Global Analysis of Locally Symmetric Spaces with Indefinite-Metric

Lecture 3
Global Analysis on Locally Symmetric Spaces Beyond the Riemannian Case

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Reminder: Locally homogeneous space $\Gamma\backslash X = \Gamma\backslash G/H$

Setting $\underset{\text{discrete}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{closed subgroup}}{H}$

$$\begin{tikzcd}
& G \arrow[dl, "\text{covering}"'] \arrow[dr] & \\
\Gamma\backslash G \arrow[dr] & & G/H = X \arrow[dl, "p_\Gamma"'] \\
& \Gamma\backslash G/H = \Gamma\backslash X &
\end{tikzcd}$$

- The quotient $\Gamma\backslash X = \Gamma\backslash G/H$ is not necessarily Hausdorff, but becomes a Hausdorff C^∞ manifold with p_Γ being a covering, if Γ acts properly discontinuously and freely on X.

Definition Such Γ is called a discontinuous group for X.

- *e.g.*, {Standard Quotients} $\subset$ {Sharp Quotients}.

Global Analysis of Locally Symmetric Spaces with Indefinite Metric

Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (Jan 1st)

— Introduction to pseudo-Riemannian space forms

— Construction/Obstruction of compact quotients $\Gamma\backslash X$

— Digression: “Tangential homogeneous space X_θ”

Lecture 2

Properness Criterion and its Quantification (Jan 2nd)

Lecture 3

Global Analysis on Locally Symmetric Spaces
Beyond the Riemannian Case (Jan 3rd)

Global Analysis of Locally Symmetric Spaces with Indefinite Metric

Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (Jan 1st)

Lecture 2

Properness Criterion and its Quantification (Jan 2nd)

- Proper Actions and Discontinuous Groups
- Properness Criterion
- Deformation *vs* local rigidity
- Quantifying Properness (“sharp” action)
- Counting of Γ-orbits

Lecture 3

Global Analysis on Locally Symmetric Spaces
Beyond the Riemannian Case (Jan 3rd)

Intrinsic differential operators in $\Gamma\backslash X = \Gamma\backslash G/H$

Let $X = G/H$ be a homogeneous space. We set

$\mathbb{D}_G(X)$:= the ring of G-invariant differential operators on X.

Example For a reductive homogeneous space $X = G/H$, the Laplacian Δ is defined (Lecture 1) and belongs to $\mathbb{D}_G(X)$.

Let Γ be a discontinuous group for X. Via the covering

$$p_\Gamma : X \to \Gamma\backslash X,$$

any $D \in \mathbb{D}_G(X)$ can be pushed forward to $\Gamma\backslash X$, and thus the ring $\mathbb{D}_G(X)$ acts on $C^\infty(\Gamma\backslash X)$.

The resulting differential operators on $\Gamma\backslash X$ are referred to as intrinsic differential operators on $\Gamma\backslash X$.

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Lecture 1

Local to Global in Non-Riemannian Geometry (Jan 1st)

Lecture 2

Properness criterion and its quantification (Jan 2nd)

Lecture 3

Global Analysis on Locally Symmetric Spaces (Jan 3rd)

— Intrinsic differential operators on $\Gamma\backslash G/H$

— Construction of discrete spectrum

— Expansions into eigenfunctions

Joint eigenfunctions on locally homogeneous space $\Gamma\backslash G/H$

Let $G \supset H$ be a pair of reductive Lie groups, and $X = G/H$.

We consider eigenfunctions f of the Laplacian Δ on $\Gamma\backslash X = \Gamma\backslash G/H$:

$$\Delta f = \lambda f,$$

or more generally, joint eigenfunctions on $\Gamma\backslash X$

$$\mathcal{M}_\lambda : \quad D_j f = \lambda_j f \qquad \text{for } D_j \in \mathbb{D}_G(X) \qquad (j = 1, \cdots, n).$$

Fact The ring $\mathbb{D}_G(X)$ is commutative,

$\Longleftrightarrow$ $\mathbb{D}_G(X)$ is a polynomial ring,

$\Longleftrightarrow$ $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical, *i.e.*, a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

Joint eigenfunctions for $\mathbb{D}_G(X) \curvearrowright C^\infty(\Gamma\backslash X)$

$$X = G/H$$

$$\mathbb{D}_G(X)^\wedge := \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}).$$

Definition (Joint eigenfunctions for intrinsic differential operators)
Let $\lambda \in \mathbb{D}_G(X)^\wedge$. A joint eigenfunction is a function f on $\Gamma\backslash X$ satisfying the following system of PDEs

$$\mathcal{M}_\lambda : Df = \lambda(D)f \quad {}^\forall D \in \mathbb{D}_G(X).$$

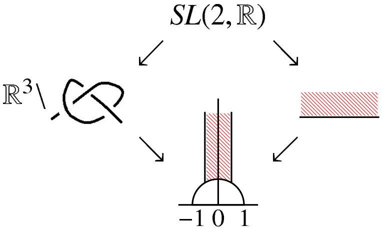
General Problem 4

- Construct Γ-periodic joint eigenfunctions;
- Expand arbitrary functions on $\Gamma\backslash G/H$ into joint eigenfunctions;
- Understand how L^2-eigenvalues are distributed.

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\begin{array}{ccccc} \Gamma & \subset & G & \supset & H \\ \text{discrete subgp} & & \text{Lie group} & & \text{subgroup} \\ SL(2,\mathbb{Z}) & \subset & SL(2,\mathbb{R}) & \supset & SO(2) \end{array}, \qquad X := G/H$$

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[dr] & \\
\Gamma\backslash G \arrow[dr] & & G/H \arrow[dl] \\
 & \Gamma\backslash G/H &
\end{tikzcd}$$



Special cases (classical cases) are already deep and rich.

- $\Gamma = \{e\}$
- H compact
- $G = \mathbb{R}^{p,q}$

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

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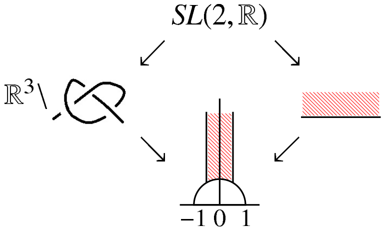

Special cases (classical cases) are already deep and rich.

- $\Gamma = \{e\} \cdots$ non-commutative harmonic analysis on $L^2(G/H)$

  Gelfand, Harish-Chandra, S. Helgason, Flensted-Jensen, T. Oshima, Delorme, . . .
- H compact, Γ arithmetic $\cdots$ automorphic forms (local theory)

  Siegel, Selberg, Piateski-Shapiro, Langlands, Arthur, Sarnak, Müller, . . .
- $G = \mathbb{R}^{p,q}$ (abelian, but non-Riemannian), $\Gamma \simeq \mathbb{Z}^{p+q}$, $H = \{e\}$

  Oppenheim conjecture, Dani, Margulis, Ratner, Eskin, Mozes, . . .

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}, \quad X := G/H$

New challenge: Spectral analysis on $\Gamma\backslash X$ by $\mathbb{D}_G(X)$
for non-trivial Γ, non-abelian G and non-compact H.

Further difficulties for non-compact H arise:

- (geometry)
- (analysis)
- (representation theory)

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

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New challenge: Spectral analysis on $\Gamma\backslash X$ by $\mathbb{D}_G(X)$
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Further difficulties for non-compact H arise:

- (geometry) Does "good" geometry $\Gamma\backslash X$ exist ?
  $\cdots$ "local to global" beyond Riemannian setting (Lectures 1 & 2).
- (analysis)
- (representation theory)

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}, \quad X := G/H$$

New challenge: Spectral analysis on $\Gamma\backslash X$ by $\mathbb{D}_G(X)$
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Further difficulties for non-compact H arise:

- (geometry) Does "good" geometry $\Gamma\backslash X$ exist ?
  - $\cdots$ "local to global" beyond Riemannian setting (Lectures 1 & 2).
- (analysis) The Laplacian Δ is no more elliptic.
  - $\cdots$ Is Δ essentially self-adjoint on $L^2(\Gamma\backslash X)$?
  - $\cdots$ Does Δ has point spectrum when $\Gamma\backslash X$ is compact?
- (representation theory) $\mathrm{vol}(\Gamma\backslash G) = \infty$ even when $\Gamma\backslash X$ is compact

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}, \quad X := G/H$$

New challenge: Spectral analysis on $\Gamma\backslash X$ by $\mathbb{D}_G(X)$
for non-trivial Γ, non-abelian G and non-compact H.

Further difficulties for non-compact H arise:

$\rightsquigarrow$ Let us develop new methods for the study!

- Geometric part

— “Quantitative” proper actions

(*e.g.,* “Sharpness”, “Counting” in Lecture 2)

- Analytic part

— Unitary representations of reductive groups,

(*e.g.,* non-commutative Harmonic Analysis, Branching Problem).

Plan of Lecture 3

Global Analysis on Locally Symmetric Spaces (Jan 3rd)

— Intrinsic differential operators on $\Gamma\backslash G/H$

— Construction of discrete spectrum

— Expansions into eigenfunctions

Main Setting

- $X = G/H$ reductie symmetric space with rank condition,
- $\Gamma \curvearrowright X$ sharp discontinuous group
  allows "non-standard" case, *e.g.*, Zariski dense subgroups,
  allows also the case $\mathrm{vol}(\Gamma\backslash X) = \infty$.

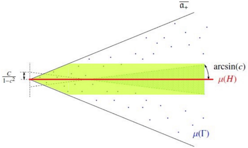
Point spectrum for locally symmetric space $\Gamma\backslash G/H$

$X = G/H$ a reductive symmetric space with $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$.

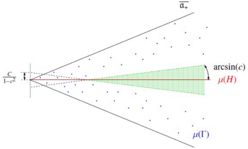
Example $G/H = GL(p+q,\mathbb{R})/GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$,
$O(p+1,q)/O(p,q)$ $(p > 0)$.

Theorem L (Kassel–TK, 2016)* For any sharp discontinuous group Γ for X, there exist infinitely many joint L^2-eigenvalues on $\Gamma\backslash X$.

- properness criterion uses



- sharpness definition uses



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[*] F. Kassel–Kobayashi, “Poincaré series for non-Riemannian locally symmetric spaces”, Adv. Math. **287** (2016), 123–236.

[**] M. Flensted-Jensen, Ann. Math., 1980; Matsuki–Oshima, 1984.

Point spectrum for locally symmetric space $\Gamma\backslash G/H$

$X = G/H$ a reductive symmetric space with $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$.

Example $G/H = GL(p+q,\mathbb{R})/GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$,
$O(p+1,q)/O(p,q)$ $(p > 0)$.

Theorem L (Kassel–TK, 2016)* For any sharp discontinuous group Γ for X, there exist infinitely many joint L^2-eigenvalues on $\Gamma\backslash X$.

Remark Theorem L allows non-standard Γ such as a Zariski dense subgroup, and also the case where $\operatorname{vol}(\Gamma\backslash X) = \infty$.
For $\Gamma = \{e\}$, Flensted-Jensen and Matsuki–Oshima proved**
$\operatorname{rank} G/H = \operatorname{rank} K/H \cap K \iff G/H$ admits a discrete series rep.

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Special case: application to the group case

A group manifold $G \simeq (G \times G)/\operatorname{diag}(G)$ is an example of symmetric spaces. In this very special case, our proof of Theorem L implies:

Corollary M Assume $\operatorname{rank} G = \operatorname{rank} K$.
Then, for any torsion-free discrete subgp Γ of G,

$$\operatorname{Hom}_G(\pi_\lambda, L^2(\Gamma\backslash G)) \neq 0$$

for all discrete series rep π_λ of G with sufficiently regular λ.

- We allow the case when $\operatorname{vol}(\Gamma\backslash G) = \infty$.
- We do not need to replace Γ with a subgroup Γ' of finite index.

Special case: application to the group case

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Remark Corollary M sharpens earlier works for arithmetic Γ:

Kazhdan '77, J.-S. Li '92,
De George–Wallach '78 (Γ: cocompact),
Clozel '86, Rohlfs–Speh '87 (Γ: non-compact lattice),

which treated an arithmetic subgroup Γ replaced by a congruence subgp Γ' (possibly depending on π_λ).

Special case: application to the group case

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$$\operatorname{Hom}_G(\pi_\lambda, L^2(\Gamma\backslash G)) \neq 0$$

for all discrete series rep π_λ of G with sufficiently regular λ.

- Theorem L applies also construction of discrete spectrum for

$$L^2(\Gamma_1\backslash G/\Gamma_2)$$

where $\Gamma_1 \times \Gamma_2 \subset G \times G \curvearrowright G$ sharp.

“Universal Spectrum” of anti-de Sitter manifold

A classical theorem for compact Riemann surfaces says that eigenvalues of the Laplacian vary as functions on Teichmüller space.

In contrast, a new phenomenon occurs in anti-de Sitter manifolds*:

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* T. Kobayashi, Intrinsic sound of anti-de Sitter manifolds, PROMS, **191** (2016), 83–99.

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"Universal Spectrum" of anti-de Sitter manifold

A classical theorem for compact Riemann surfaces says that eigenvalues of the Laplacian vary as functions on Teichmüller space.

In contrast, a new phenomenon occurs in anti-de Sitter manifolds*:

Corollary O** Let M be any 3-dimensional compact anti-de Sitter manifold. Then there is $C_M > 0$ such that the hyperbolic Laplacian Δ_M has countably many L^2-eigenvalues

$$\{\ell(\ell-2) : 2\mathbb{Z} \ni \ell \geq C_M\}$$

which are stable locally as functions on "Teichmüller space".

Reminder: There exist plentiful deformations of discontinuous groups for $\mathrm{Ad}S^3$ that yield 3-dimensional compact anti-de Sitter manifolds by Goldman, T. Kobayashi, F. Kassel, … (Lecture 2).

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Illustration of the proof by an example $X = \mathrm{AdS}^3$

$$X = \mathrm{AdS}^3 = \left\{ x := \begin{pmatrix} x_1 + x_4 & -x_2 + x_3 \\ x_2 + x_3 & x_1 - x_4 \end{pmatrix} : \det x = 1 \right\}$$
$$\simeq \{ x \in \mathbb{R}^4 : x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1 \} \subset \mathbb{R}^{2,2}$$

The pseudo-distance $\|x\|$ from the origin $o := (1, 0, 0, 0)$ is given by

$$\cosh \|x\| = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Step 1   Find (non-periodic) eigenfunctions with rapid decay.

$$\boxed{\begin{aligned} \Delta \psi_\ell &= \ell(\ell - 2)\psi_\ell, \\ |\psi_\ell(x)| &\sim e^{-\frac{1}{2}\ell\|x\|} \quad \text{as } \|x\| \to \infty. \end{aligned}}$$

For example, one considers

$$\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell} \quad \text{for } \ell \in \mathbb{N}.$$

Step 2: Averaging over $\Gamma \curvearrowright X$ sharp (Example)

Step 2: Construction of Γ-periodic eigenfunctions.

— Check the convergence of generalized Poincaré series.

$$\psi_\ell^\Gamma(x) := \sum_{\gamma\in\Gamma} \psi_\ell(\gamma \cdot x) = \sum_{R=1}^{\infty} \sum_{\gamma\in\Gamma_R} \psi_\ell(\gamma \cdot x).$$

$$\boxed{\Gamma = \coprod_{R=1}^{\infty} \Gamma_R, \qquad \Gamma_R := \{\gamma \in \Gamma : R-1 \le \|\gamma \cdot x\| < R\}.}$$

$\#\Gamma_R \le N_\Gamma(x; R) \sim Ae^{aR}$ (“Counting”, Lecture 2)

For $\gamma \in \Gamma_R$, $|\psi_\ell(\gamma \cdot x)| \sim e^{-\frac{\ell}{2}R}$ (rapid decay)

$\rightsquigarrow \psi_\ell^\Gamma$ converges uniformly on every compacta in $\Gamma\backslash X$ if $\ell \gg 2a$.

Step 3: Non-vanishing of Γ-averaging

$$\psi_\ell^\Gamma(x) = \sum_{\gamma\in\Gamma} \psi_\ell(\gamma \cdot x)$$

- Proof of $\psi^\Gamma \neq 0$ is much more involved.

Enough to show

$$1 = \psi_\ell(o) > \sum_{\gamma\in\Gamma\setminus\{e\}} |\psi_\ell(\gamma \cdot o)|$$

We recall $\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell}$ and $o = (1, 0, 0, 0)$.

The proof can be performed uniformly with respect to a small deformation of Γ.

How to find $\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell}$

M
compact 3-dim'l
anti-de Sitter mfd

How to find $\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell}$

covering

$\mathrm{AdS}^3 \quad \rightarrow M = \mathrm{AdS}^3/\Gamma$

compact 3-dim'l
anti-de Sitter mfd

How to find $\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell}$

complexification

covering

$S^3_{\mathbb{C}}$

↺ real forms ↻

$\mathbb{H}^3$
hyperbolic
space

AdS^3
anti-de Sitter
space

$\to M = \mathrm{AdS}^3/\Gamma$
compact 3-dim'l
anti-de Sitter mfd

How to find $\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell}$

Chern's double filtration

complexification

covering

$$\begin{tikzcd} & Z \arrow[dl, "q"'] \arrow[dr, "p"] & & S^3_{\mathbb{C}} \arrow[dl, bend right, "\text{real forms}"'] \arrow[dr, bend left] & & \\ \Xi & & \mathbb{H}^3 & & \mathrm{AdS}^3 \arrow[r] & M = \mathrm{AdS}^3/\Gamma \end{tikzcd}$$

horosphere

hyperbolic space

anti-de Sitter space

compact 3-dim'l anti-de Sitter mfd

How to find $\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell}$

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Ξ
horosphere

$\mathbb{H}^3$
hyperbolic space

AdS^3
anti-de Sitter space

$M = \mathrm{AdS}^3/\Gamma$
compact 3-dim'l anti-de Sitter mfd

$$\begin{array}{c} \{{y_1}^2 - {y_2}^2 - {y_3}^2 - {y_4}^2 = 0\} \\ \cap \\ \mathbb{R}^{1,3}\setminus\{0\} \end{array} \qquad \begin{array}{c} \{{y_1}^2 - {y_2}^2 - {y_3}^2 - {y_4}^2 = 1\} \\ \cap \\ \mathbb{R}^{1,3} \end{array} \qquad \begin{array}{c} \{{x_1}^2 + {x_2}^2 - {x_3}^2 - {x_4}^2 = 1\} \\ \cap \\ \mathbb{R}^{2,2} \end{array}$$

How to find $\psi_\ell(x) := (x_1 + \sqrt{-1}x_2)^{-\ell}$

Chern's double filtration complexification covering

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horosphere

hyperbolic space

anti-de Sitter space

compact 3-dim'l anti-de Sitter mfd

$\{y_1{}^2 - y_2{}^2 - y_3{}^2 - y_4{}^2 = 0\}$ $\{y_1{}^2 - y_2{}^2 - y_3{}^2 - y_4{}^2 = 1\}$ $\{x_1{}^2 + x_2{}^2 - x_3{}^2 - x_4{}^2 = 1\}$

$$\mathcal{D}'(\Xi) \xrightarrow[\text{Radon–Fourier}]{p_! \circ q^*} \mathcal{A}(\mathbb{H}^3), \quad g \mapsto f := p_! \circ q^*(g)$$

$\Delta f = \lambda f$ if $g \in \mathcal{D}'(\Xi)$ is homogeneous of degree $-\ell$

$$\rightsquigarrow \lambda = \ell(\ell - 2)$$

How to find $\psi_\ell(x)$ $:= (x_1 + \sqrt{-1}x_2)^{-\ell}$

Chern's double filtration — complexification — covering

$$\begin{tikzcd} & Z \arrow[dl, "q"'] \arrow[dr, "p"] & & S^3_{\mathbb{C}} \arrow[dl, bend right, "\text{real forms}"'] \arrow[dr, bend left] & & \\ \Xi & & \mathbb{H}^3 & & \mathrm{AdS}^3 \arrow[r] & M = \mathrm{AdS}^3/\Gamma \end{tikzcd}$$

Ξ horosphere

$\mathbb{H}^3$ hyperbolic space

AdS^3 anti-de Sitter space

$M = \mathrm{AdS}^3/\Gamma$ compact 3-dim'l anti-de Sitter mfd

$\{y_1{}^2 - y_2{}^2 - y_3{}^2 - y_4{}^2 = 0\}$ $\{y_1{}^2 - y_2{}^2 - y_3{}^2 - y_4{}^2 = 1\}$ $\{x_1{}^2 + x_2{}^2 - x_3{}^2 - x_4{}^2 = 1\}$

$$\begin{tikzcd} & & \mathcal{O}(S^3_{\mathbb{C}}) \arrow[dl] \arrow[dr] & \\ \mathcal{D}'(\Xi) \arrow[r, "p_! \circ q^*", "\text{Radon–Fourier}"'] & \mathcal{A}(\mathbb{H}^3) \arrow[rr, dashed, "\eta"] & & \mathcal{A}(\mathrm{AdS}^3) \end{tikzcd}$$

$g \mapsto f := p_! \circ q^*(g)$, $\Delta f = \lambda f$ $\dashrightarrow$ $\square \widetilde{f} = \lambda \widetilde{f}$ if $\widetilde{f} = \eta f$

ψ_ℓ $= \eta \circ p_! \circ q^*(g)$

if we take g is the delta function supported on $y_1 = y_2$ and of homogeneous degree $-\ell$.

Idea: Construction of L^2-eigenfunctions on $\Gamma\backslash G/H$

Step 1 Find (non-periodic) eigenfunctions with rapid decay

Step 2 Construction of periodic eigenfunctions (Poincaré series)

Step 3 Proof of non-vanishing

- Geometric estimate

- Analytic estimate

Idea: Construction of L^2-eigenfunctions on $\Gamma\backslash G/H$

$$\Delta f = \lambda f \text{ on } \Gamma\backslash G/H$$

Step 1 Find (non-periodic) eigenfunctions with rapid decay
(Poisson transform, Flensted-Jensen's duality)

Step 2 Construction of periodic eigenfunctions (Poincaré series)

Step 3 Proof of non-vanishing

- Geometric estimate for proper actions $\Gamma \curvearrowright G/H$
  (Kazhdan–Margulis, K– , Benoist, Kassel–K, . . .)
- Analytic estimate of eigenfunctions on G/H
  (systems of PDEs, micro-local analysis)
  (Sato–Kashiwara–Kawai, Oshima, . . .)

Idea: Construction of L^2-eigenfunctions on $\Gamma\backslash G/H$

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Steps 2 and 3 are carried out by "uniform estimates".

Plan of Lecture 3

Global Analysis on Locally Symmetric Spaces (Jan 3rd)

- Intrinsic differential operators on $\Gamma\backslash G/H$
- Construction of discrete spectrum
- Expansions into eigenfunctions

Main Setting

- $X = G/H$ reductive symmetric space (without rank assumption),
- $\Gamma\backslash X$ standard quotient with $\Gamma \subset G' \curvearrowright X$ proper and spherical allowing the case $\mathrm{vol}(\Gamma\backslash X) = \infty$.

Spectral analysis of standard locally symmetric space $\Gamma\backslash G/H$

General Problem 5 Can we expand any function on $\Gamma\backslash G/H$ into eigenfunctions?

Reminder: For any complete Riemannian manifolds M, it is well-known that the Laplacian Δ is essentially self-adjoint.

Difficulties Such self-adjointness of the Laplacian is non-trivial for complete pseudo-Riemannian manifolds where the metric tensor is indefinite.

Spectral analysis of standard locally symmetric space $\Gamma\backslash G/H$

Let $X = G/H$ be a reductive symmetric space, and G' reductive.

$$\begin{tikzcd}
\Gamma \subset G' \arrow[d, phantom, "\cap"] \arrow[drr, bend left, "\text{proper}"] & & & \\
G \arrow[rr, phantom, "\overset{\text{isometry}}{\curvearrowright}"] & & X = G/H \arrow[r, rightsquigarrow] & \Gamma\backslash X = \Gamma\backslash G/H
\end{tikzcd}$$

pseudo-Riemannian

standard quotient

Theorem P (Kassel–K, preprint[*]) Assume $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

Then there exist measure μ on $\mathbb{D}_G(X)^\wedge$ and a family of maps

$$\mathcal{F}_\lambda : C_c^\infty(\Gamma\backslash X) \to C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda)$$

s.t. any $f \in C_c^\infty(\Gamma\backslash X)$ is expanded into joint eigenfunctions on $\Gamma\backslash X$:

$$f = \int_{\mathbb{D}_G(X)^\wedge} \mathcal{F}_\lambda f \, d\mu(\lambda),$$

$$\|f\|^2_{L^2(\Gamma\backslash X)} = \int_{\mathbb{D}_G(X)^\wedge} \|\mathcal{F}_\lambda f\|^2_{L^2(\Gamma\backslash X)} d\mu(\lambda).$$

[*] F. Kassel–TK, Spectral analysis on standard locally homogeneous spaces. preprint (arXiv:1912.12601), 98 pages.

Self-adjoint extension of the Laplacian on $\Gamma\backslash X$

Let $X = G/H$ be a reductive symmetric space, and G' reductive.

$$\begin{tikzcd}
\Gamma \subset & G' \arrow[rrdd, bend left, "\text{proper}"] & & & \\
& \cap & & & \\
& G & \overset{\text{isometry}}{\curvearrowright} & \underset{\text{pseudo-Riemannian}}{X = G/H} & \rightsquigarrow \ \underset{\underline{\text{standard quotient}}}{\Gamma\backslash X = \Gamma\backslash G/H}
\end{tikzcd}$$

<u>Corollary Q</u> If $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical, then the Laplacian is essentially self-adjoint on $L^2(\Gamma\backslash X)$.

<u>Remark</u> In Theorem P and Corollary Q,
We allow the case where $\mathrm{vol}(\Gamma\backslash X) = \infty$.

Strategy for spectral analysis on $\Gamma\backslash G/H$

Reminder: Standard quotient $\Gamma\backslash X = \Gamma\backslash G/H$.

$$\begin{tikzcd}
(\Gamma \subset) G' \arrow[d, phantom, "\cap"] \arrow[rr, bend left, "\text{properly}"] & & X \\
G \arrow[r, phantom, "\curvearrowright"] & & X = G/H
\end{tikzcd}$$

In order to develop spectral analysis on $\Gamma\backslash G/H$, we investigate the G-irreducible representations π of G realized in $\mathcal{D}'(X)$, and how the restriction $\pi|_{G'}$ behaves.

$$H \nearrow G \searrow G' (\supset \Gamma)$$

- Harmonic Analysis on G/H + Branching Problem $G \downarrow G'$.

Compact-like geometric/linear actions (from Lecture 1)

Geometric actions on topological space $X = G/H$

$G' \curvearrowright X$ proper actions

L behaves nicely in $\mathrm{Homeo}(X)$

as if it were a compact group.

$\rightsquigarrow$??

Linear actions on Hilbert space $\mathcal{H}$

$G' \curvearrowright \mathcal{H} = L^2(X)$

G' behaves nicely in $U(\mathcal{H})$ (unitary operators)

as if it were a compact group.

Proper action *vs* discrete decomposability

Setting: $G' \subset G \supset H$ real reductive, $X := G/H$.

reductive symmetric space

Assme $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical, namely, a Borel subgroup of $G'_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

Theorem R (2019)* In the above setting, if $G' \curvearrowright X$ proper, then the restriction $\pi|_{G'}$ is discretely decomposable for any irreducible unitary representation π of G which is realized in $\mathcal{D}'(X)$.

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* T. Kobayashi, Global analysis by hidden symmetry, Progr. Math., **323** (2017), 359–397; F. Kassel – T. Kobayashi, JLT2019, pp. 663–754.

Summary: Strategy of the proof for Theorem P

1. Standard quotient (*cf.* Theorem B in Lecture 1)

$$\begin{tikzcd} \Gamma \subset & G' \arrow[rr, bend left, "\text{proper}"] & \subset G \curvearrowright & X & \rightsquigarrow & \Gamma\backslash X \end{tikzcd}$$

2. (Hidden symmetry)[*] If $G'_{\mathbb{C}} \curvearrowright X_{\mathbb{C}}$ is spherical, one has

$$\begin{tikzcd} \mathbb{D}_{G'}(X) \arrow[rr, bend left, "\text{hidden symmetry}"] & \supset \mathbb{D}_G(X) \curvearrowright & C^\infty(X; \mathcal{M}_\lambda). \end{tikzcd}$$

3. (Branching law $G \downarrow G'$) If $G' \curvearrowright X$ proper, then any irred G-rep π in $\mathcal{D}'(X)$ splits discretely when restricted to G' (Theorem R).

[*] T. Kobayashi, Global analysis by hidden symmetry, Progr. Math., **323** (2017), 359–397; Kassel–T. Kobayashi, “Invariant differential operators on spherical homogeneous spaces”, JLT (2019).

Global Analysis of Locally Symmetric Spaces with Indefinite Metric

Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (Jan 1st)

Lecture 2

Properness criterion and its quantification (Jan 2nd)

Lecture 3

Global Analysis on Locally Symmetric Spaces (Jan 3rd)

— Intrinsic differential operators on $\Gamma\backslash G/H$

— Construction of discrete spectrum

— Expansions into eigenfunctions

Expository papers and Open Problems for Lecture 3

Expository paper

- T. Kobayashi, Intrinsic sound of anti-de Sitter manifolds, PROMS, **191** (2016), 83–99, Springer.

Open Problems

- T. Kobayashi, Chapter 5 in Conjectures on Reductive Homogeneous Spaces, Mathematics Going Forward, Lecture Notes in Mathematics **2313**, pp. 213–231 (2023), Springer-Nature.

References for Lecture 3

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  Adv. Math. **287**, (2016), pp.123–236.

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  J. Lie Theory **29** (2019), 663–754.
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  Spectral analysis on standard locally homogeneous spaces,
  preprint, 98 pages.

Thank you very much for your attention!