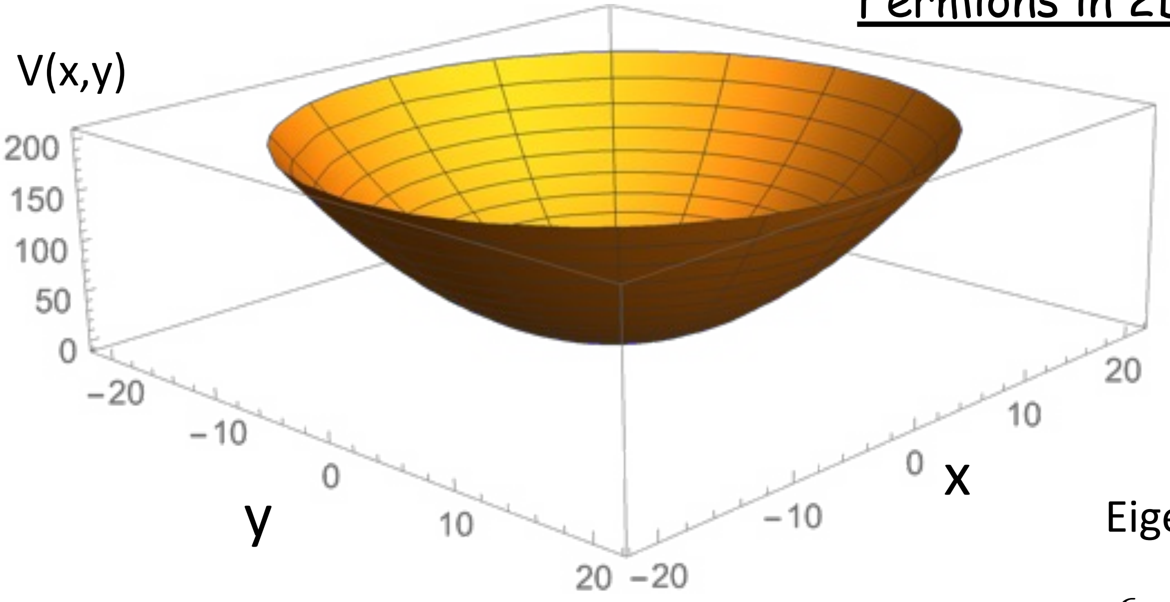
- Single particle eigenfunctions

$$\varphi_{n_1,n_2}(x,y) \propto H_{n_1}(\alpha x)H_{n_2}(\alpha y)e^{-\frac{\alpha^2}{2}(x^2+y^2)}$$

Eigenvalues

$$\epsilon_{n_1,n_2} = (n_1+n_2+1)\hbar\omega, \quad n_1 = 0,1,2\ldots \text{ and } \quad n_2 = 0,1,2\ldots$$

Fermions in 2D Trap



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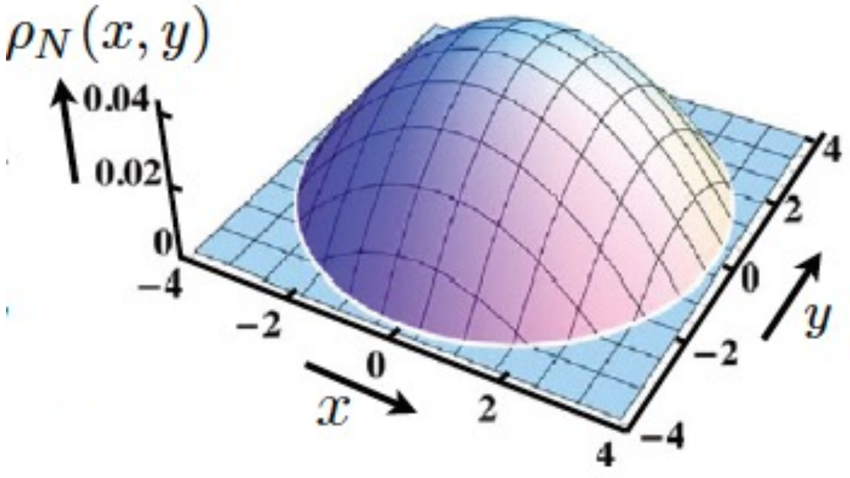


$N \to \infty$

$$\rho_N(\mathrm{x}) \approx \frac{1}{N}\left(\frac{m}{2\pi\hbar^2}\right)^{d/2} \frac{[\mu - V(\mathrm{x})]^{d/2}}{\Gamma(d/2+1)}$$

Dean, Le Doussal, Majumdar, Schehr, EPL (2015)

- Interestingly, there is no non-Hermitian random matrix whose joint distribution is same as this Fermionic system

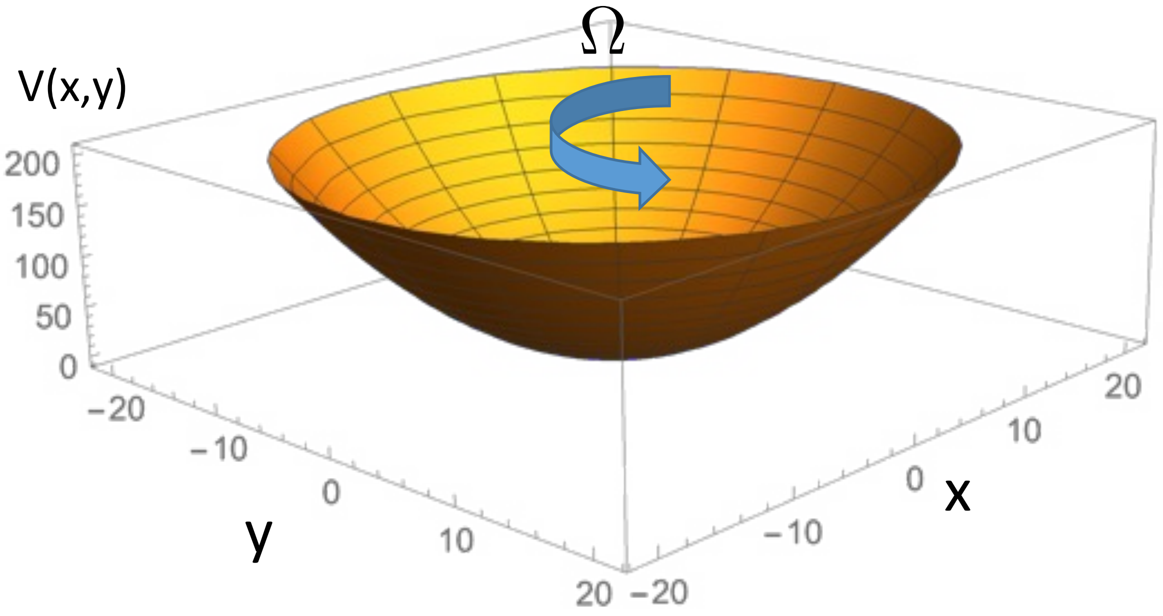
- The hope is to make connection with Ginibre Random Matrix (GinUE)

- But, eigenvalue distribution of GinUE is a uniform disk of radius $\sqrt{N}$

Fermions in 2D Trap: Rotatating it to flatten the dome

- To find 2D systems whose ground state (modulus square) corresponds to join distribution of GinUE eigenvalues

$$|\Psi_0(z_1, z_2 \ldots z_N)|^2 \equiv P_{\mathrm{Gin}}(z_1, z_2 \ldots z_N) = \frac{1}{Z_N} e^{-\sum_{i=1}^N |z_i|^2} \prod_{j<k} |z_j - z_k|^2$$



In the rotating frame: $\hat{H}_N = \sum_{i=1}^N \hat{h}_i$

$$\hat{h} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{r}^2 - \Omega \hat{L}_z$$

$$\hat{L}_z = i\hbar(y\partial_x - x\partial_y)$$

$\Omega = 0$ (2D Harmonic trap)

$\Omega = \omega$ (free Fermions in a uniform perpendicular Magnetic field)

Related Experiment: Schweikhard et al, PRL (2004), *"Rapidly Rotating Bose-Einstein Condensates in and near the Lowest Landau Level"*

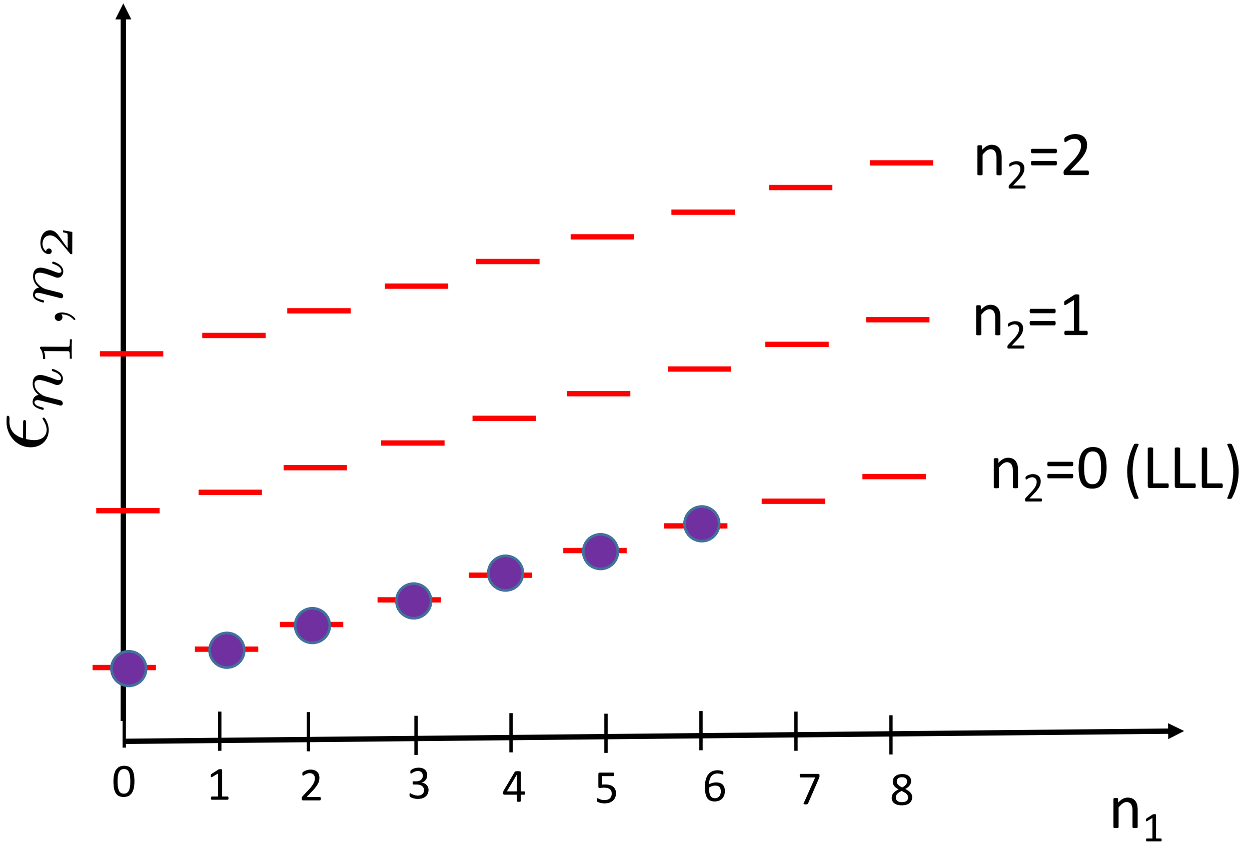
Review: N. Cooper (Advances in Physics, 2008), *"Rapidly Rotating Atomic Gases"*

Lacroix-A-Chez-Toine, Majumdar, Schehr (PRA 2019)

Single particle spectrum

$$\phi_{n_1,n_2}(x,y) \propto e^{\frac{r^2}{2}} (\partial_x + i\partial_y)^{n_1} (\partial_x - i\partial_y)^{n_2} e^{-r^2}$$

$$\epsilon_{n_1,n_2} = \hbar[\omega + (\omega - \Omega)n_1 + (\omega + \Omega)n_2] \quad n_1, n_2 = 0, 1, \ldots$$



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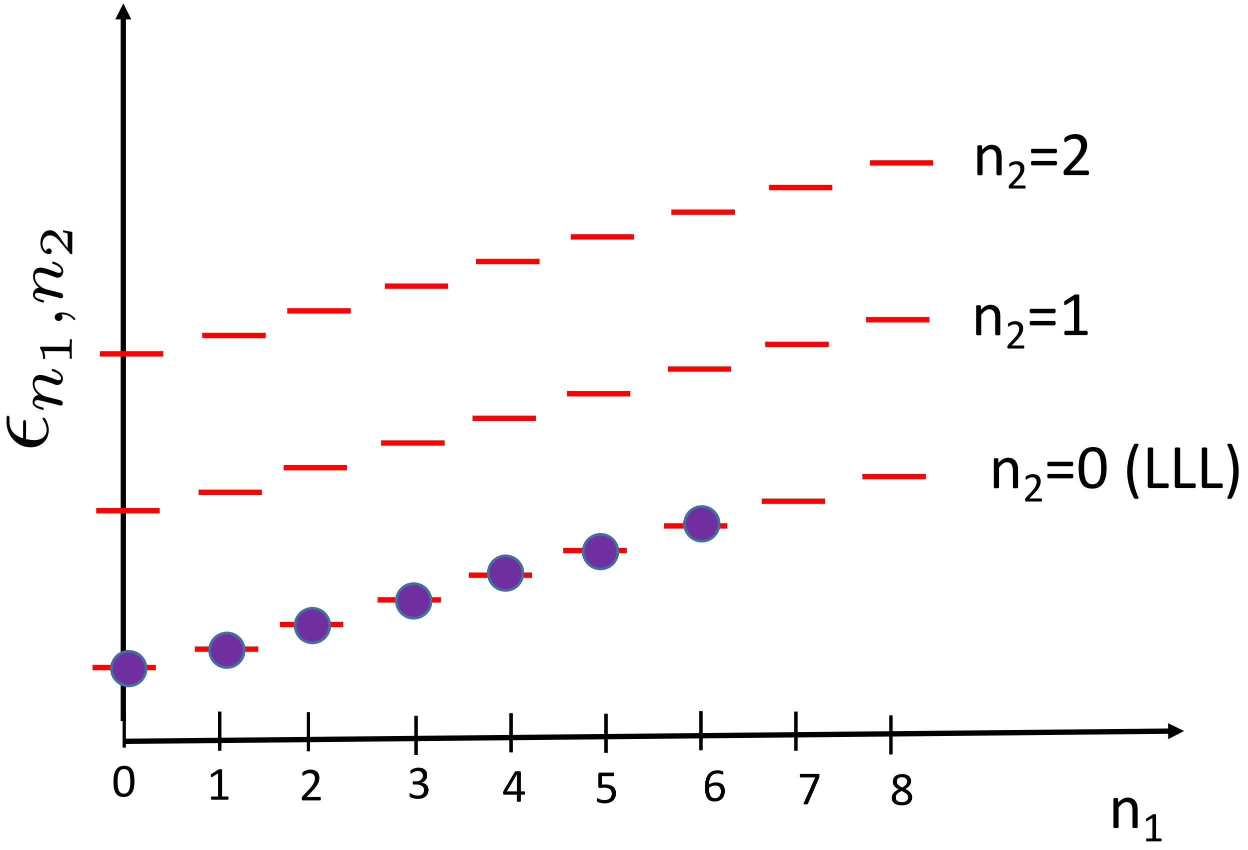

Highest occupied level in $n_2=0$ sector should be lower than lowest energy of the next $n_2=1$ sector

$$\Longrightarrow \quad \omega(1 - 2/N) < \Omega < \omega$$

If we stick to this bound and fill Fermions respecting Pauli exclusion we finally get

$$|\Psi_0(z_1, z_2 \ldots z_N)|^2 \propto e^{-\sum_{i=1}^N |z_i|^2} \prod_{j<k} |z_j - z_k|^2$$

$$= P_{\text{Gin}}(z_1, z_2 \ldots z_N)$$

Physical realization of GinUE

Lacroix-A-Chez-Toine, Majumdar, Schehr (PRA 2019)

Fermions in 2D Trap with additional repulsive central potential

M.K, Majumdar, Schehr, PRA (2021)

Solve Schrodinger Equation
(Eignevalues and eigenfunctions)

↓

Start filling N Fermions
respecting Pauli exclusion

↓

Ask how they would look
in space for any finite - N

↓

Derive analytical results for
large-N (both bulk and edges)

↓

Find deep connections with RMT,
orthogonal polynomials and other
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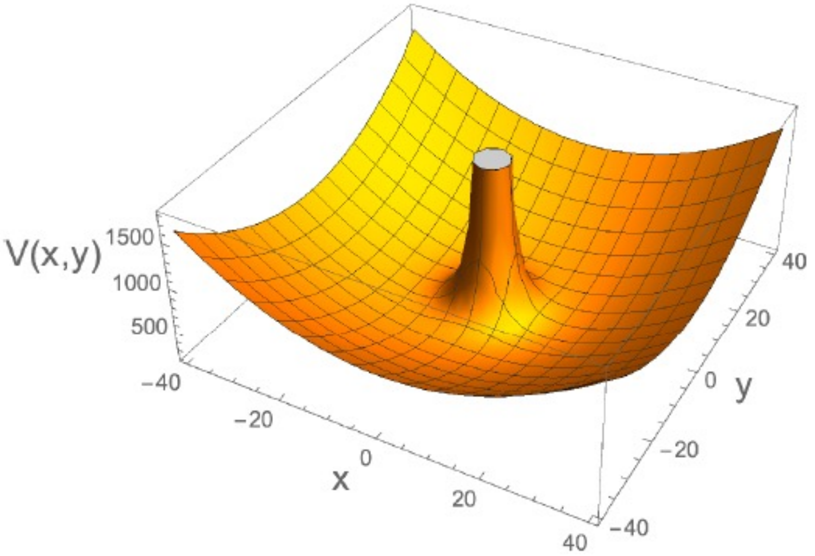
↓

Find deep connections with RMT, orthogonal polynomials and other quantum models

$$\hat{H} = \frac{p^2}{2m} + V(r) - \Omega L_z$$

$$V(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\gamma}{2r^2}, \quad \gamma \geqslant 0$$



Two important parameters:

$\nu = \Omega/\omega \qquad 0 < \nu < 1$

$\gamma \geqslant 0$

Eigenfunctions

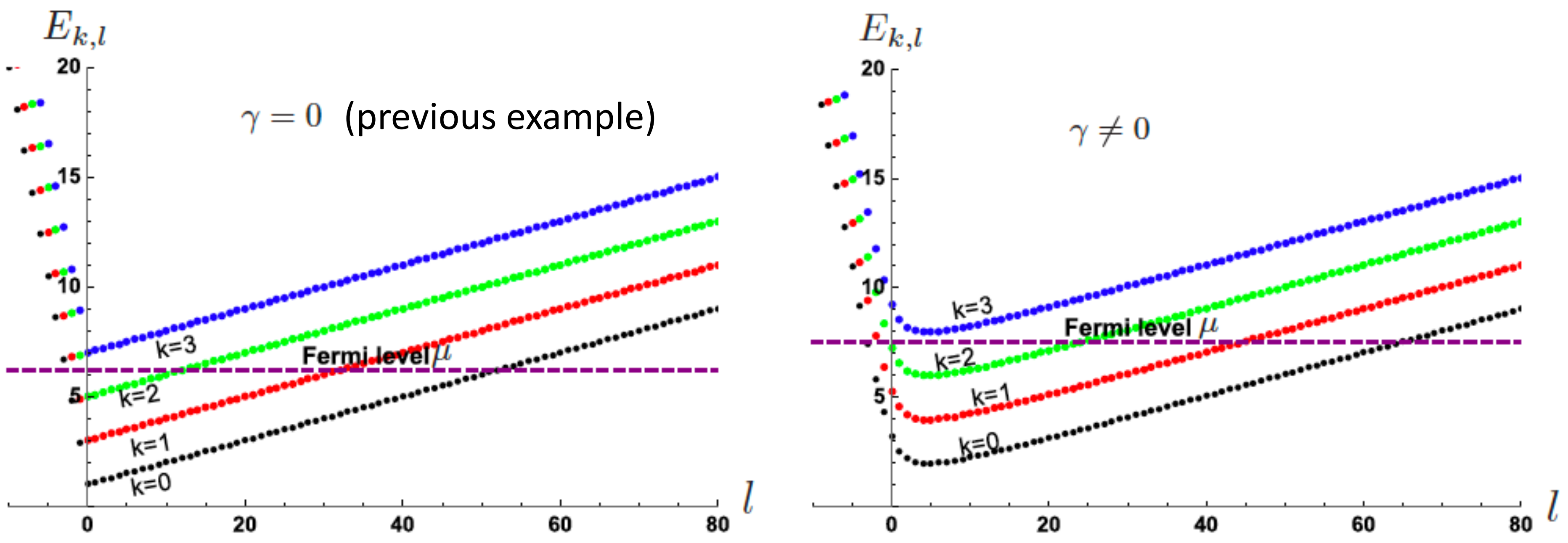
$$\psi_{k,l}(r,\theta) = a_{k,l} L_k^{\lambda}(r^2) r^{\lambda} e^{-r^2/2} e^{il\theta}, \quad \text{with} \quad \lambda = \sqrt{\gamma + l^2}$$

generalized Laguerre polynomials

The single-particle states are labeled by a pair of integers (k, l) with $k = 0, 1, 2...$ and $l = 0, \pm 1, \pm 2, ...$

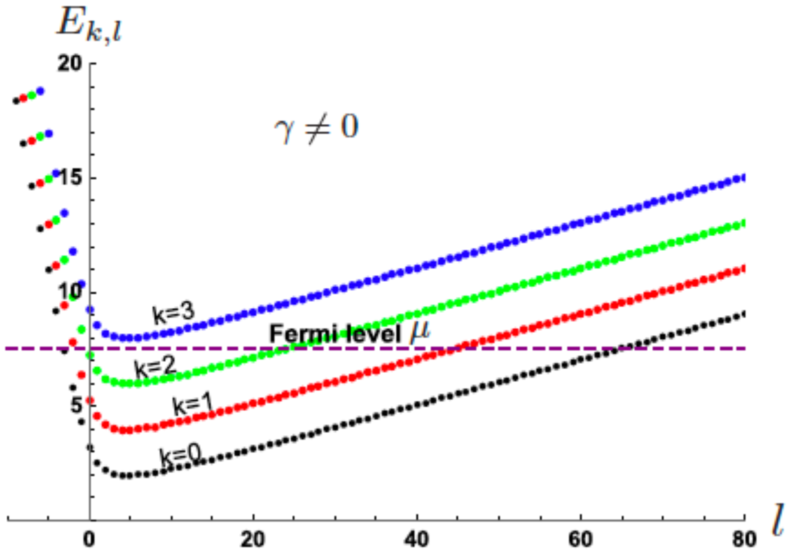
Eigenvalues

$$E_{k,l} = 2k + 1 + \sqrt{\gamma + l^2} - \nu l$$



- Consider N spinless noninteracting fermions in their ground state.
- Many-body ground state is thus given by a Slater determinant constructed from N single-particle eigenfunctions associated to the lowest N eigenvalues
- For a given N, the eigenfunctions participating in the Slater determinant may belong to multiple bands of the spectrum

Average number density (finite-N)



- The average number density (normalized to N)

$$\rho(r,\theta,N) = \sum_{k,l} |\psi_{k,l}(r,\theta)|^2 = \sum_{k=0}^{k^*} \rho_k(r,\theta,N)$$

where $$\rho_k(r,\theta,N) = \frac{\Gamma(k+1)\,e^{-r^2}}{\pi} \sum_{l=l_-(k)}^{l_+(k)} \frac{\left[L_k^\lambda(r^2)\right]^2 r^{2\lambda}}{\Gamma(\lambda+k+1)}$$

Here $l_\mp(k)$ are the locations where the Fermi level μ intersects the kth band, i.e., $E_{k,l_\pm(k)} = \mu$.

$$l_\pm(k) = \frac{\nu(\mu-2k) \pm \sqrt{(\mu-2k)^2 - \gamma(1-\nu^2)}}{1-\nu^2}$$

For a given k, $E_{k,l}$ minimum occurs at $l^* = \frac{\nu}{\sqrt{1-\nu^2}}\sqrt{\gamma}$

Energy of the k^{th} band at minimum is $E_{k,l^*} = 2k + \sqrt{(1-\nu^2)\gamma}$

$k^* = \mathrm{Int}[\frac{\mu-\sqrt{(1-\nu^2)\gamma}}{2}]$ (k* is the number of bands below Fermi level)

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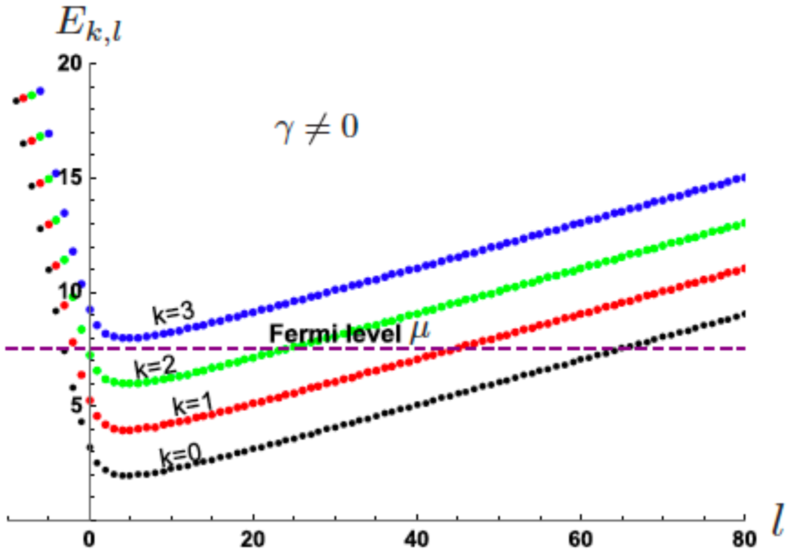
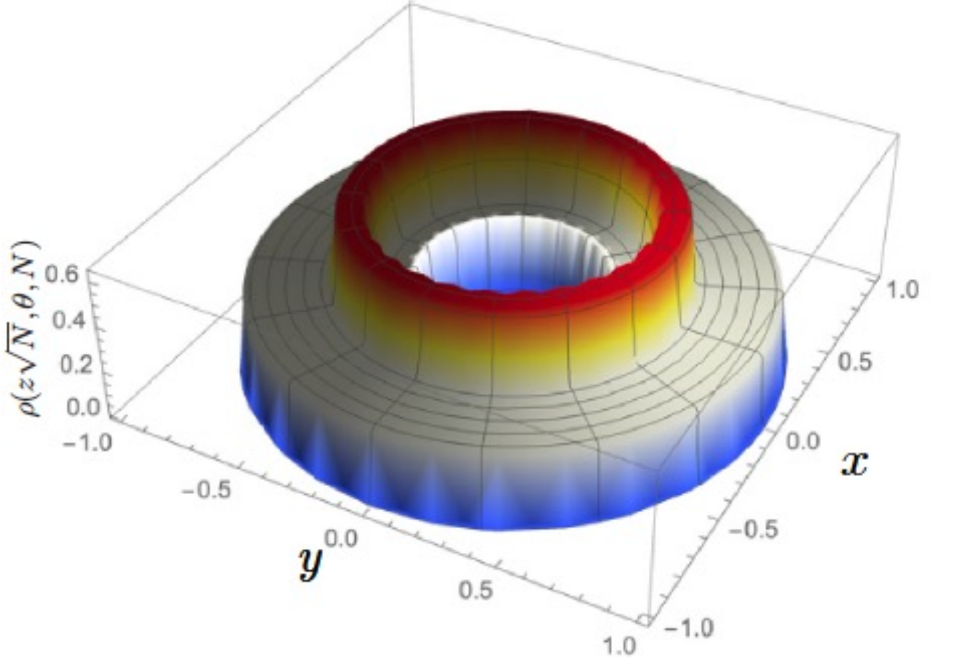
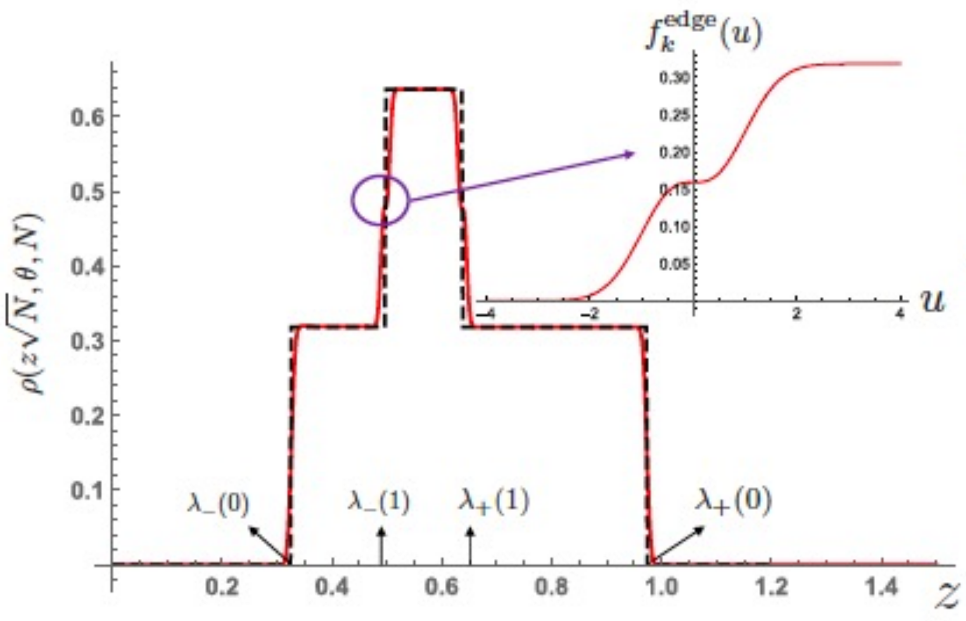
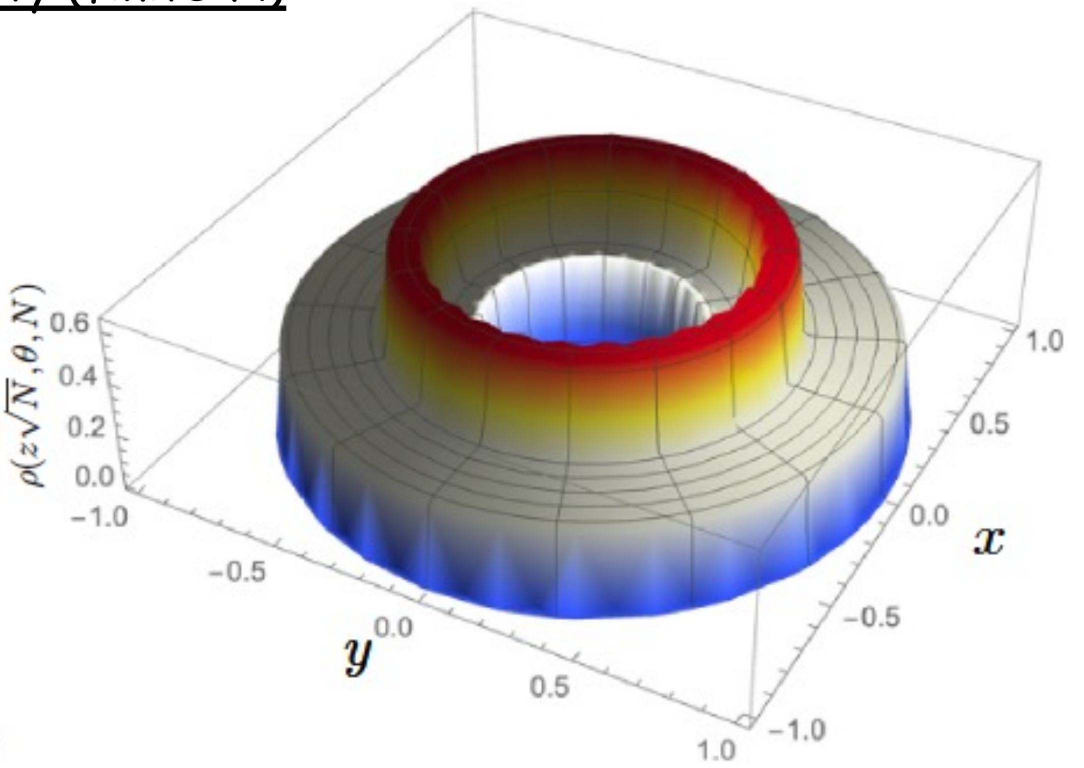




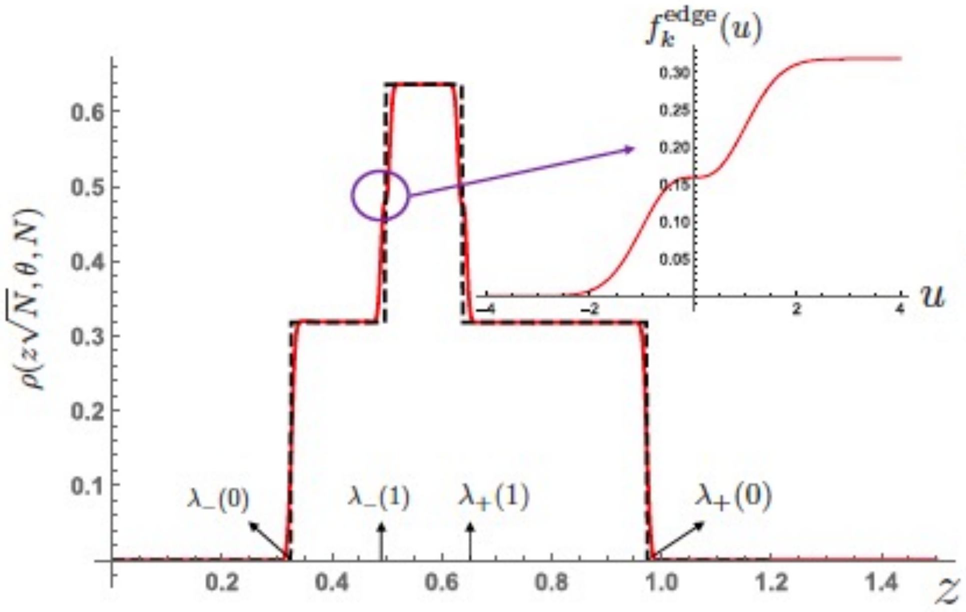
$r=z\sqrt{N}$

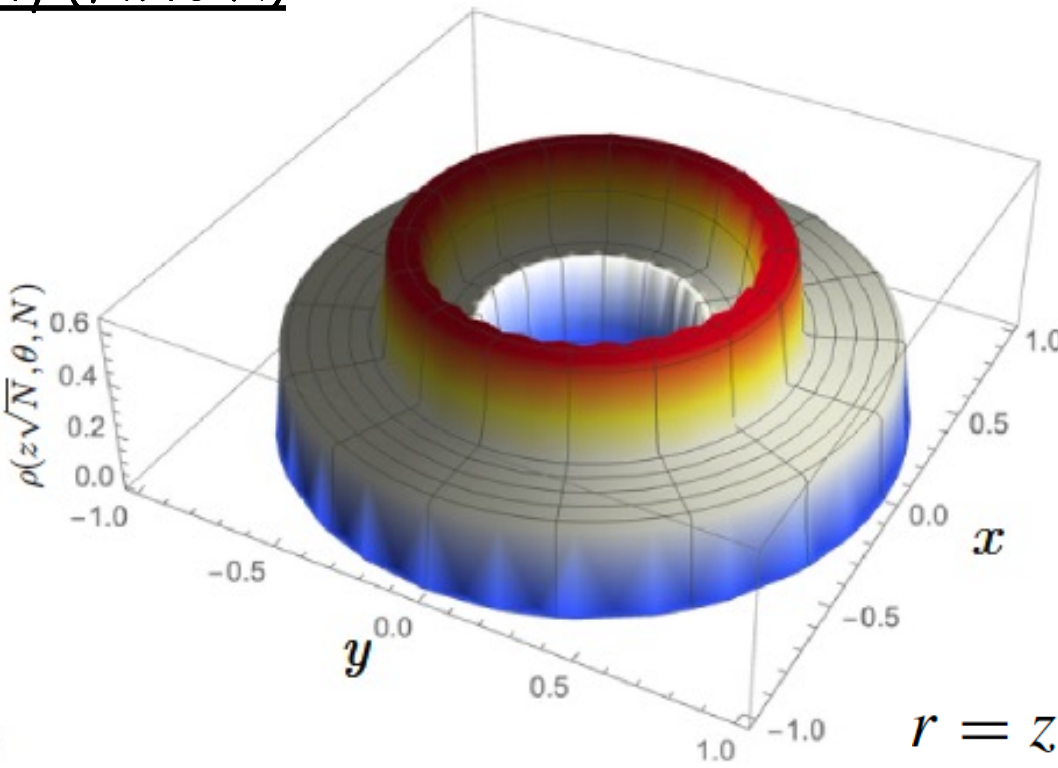
Average number density (finite-N)





Average number density (finite-N)





$$r = z\sqrt{N}$$
$$l_\pm(k) = \lambda_\pm(k)N$$

- For a given k*, the bulk density vanishes as $r < \sqrt{l_-(0)}$ creating a hole around the origin
- Outside the hole, the density of non-zero over an annulus $\sqrt{l_-(0)} < r < \sqrt{l_+(0)}$.
- On top of this annulus, there is a "wedding cake" structure with k* layers with progressively smaller supports but with equal heights $1/\pi$
- For e.g., k^{th} layer has a support $\sqrt{l_-(k)} < r < \sqrt{l_+(k)}$

Large-N Results

Recap

$$\hat{H} = \frac{p^2}{2m} + V(r) - \Omega L_z$$

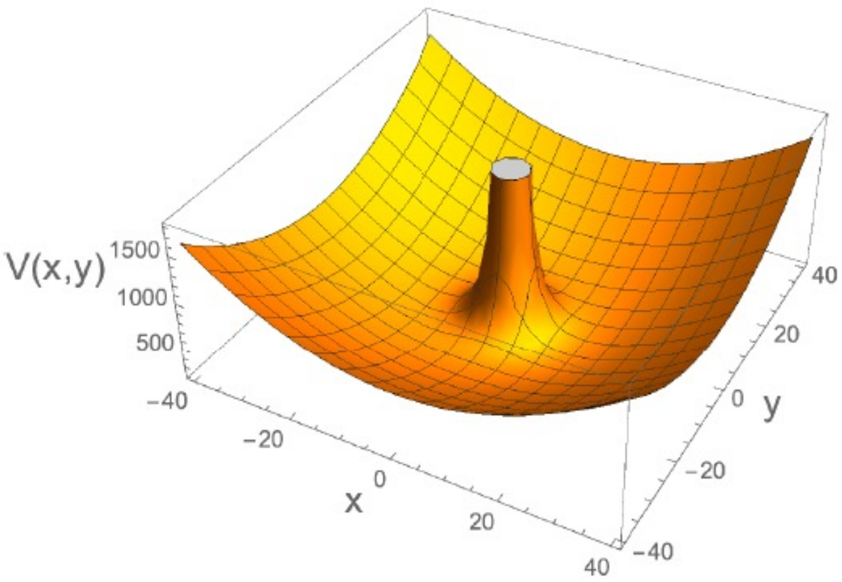
$$V(r) = \frac{1}{2} m\omega^2 r^2 + \frac{\gamma}{2r^2}, \quad \gamma \geqslant 0$$

Two important parameters:

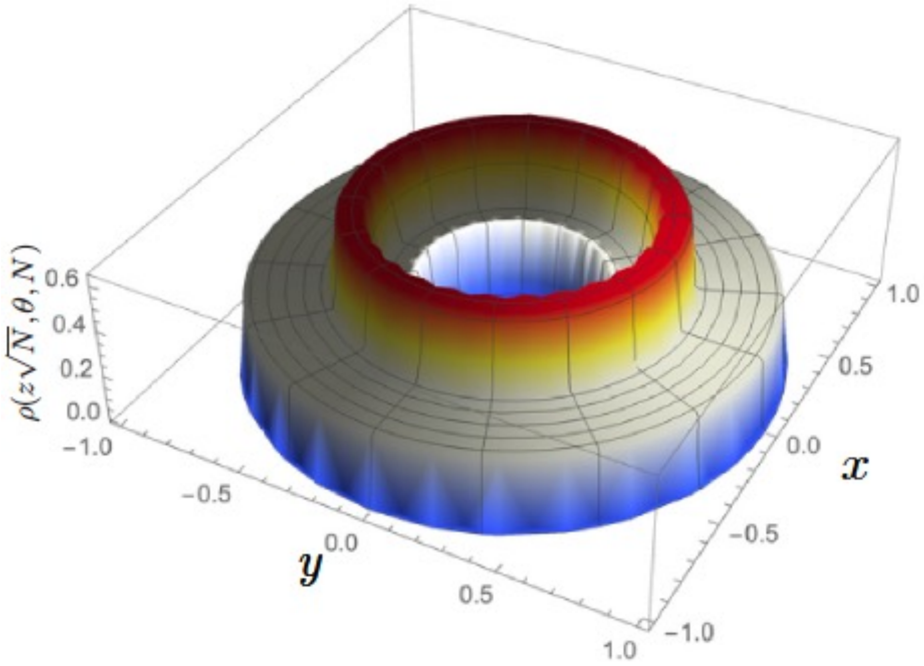
$\nu = \Omega/\omega \qquad 0 < \nu < 1$

$\gamma \geqslant 0$

Potential

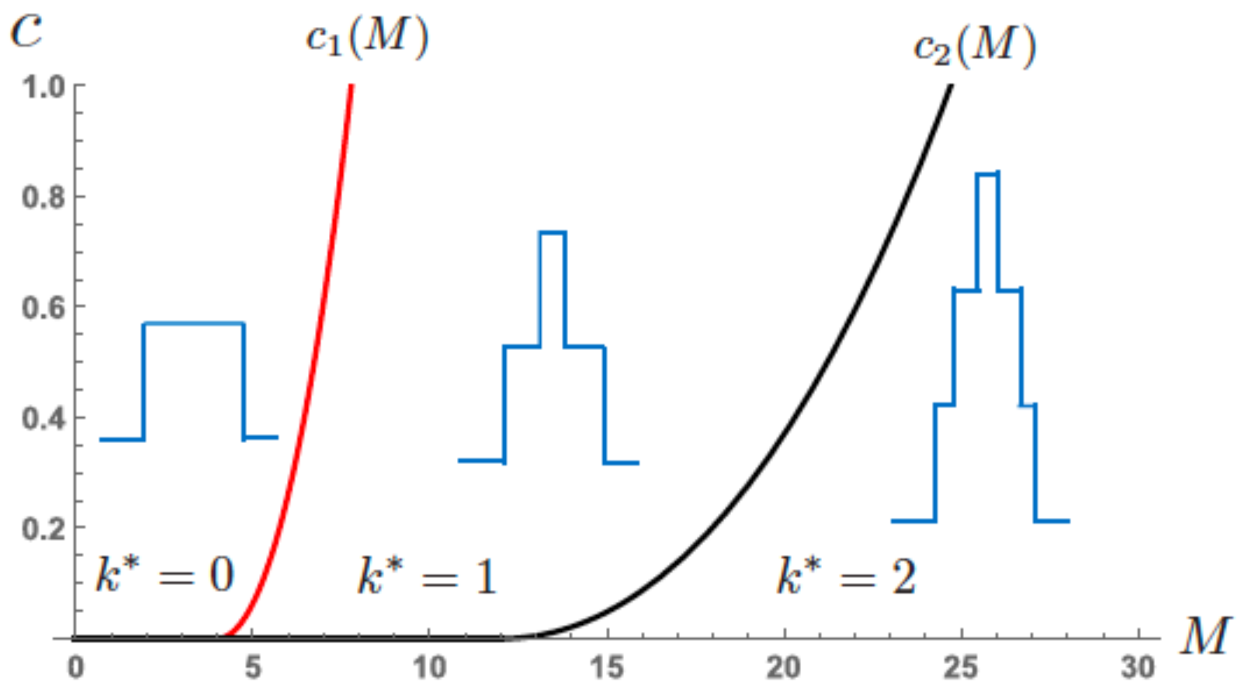


Resulting Density



- In the large-N limit the appropriate rescaled parameters both of O(1) are $c = \frac{\gamma}{N}$ and $M = (1-\nu^2)N$
- This scaling is necessary to keep the average density of fermions of order O(1) as N $\rightarrow \infty$.

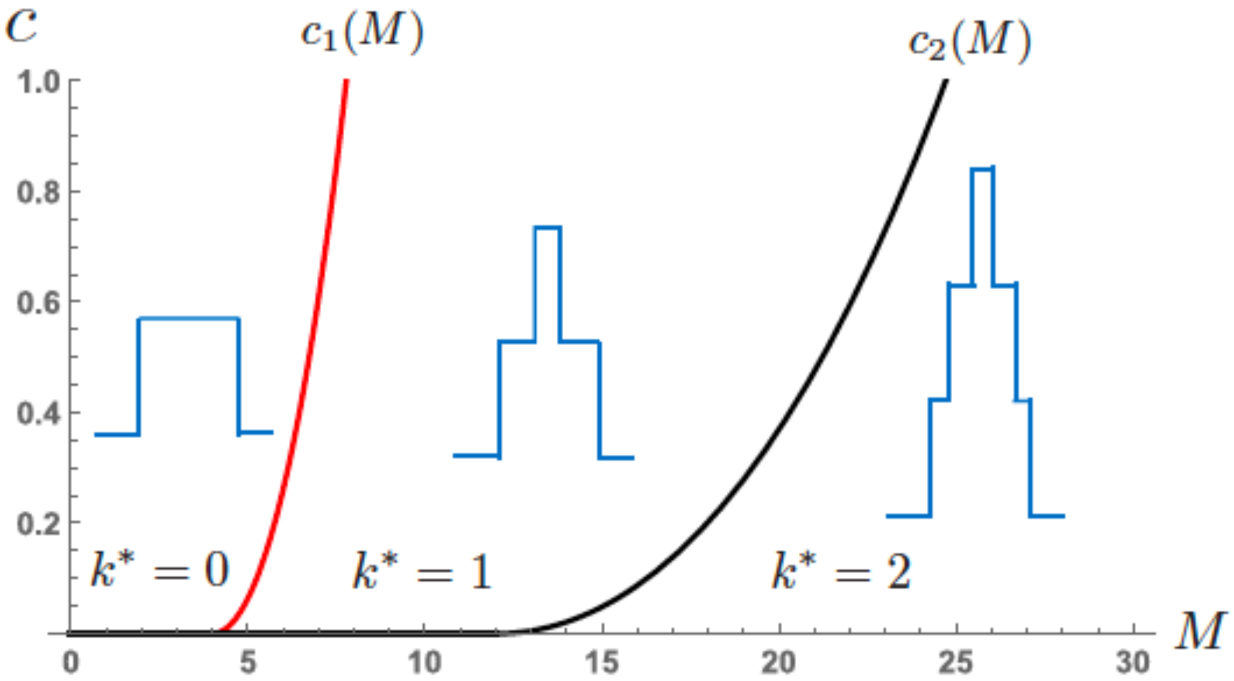
Phase diagram in (c,M) plane



- These are series of critical lines $c_1(M)$, $c_1(M)$ … separates the regions labelled by k^* where $k^* + 1$ is the number of Landau levels included in the ground state

- As one crosses these critical lines, the density profile undergoes abrupt changes

Phase diagram in (c,M) plane



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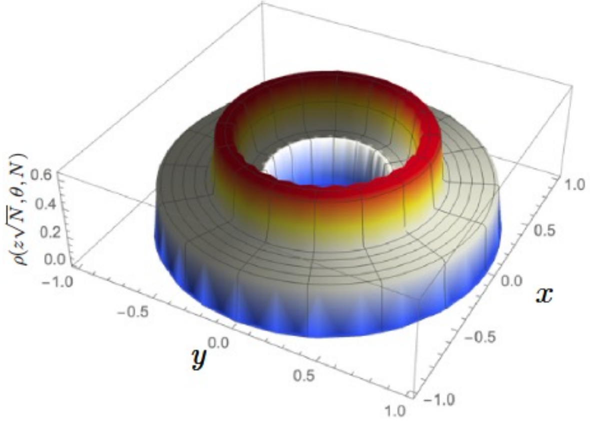
$$c_1(M) = \begin{cases} \frac{1}{M}\left(\frac{M^2}{16} - 1\right)^2, & M \geqslant 4 \\ 0 & M < 4. \end{cases}$$



$$c_2(M) = \begin{cases} \frac{17M^5 - 416M^3 - 12\sqrt{2}\sqrt{M^{10} - 48M^8 + 768M^6 - 4096M^4} + 2304M}{256M^2}, & M > 12 \\ 0, & 4 < M < 12. \end{cases}$$

- More generally, solving $\frac{4}{M}\sum_{q=1}^{k^*+1}\sqrt{q(q+\sqrt{cM})} = 1$ gives the critical line $c_{k^*+1}(M)$

Density in the large-N limit

Recap

- The average number density (normalized to N)

$$\rho(r,\theta,N) = \sum_{k,l} |\psi_{k,l}(r,\theta)|^2 = \sum_{k=0}^{k^*} \rho_k(r,\theta,N)$$

where $$\rho_k(r,\theta,N) = \frac{\Gamma(k+1)\,e^{-r^2}}{\pi} \sum_{l=l_-(k)}^{l_+(k)} \frac{\left[L_k^{\lambda}(r^2)\right]^2 r^{2\lambda}}{\Gamma(\lambda+k+1)}$$

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- Summation to integral
- Saddle point calculation
- Limiting identities of generalized Laguerre polynomials

$$\rho_k(z\sqrt{N},\theta,N) \approx \frac{2^{-k}}{\pi^{3/2} k!} \int_{a_-(k)}^{a_+(k)} dx\, e^{-x^2} \left[H_k(x)\right]^2 \quad \text{where } a_\pm(k) = \frac{(\lambda_\pm(k) - z^2)\sqrt{N}}{z\sqrt{2}}$$

k^{th} Hermite Polynomial

Density in the large-N limit: Bulk

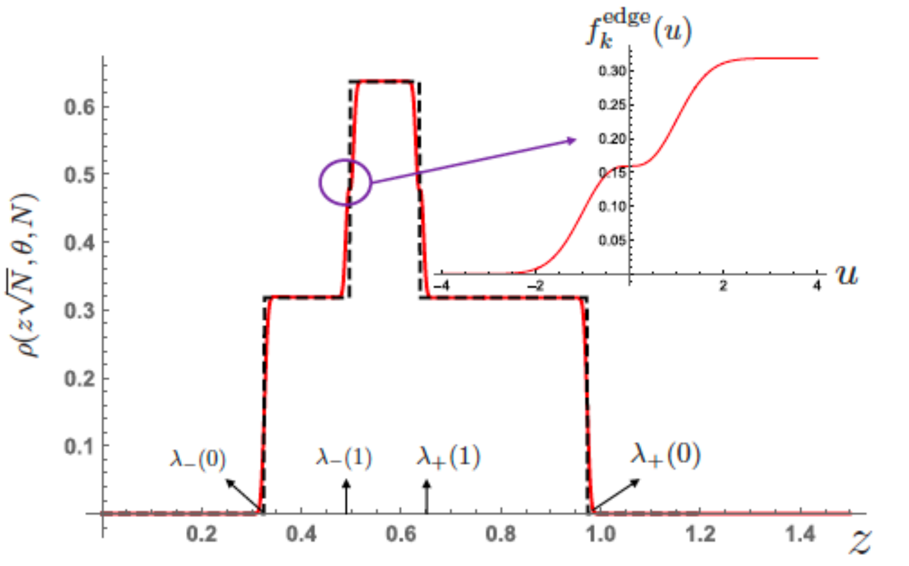
Recap

$$\rho_k(z\sqrt{N},\theta,N) \approx \frac{2^{-k}}{\pi^{3/2}k!}\int_{a_-(k)}^{a_+(k)} dx\, e^{-x^2}\,[H_k(x)]^2 \quad \text{where } a_\pm(k) = \frac{(\lambda_\pm(k)-z^2)\sqrt{N}}{z\sqrt{2}}$$

k^{th} Hermite Polynomial

$$\rho_k^{\text{bulk}}(r,\theta,N) \approx \frac{1}{\pi}\mathcal{I}_{\sqrt{\lambda_-(k)})<z<\sqrt{\lambda_+(k)}}$$

$$\rho(r,\theta,N) \sim f\left(\frac{r}{\sqrt{N}}\right) \quad \text{where} \quad f(z) = \frac{1}{\pi}\sum_{k=0}^{k^*}\mathcal{I}_{\sqrt{\lambda_-(k)}<z<\sqrt{\lambda_+(k)}}$$



(Analytical expression - black-dashed line)

- The bulk (when z is far away from edges) is characterized by multiple indicator functions.
- Next, we study the edge properties

Density in the large-N limit: Edge

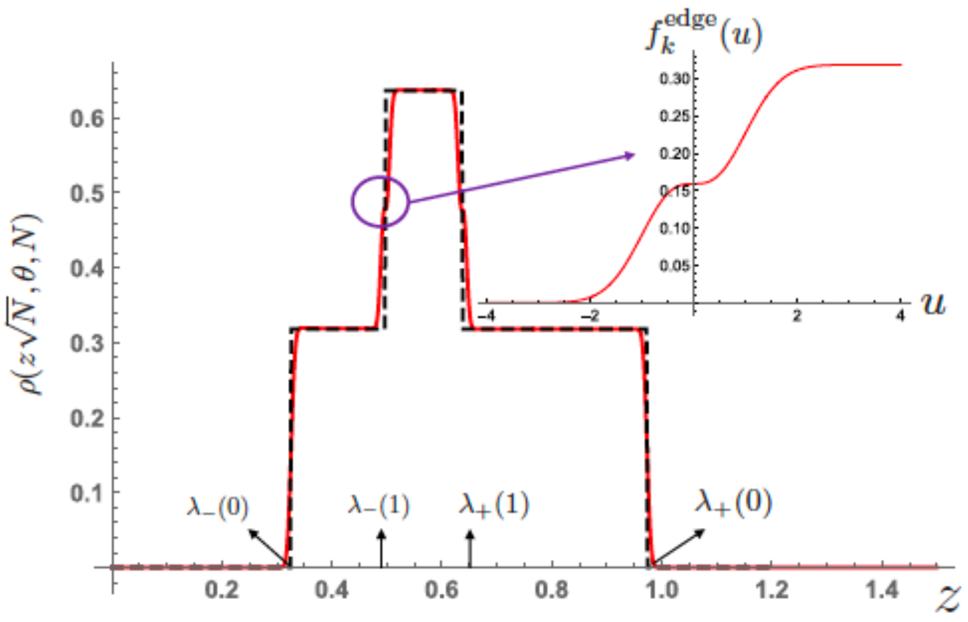
- We discuss the situation when z is close to one of the two edges, say the left edge $\sqrt{\lambda_-(k)}$

$$\rho_k^{\text{edge}}(r,\theta,N) \to f_k^{\text{edge}}(u)$$

where

$$f_k^{\text{edge}}(u) = \frac{2^{-k}}{\pi^{3/2}\Gamma(k+1)} \int_{-u}^{\infty} dx\, e^{-x^2}\, [H_k(x)]^2,$$

and $$u = \sqrt{\frac{N}{2\lambda_-(k)}}\left(\frac{r^2}{N} - \lambda_-(k)\right)$$



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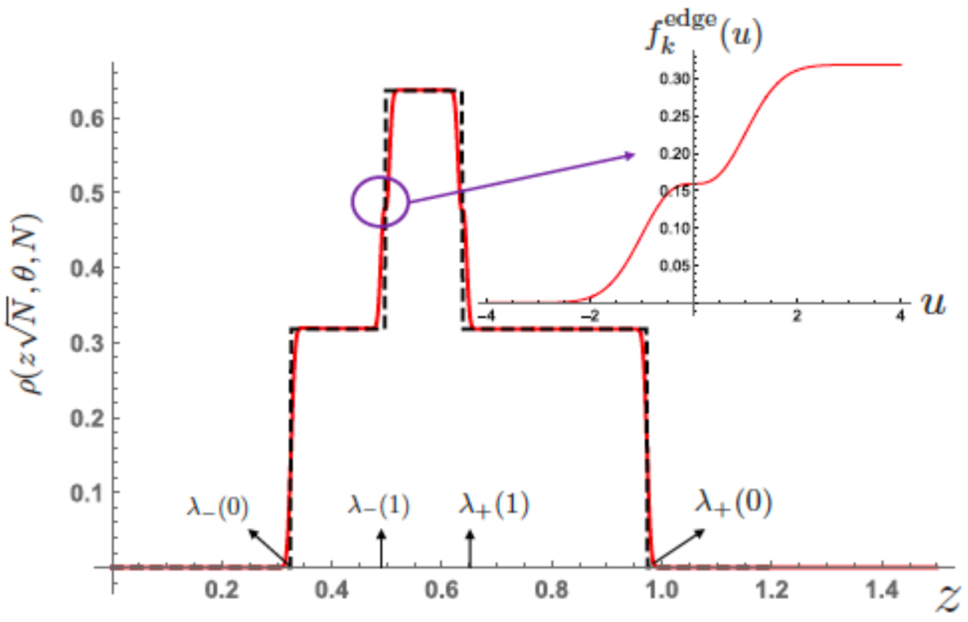
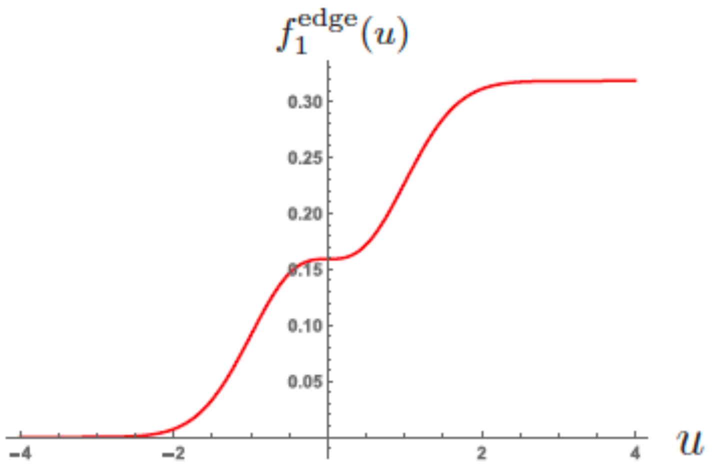
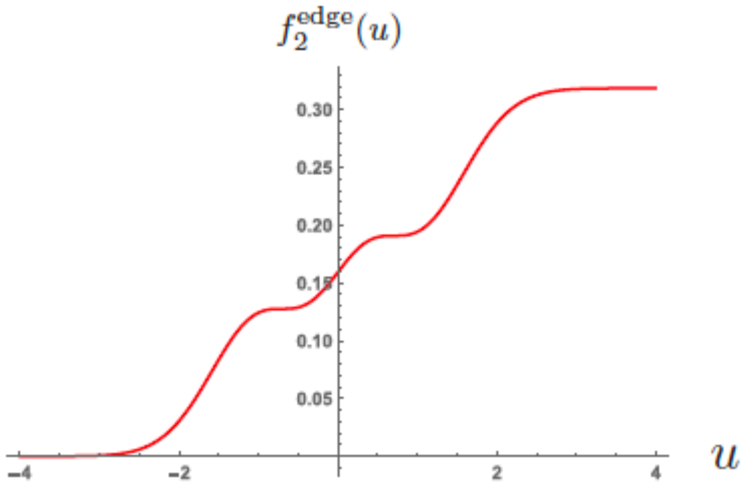




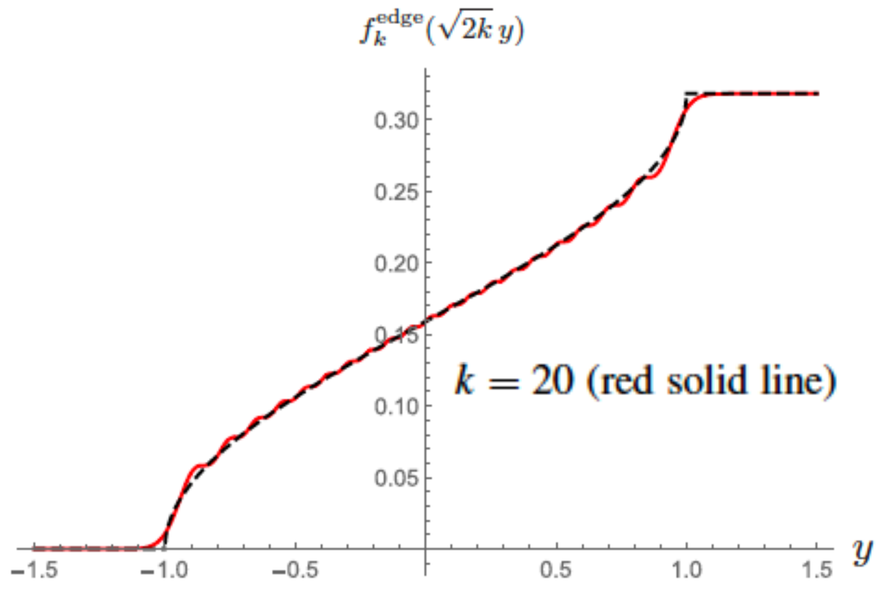


- $f_k(u)$ has k kinks as a function of u
- The location of the kinks in the edge density of the k^{th} band coincide with the zeros of k^{th} Hermite polynomial
- Kink positions are related to Steiltjes problem and kink density is a Wigner Semi-circle

Edge density in the limit of high Landau levels ($k \gg 1$) and connection to random matrix theory

- Let us set $u = \sqrt{2k}y$, with $y \sim O(1)$

$$\lim_{k\to\infty} f_k^{\text{edge}}(u = \sqrt{2k}\, y) = \begin{cases} 0\,, & y < -1 \\ \dfrac{1}{\pi^2}\left(\dfrac{\pi}{2} + \sin^{-1}(y)\right), & -1 < y < 1 \\ \dfrac{1}{\pi}\,, & y > 1 \end{cases}$$



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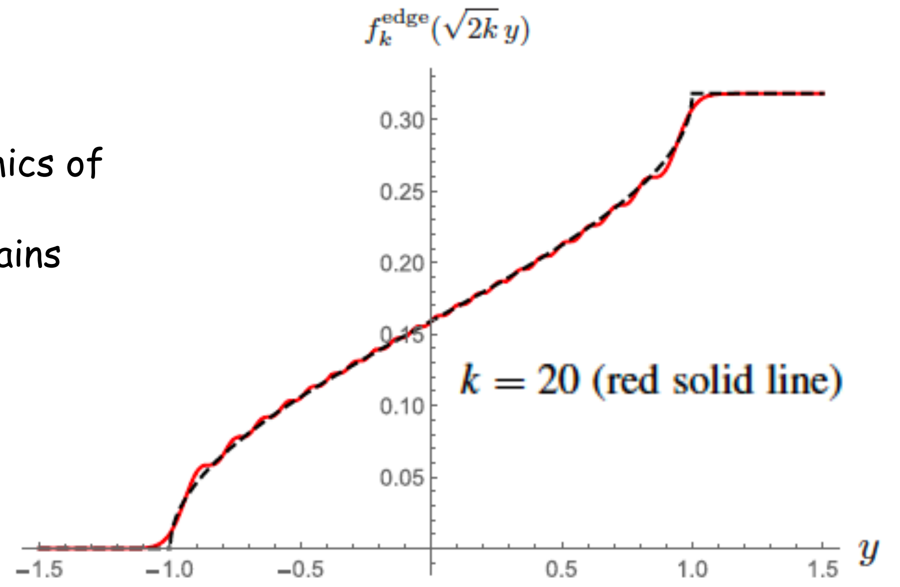
- Let us set $u = \sqrt{2k}y$, with $y \sim O(1)$

$$\lim_{k\to\infty} f_k^{\rm edge}(u=\sqrt{2k}\,y) = \begin{cases} 0\,, & y<-1 \\ \frac{1}{\pi^2}\left(\frac{\pi}{2}+\sin^{-1}(y)\right), & -1<y<1 \\ \frac{1}{\pi}\,, & y>1 \end{cases}$$

→ Scaling forms of dynamics of magnetization profile for XX quantum spin chains [Antal et al, PRE 1999]



Close to $u = \pm\sqrt{2k}$ there is an interesting edge region of width $O(k^{-1/6})$ where the density is described by Airy functions very similar to the well-known "Tracy-Widom" regime at the edge of the Wigner semicircle in RMT

$$u = -\sqrt{2k} + \frac{w}{\sqrt{2}k^{1/6}} \text{ with } w = O(1)$$

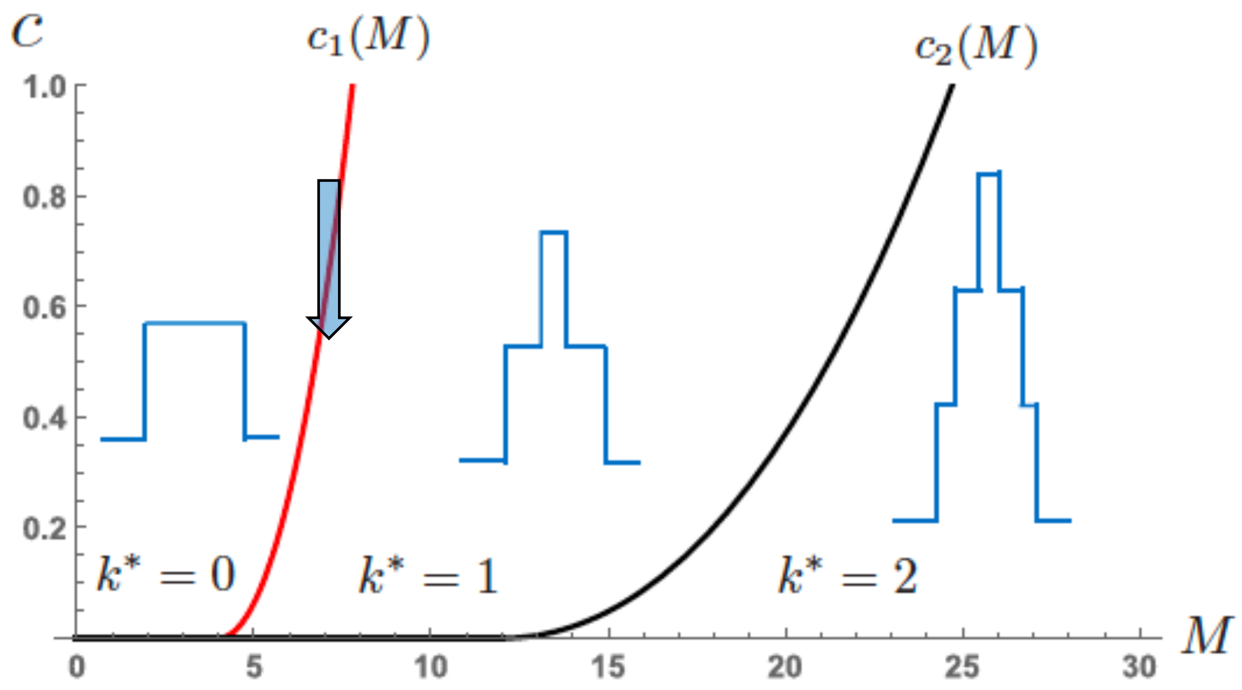
$$f_k\left(-\sqrt{2k} + \frac{w}{\sqrt{2}k^{1/6}}\right) \sim \frac{1}{k^{1/3}}\mathcal{F}(w)$$

$$\mathcal{F}(w) = \frac{1}{\pi}\int_0^\infty {\rm Ai}^2(v-w)\,dv = \frac{1}{\pi}([{\rm Ai}'(-w)]^2 + w{\rm Ai}^2(-w)),$$

Analog of Bowick, Brezin (1991), Forrester (1993)

Standard Airy Function

Emergence of new droplet as one crosses the critical line



- When k^* changes from $k^* = 0$ to $k^* = 1$ (i.e., a new band is included below the Fermi level), how does the density profile change from one layered to two layered ?

- Let us denote $c = c_1(M) \equiv c_1$

$$c = c_1 - \Delta \quad \text{where} \quad 0 < \Delta \ll 1$$

Measures the location the distance in the phase diagram with respect to the critical line

- The question is given Δ how does the second droplet emerge ?

- Large N calculations along with several reparameterizations

Emergence of new droplet as one crosses the critical line

$$\rho_1(r = z\sqrt{N}, \theta, N) \approx \frac{1}{\pi}[F_1(s + v_1 t) - F_1(s - v_1 t)]$$

$$v_1 = \frac{\sqrt{2}}{M}(c_1 M)^{1/4} \qquad t = \left(\frac{M}{4c_1}\right)^{1/4} \frac{64\sqrt{2}}{(M^2 - 16)^{3/2}}[c_1 - c]\sqrt{N} \qquad s = \left(\frac{M}{4c_1}\right)^{1/4}\left(z^2 - \frac{c_1}{M}\right)\sqrt{N}$$

$$F_1(z) = \frac{1}{2}\left[\mathrm{erf}(z) - \frac{2}{\sqrt{\pi}} z\, e^{-z^2}\right]$$

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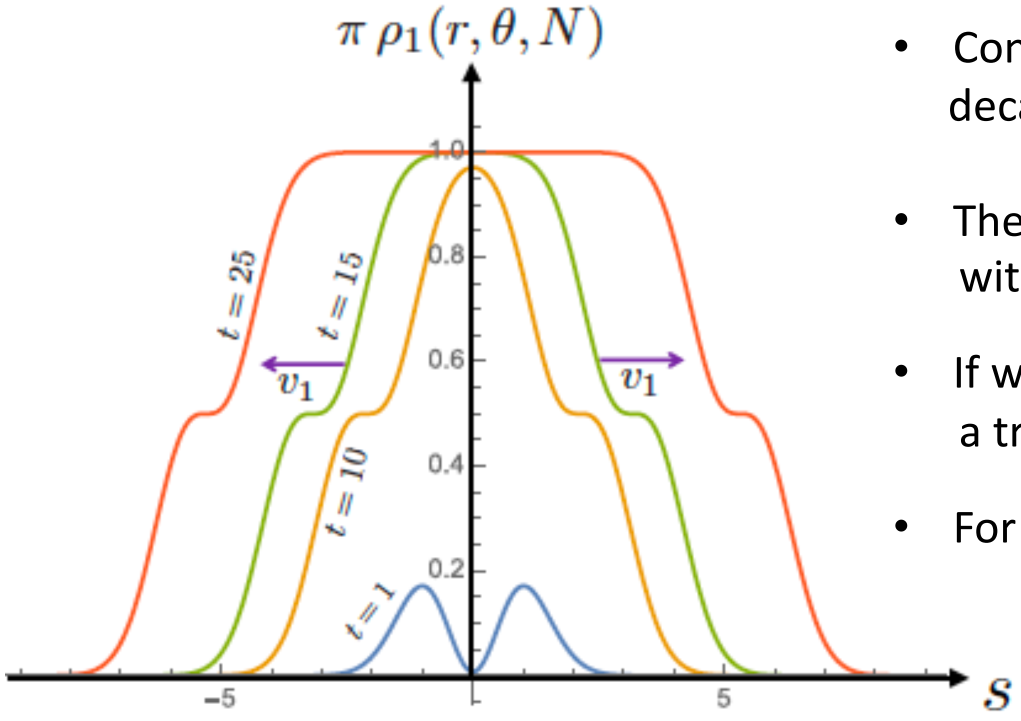


- The scaled density has an interesting travelling front structure
- Consider the density as a function of s for a fixed t . The density decays to 0 very rapidly as |s} >> $v_1 t$
- Therefore, the two edges of this profile move "ballistically" with increasing t with a "speed" given by v_1
- If we interpret t as a "time," then at late times the density profile develops a traveling front structure with velocity v_1
- For large t , the density has a constant value $1/\pi$ for all $|s| < v1t$

We did similar analysis for transition in higher bands

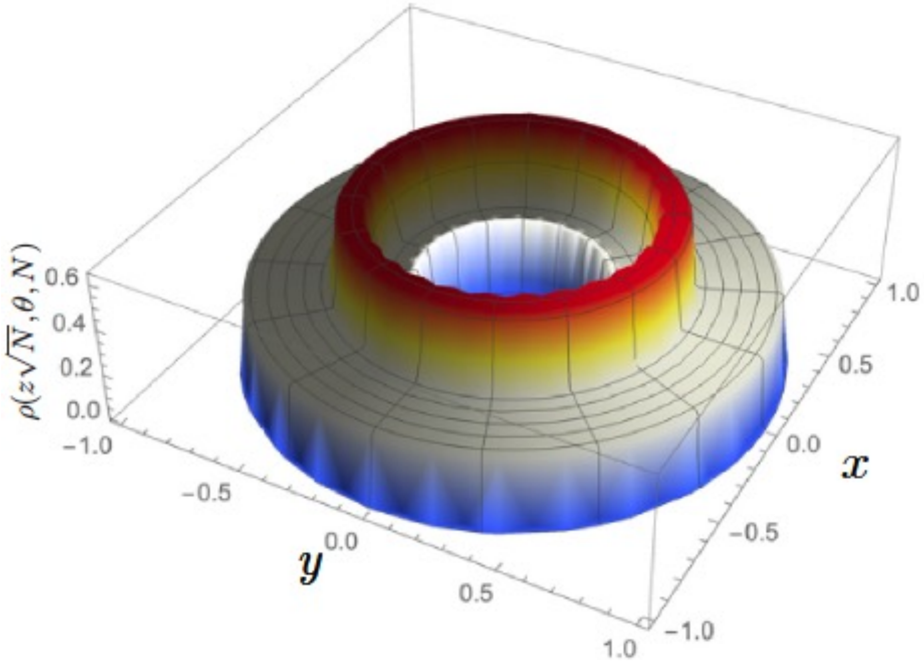
M.K, S. N. Majumdar, G. Schehr, Phys. Rev. A (2021)

Conclusions

- Density profile in the ground state of N non-interacting fermions in a rotating trap exhibits a rich multi-layered “wedding cake” structure

- Interesting phase diagram in the parameter space

- Connections to Random Matrices, Spin Chains , orthogonal polynomials, Stieltjes algebraic equations.



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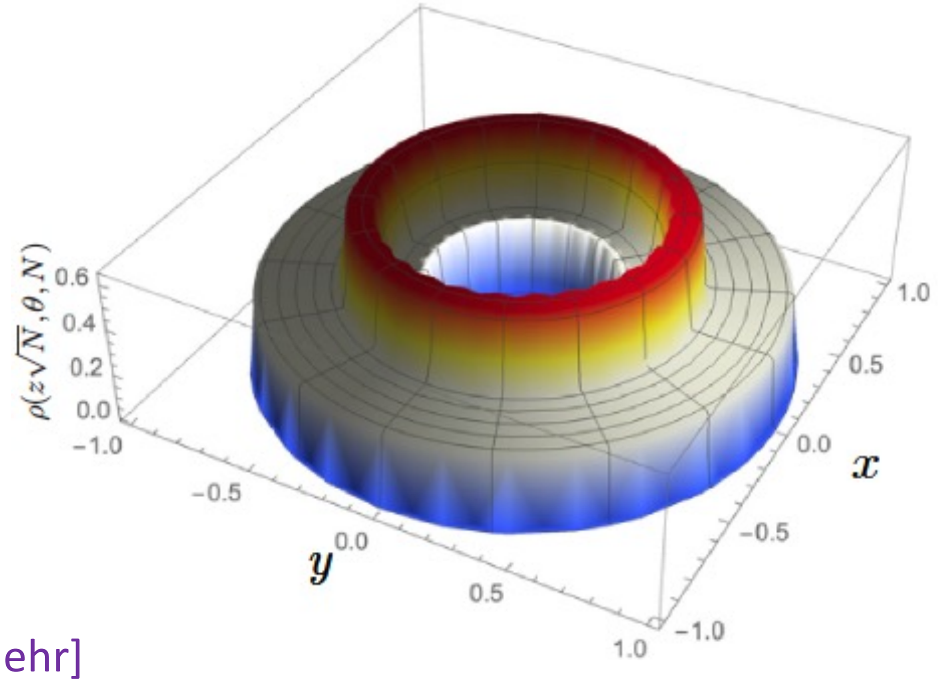

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Future

- Finite Temperature excitations [ongoing work with Le Doussal, Majumdar, Schehr]

- Banded filing of Fermions [1D was related to α-determinant process, Cunden, Majumdar ,O'Connell 2019]

- Number variance , entanglement entropy, correlations, generalized gap statistics

- Field Theory Descriptions, Phase space hydrodynamics [Generalizing 1D – M.K. , Mandal, Morita, PRA 2018]