Global Analysis of Locally Symmetric Spaces with Indefinite-Metric

Lecture 2

Properness Criterion and its Quantification

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Global Analysis of Locally Symmetric Spaces with Indefinite Metric

Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (Jan 1st)

- Introduction to pseudo-Riemannian space forms
- Construction/Obstruction of compact quotients $\Gamma\backslash X$
- Digression: “Tangential homogeneous space X_θ”

Lecture 2

Properness Criterion and its Quantification (Jan 2nd)

Lecture 3

Global Analysis on Locally Symmetric Spaces
Beyond the Riemannian Case (Jan 3rd)

Global Analysis of Locally Symmetric Spaces with Indefinite Metric

Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (Jan 1st)

Lecture 2

Properness Criterion and its Quantification (Jan 2nd)

- Proper Actions and Discontinuous Groups
- Properness Criterion
- Deformation *vs* local rigidity
- Quantifying Properness (“sharp” action)
- Counting of Γ-orbits

Lecture 3

Global Analysis on Locally Symmetric Spaces
Beyond the Riemannian Case (Jan 3rd)

Reminder· · · proper [properly discontinuous, free] action

$$\begin{array}{ccc} L & \overset{\text{action}}{\curvearrowright} & X \\ \text{loc. compact group} & & \text{loc. compact Hausdorff space} \end{array}$$

$$\begin{array}{rcl} X & & L \\ \text{subset} \cup & \rightsquigarrow & \cup \\ S & & L_S := \{\gamma \in L : \gamma S \cap S \neq \phi\} \end{array}$$

$$S = \{x\} \quad \rightsquigarrow \quad L_{\{x\}} \equiv L_x = \text{stabilizer of } x$$

Reminder··· proper [properly discontinuous, free] action

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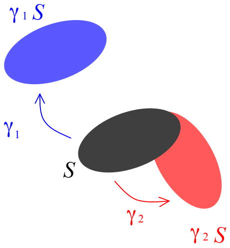

$$\Longrightarrow \begin{array}{l} \gamma_1 \not\in L_S \\ \gamma_2 \in L_S \end{array}$$

Reminder$\cdots$ proper [properly discontinuous, free] action

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$$S = \{x\} \quad \leadsto \quad L_{\{x\}} \equiv L_x = \text{stabilizer of } x$$

Definition

$L \curvearrowright X$ is proper $\Longleftrightarrow$ L_S is compact ${}^\forall S$: compact.

$L \curvearrowright X$ is properly discont $\Longleftrightarrow$ L_S is finite ${}^\forall S$: compact.

$L \curvearrowright X$ is free $\Longleftrightarrow$ $\# L_x = 1$ $\forall x \in X$.

Covering transformation and properly discontinuously action

Reminder from Lecture 1

$\Gamma \curvearrowright \underset{C^\infty \text{ manifold}}{X}$ properly discontinuously and freely

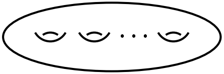
$\Longrightarrow$ The quotient $\Gamma\backslash X$ carries a C^∞-manifold structure such that the covering $X \to \Gamma\backslash X$ is smooth.

Example (Riemann surface Σ_g of genus $g \geq 2$)

$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

$\downarrow$ covering

$\Sigma_g =$



$\simeq \underset{\text{surface group}}{\pi_1(\Sigma_g)}\backslash\mathbb{H}$

Properly discontinuous actions: Riemannian geometry

(X, g) : a complete Riemannian manifold,

$G = \mathrm{Isom}(X)$: the group of isometries,

$\Gamma \subset G$ subgroup.

Proposition 2 (Riemannian geometry) Equivalent (i) $\Longleftrightarrow$ (ii):

(i) Γ is a discrete subgroup of G.

(ii) Γ acts properly discontinuously on X.

(ii) $\Rightarrow$ (i) easy.

(i) $\Rightarrow$ (ii) (non-trivial) Use an Ascoli–Arzela type argument to the metric space (X, g).

This proof depends heavily on the positivity of g. □

Question What if X is a pseudo-Riemannian manifold?

Calabi–Markus phenomenon (1962) from Lecture 1

Riemannian geometry

Actions of discrete subgroups of isometries

$\Longleftrightarrow$ isometric properly discontinuous actions

pseudo-Riemannian geometry

Actions of discrete subgroups of isometries

$\substack{\Longleftarrow \\ \not\Longrightarrow}$ isometric properly discontinuous actions

Let $(G, H) = (O(n,1), O(n-1,1))$.

$$\Gamma \underset{\text{discrete}}{\subset} G \underset{\text{isometry}}{\curvearrowright} \underset{\text{de Sitter space}}{G/H \simeq \{x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\}} \subset \mathbb{R}^{n,1}$$

Theorem A (Lect. 1) (Calabi–Markus)* Only a finite subgroup can act properly discontinuously on G/H.

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* E. Calabi–L. Markus, Relativistic space forms, Ann. Math., **75**, (1962), 63–76.

Properness criterion

General Problem 2 (Lecture 1)

(1) Given an action of a discrete group Γ on X,
find a "useful" criterion for the action to be
properly discontinuous.

More generally

(2) Given an action of a Lie group L on X,
find a "useful" criterion for the action to be proper.

Plan of Lecture 2

Properness Criterion and its Quantification

- Proper Actions
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Elementary consequences of proper actions

L : locally compact group.

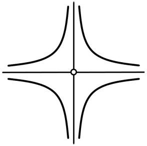
X : locally compact, Hausdorff space.

Proposition If L acts properly on X, then one has
(1) $L\backslash X$ is Hausdorff in the quotient topology;
(2) Any orbit $L \cdot x$ is closed in X;
(3) Any isotropy subgroup L_x is compact.

- Hausdorff $\Longrightarrow$ T_1 Obvious
  $$\underset{\text{global}}{(1)} \Longrightarrow \underset{\text{local}}{(2)}$$
- The conditions (2) and (3) are “local” and easily verified.
- However, (1) $\sim$ (3) do not guarantee the properness of the action.

Delicate examples (Hausdorff $\neq (T_1)$)

$$a \in \mathbb{R}_{>0} \curvearrowright X = \mathbb{R}^2 \setminus \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax \\ \frac{1}{a}y \end{pmatrix}$$



This action is free, and any orbit is closed.

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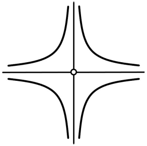
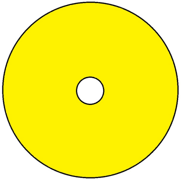
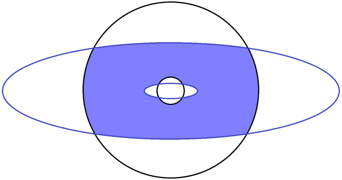


This action is free, and any orbit is closed.
But the action is not proper.

S

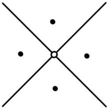


$S \cap a \cdot S$



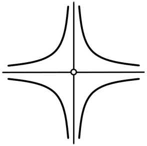
The quotient space $\mathbb{R}_{>0} \backslash X$ is (T_1) but not Hausdorff.

$\mathbb{R}_{>0} \backslash X \simeq$



Delicate examples (Hausdorff ≠ (T_1))

$a \in \mathbb{R}_{>0} \curvearrowright X = \mathbb{R}^2 \setminus \{\begin{pmatrix}0\\0\end{pmatrix}\}$, $\quad \begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}ax\\ \frac{1}{a}y\end{pmatrix}$



This action is free, and any orbit is closed.
But the action is not proper, and $\mathbb{R}_{>0}\backslash X$ is not Hausdorff.

Interpretation in group language

$$A = \{\begin{pmatrix}a & 0\\0 & a^{-1}\end{pmatrix} : a > 0\} \subset G = SL(2,\mathbb{R}) \supset N = \{\begin{pmatrix}1 & n\\0 & 1\end{pmatrix} : n \in \mathbb{R}\}$$

$$\begin{array}{ccccc} \begin{pmatrix}a & 0\\0 & a^{-1}\end{pmatrix} \in & A & \curvearrowright & G/N & \\ \updownarrow & \wr & & \wr & \\ a \in & \mathbb{R}_{>0} & \curvearrowright & \mathbb{R}^2 \setminus \{\begin{pmatrix}0\\0\end{pmatrix}\} & \begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}ax\\ \frac{1}{a}y\end{pmatrix} \end{array}$$

$A \curvearrowright G/N$ non-proper $\iff$ $N \curvearrowright G/A$ non-proper (Lorentz isometry)

Lipsman's conjecture (1995)

Setting $X = G/H$ where $\underset{\text{closed subgp}}{L} \subset \underset{\text{Lie gp}}{G} \supset \underset{\text{closed subgp}}{H}$

Lipsman's conjecture (1995)* G: 1-conn nilpotent Lie group

$$L \curvearrowright X \text{ free } \overset{?}{\iff} L \curvearrowright X \text{ proper}$$

True : G: 2-step nilpotent Lie group (Nasrin '01)

G: 3-step nilpotent Lie group (Baklouti '05, Yoshino '07)**

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[*] R. Lipsman, Proper actions and a cocompactness condition, J. Lie Theory **5** (1995), 25–39.

[**] A. Baklouti, Internat. J. Math. **16** (2005); T. Yoshino, Internat. J. Math. **18** (2007).

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True : G: 2-step nilpotent Lie group (Nasrin '01)

G: 3-step nilpotent Lie group (Baklouti '05, Yoshino '07)[**]

False : G: 4-step nilpotent Lie group (Yoshino '05)[***]

$$L \simeq \mathbb{R}^2 \curvearrowright X \simeq \mathbb{R}^5 \quad \text{(nilmanifold)}$$

This is a free action on a nilpotent homogeneous space $X = G/H$ such that all L-orbits are closed. However, $L \backslash X$ is not Hausdorff.

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[*] R. Lipsman, Proper actions and a cocompactness condition, J. Lie Theory **5** (1995), 25–39.

[**] A. Baklouti, Internat. J. Math. **16** (2005); T. Yoshino, Internat. J. Math. **18** (2007).

[***] T. Yoshino, A counterexample to Lipsman's conjecture, Internat. J. Math. **16** (2005), pp. 561–566.

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Properness Criterion and its Quantification

- Proper Actions
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- Deformation *vs* local rigidity
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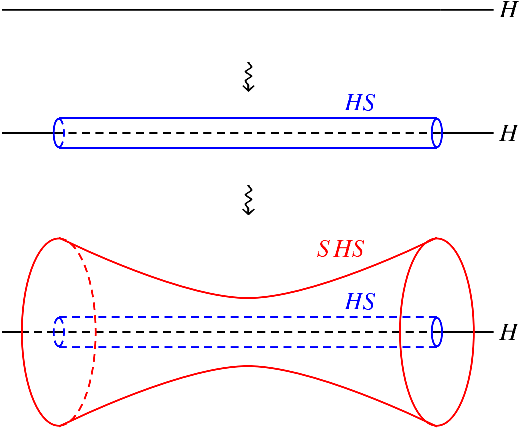
Problem For $L \subset G \supset H$,
find a criterion that $L \curvearrowright G/H$ properly.

Expanding H in a group G by compact set S

$G \supset H$

S : compact subset



$\pitchfork$ and $\sim$ for locally compact group G

$$L \quad \subset \quad G \quad \supset \quad H$$

Idea: forget even that L and H are subgroups

$\pitchfork$ and $\sim$ for locally compact group G

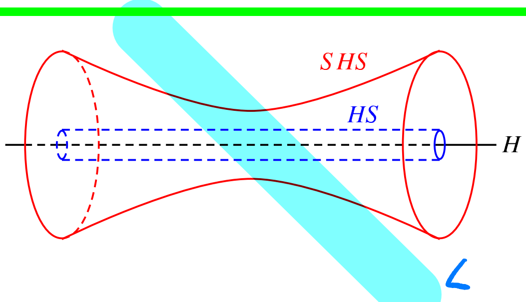
$$L \subset G \supset H$$

Idea: forget even that L and H are subgroups

Definition[*] ($\pitchfork$ and $\sim$)

1) $L \pitchfork H \iff \overline{L \cap SHS}$ is compact
for any compact subset $S \subset G$

2) $L \sim H \iff {}^{\exists}$ compact subset $S \subset G$.
such that $L \subset SHS$ and $H \subset SLS$.



[*] T. Kobayashi, Criterion for proper actions on homogeneous spaces $\cdots$, J. Lie Theory **6** (1996) 147–163.

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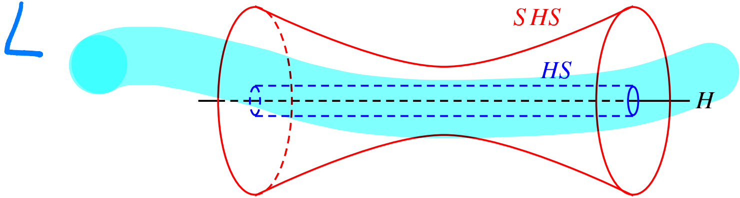

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Example (abelian case) $G = \mathbb{R}^n$; L, H subspaces

$$L \pitchfork H \iff L \cap H = \{0\}.$$

$$L \sim H \iff L = H.$$

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[*] T. Kobayashi, Criterion for proper actions on homogeneous spaces $\cdots$, J. Lie Theory **6** (1996) 147–163.

$\pitchfork$ and $\sim$ (meaning)

$$L \quad \subset \quad \underset{\text{loc compact group}}{G} \quad \supset \quad H$$

Meaning of $\pitchfork$: If both L and H are closed subgroups, then

$$\boxed{\begin{array}{lcl} L \pitchfork H & \iff & L \curvearrowright G/H \text{ proper action} \\ \quad \Updownarrow \text{ symmetric relation} & & \qquad \Updownarrow \text{ duality} \\ H \pitchfork L & \iff & H \curvearrowright G/L \text{ proper action} \end{array}}$$

$\sim$ defines an equivalence relation suitable for $\pitchfork$

$$H \sim H' \implies \boxed{H \pitchfork L \iff H' \pitchfork L}$$

Discontinuous duality theorem

G: locally compact topological group, separable

$G \supset H$ subset

$\rightsquigarrow \pitchfork (H : G) := \{L : L \pitchfork H\}$ discontinuous dual

Theorem F (discontinuous duality thm*. TK '96, Yoshino '07**)
Any subset H is determined uniquely by $\pitchfork (H : G)$
up to the equivalence relation $\sim$.

[*] T. Kobayashi, Criterion for proper actions . . . , J. Lie theory **6** (1996) 147–163. $\cdots$ reductive case

[**] T. Yoshino, Discontinuous duality theorem, Internat. J. Math. **18** (2007), pp. 887–893. $\cdots$ loc. compact gp

Properness criterion for reductive groups

We reformulate Problem 2 in this generality.

General Problem 2′ Find a handy criterion for two subsets $L, H \subset G$ to satisfy

$$L \pitchfork H \qquad \text{(properness criterion)}$$

up to the equivalence relation $H \sim H'$.

Shall explain the solution when G is a real reductive group.

Properness criterion for reductive groups

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$G = K\exp(\mathfrak{a})K$: Cartan decomposition of a real reductive group G

$W \equiv W(\Sigma(\mathfrak{g}, \mathfrak{a}))$: Weyl group.

$\mu: G \to \mathfrak{a}/W$: Cartan projection

Example $G = GL(n, \mathbb{R})$, $K = O(n)$, $\mathfrak{a} \simeq \mathbb{R}^n$, $W \simeq \mathcal{S}_n$.

$$\begin{aligned} \mu: \quad GL(n,\mathbb{R}) &\to \mathbb{R}^n/\mathfrak{S}_n \\ g &\mapsto \tfrac{1}{2}(\log\lambda_1, \cdots, \log\lambda_n) \end{aligned}$$

Here, $\lambda_1 \geq \cdots \geq \lambda_n$ (> 0) are the eigenvalues of ${}^t gg$.

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TK('89, '96) and Benoist('96) proved:

Theorem G (properness criterion) *

(1) $L \sim H$ in G $\iff$ $\mu(L) \sim \mu(H)$ in $\mathfrak{a}$.

(2) $L \pitchfork H$ in G $\iff$ $\mu(L) \pitchfork \mu(H)$ in $\mathfrak{a}$.

non-commutative abelian

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* T. Kobayashi, Math. Ann. (1989); J. Lie Theory **6** (1996) 147–163. ; Y. Benoist, Ann. Math., **144** (1996) 315–347.

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non-commutative abelian

Special cases include

⇒ in (1): Uniform error estimates of eigenvalues when a matrix is perturbed.

⇔ in (2): Criterion for proper actions.

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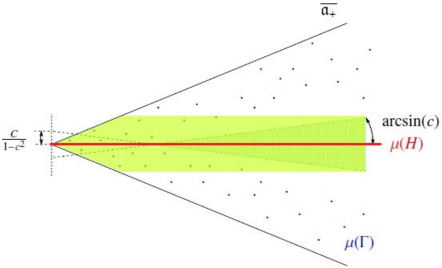
* T. Kobayashi, Math. Ann. (1989); J. Lie Theory **6** (1996) 147–163. ; Y. Benoist, Ann. Math., **144** (1996) 315–347.

Properness criterion (Theorem G)



$$\begin{tikzcd}
\Gamma \pitchfork H \arrow[r, Leftrightarrow, "\text{properness criterion}"'] \arrow[d, Leftrightarrow] & \mu(\Gamma) \pitchfork \mu(H) \text{ in } \mathfrak{a} \arrow[d, Leftrightarrow] \\
\underset{\text{properly discontinuously}}{\Gamma \curvearrowright G/H} & \#(\mu(\Gamma) \cap \varepsilon\text{-nbd of } \mu(H)) < \infty
\end{tikzcd}$$

Properness criterion — special case (H, L reductive)

For a reductive subgroup G' in G, the Cartan projection of G' takes the form $\mu(G') = W \cdot \mathfrak{a}_{G'}$ in $\mathfrak{a}$ (after conjugation of G' in G):

$$\begin{array}{ccccccccccc}
\mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} & \supset & \mathfrak{p} & \underset{\text{max abelian}}{\supset} & \mathfrak{a} & & \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
\mathfrak{g}' & = & \mathfrak{k}' & + & \mathfrak{p}' & \supset & \mathfrak{p}' & \underset{\text{max abelian}}{\supset} & \mathfrak{a}_{G'} & := & \mathfrak{a} \cap \mathfrak{g}'.
\end{array}$$

A special case of Theorem G includes:

Theorem G′(TK ’89)* Assume $H, L \subset G$ are reductive subgroups.

$L \curvearrowright G/H$ proper $\iff$ $\mathfrak{a}_H \cap W \cdot \mathfrak{a}_L = \{0\}$ in $\mathfrak{a}$.

Remark $\mu(H) \pitchfork \mu(L)$ in $\mathfrak{a}$ $\iff$ $\mathfrak{a}_H \cap W \cdot \mathfrak{a}_L = \{0\}$.

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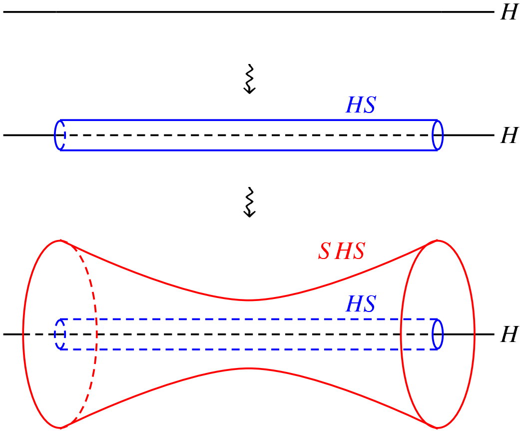
* Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989), 249–263.

Expanding H in a group G by compact set S

$G \supset H$

S : compact subset



Criterion for the Calabi–Markus phenomenon

Theorem A (Calabi–Markus, '62)[*] $(G, H) = (O(n+1,1), O(n,1))$.
Then G/H does not admit an infinite discontinuous group.

Corollary of Thm G (criterion of Calabi–Markus phenomenon) [**]
$G \supset H$ pair of real reductive Lie groups. Then one has
the following equivalences (i) ⇔ (ii) ⇔ (iii) ⇔ (iv):

(i) G/H admits a discontinuous group $\Gamma \simeq \mathbb{Z}$.
(ii) G/H admits an infinite discontinuous group Γ.
(iii) $G \not\sim H$.
(iv) $\operatorname{rank}_{\mathbb{R}} G > \operatorname{rank}_{\mathbb{R}} H$.

$$\text{(i)} \underset{\text{clear}}{\Longrightarrow} \text{(ii)} \underset{\substack{\Gamma \pitchfork H \\ \Gamma \not\pitchfork G}}{\Longrightarrow} \text{(iii)} \underset{\text{Cartan decomposition}}{\Longrightarrow} \text{(iv)} \underset{\text{Theorem G}}{\Longrightarrow} \text{(i)}$$

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[*] E. Calabi–L. Markus, Relativistic space forms, Ann. Math., **75**, (1962), 63–76.
[**] Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989), 249–263.

Another application: proper action of $SL(2,\mathbb{R})$ on G/H

- There are finitely many homomorphisms $\phi\colon SL(2,\mathbb{R}) \to G$ up to inner automorphisms.

- Okuda (2012)[*] classified all the irreducible symmetric spaces G/H and ϕ such that $\phi(SL(2,\mathbb{R}))$ acts properly on G/H.

- Proof is based on the properness criterion (Theorem G′) and on the Dynkin–Kostant classification of nilpotent orbits.

- The above symmetric spaces G/H admit a discontinuous group $\Gamma \simeq \pi_1($  $)$ $(g \geq 2)$ and vice versa[*].

[*] T. Okuda, Classification of semisimple symmetric spaces with proper $SL(2,\mathbb{R})$-actions, J. Differential Geom. **94** (2013).

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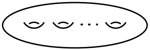
Deformation *vs* rigidity in the Riemannian setting

Reminder in the Riemannian setting

- Deformation theory

The Teichmüller space describes the variations of complex structures (or hyperbolic structures) of

the Riemannn surface $\Sigma_g =$  $\simeq \Gamma\backslash G/K$,

where

$$(\Gamma, G, K) = (\pi_1(\Sigma_g), PSL(2,\mathbb{R}), PSO(2)).$$

The dimension of (non-trivial) deformations is $6g - 6$ if $g \geq 2$.

- Rigidity theorem (Selberg, Weil, Mostow, Prasad, Margulis, Zimmer, . . .)

Deformation *vs* rigidity in the Riemannian setting

- Deformation in the Riemannian setting (previous slide)

  Teichmüller theory is for $X = G/K$ of 2-dimension.

- Rigidity (Selberg, Weil, Mostow, Prasad, Margulis, Zimmer, . . .)

  G: (non-compact) simple Lie group with Lie algebra $\mathfrak{g}$,
  $X = G/K$ irreducible Riemannian symmetric space.

Theorem H (Selberg–Weil's local rigidity in the Riemannian setting) *
If $\dim X > 2$ (*i.e.*, $\mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{R})$), then no cocompact discontinuous group Γ for X admits a continuous deformation.

In contrast, a discovery** in the non-Riemannian case: Flexibility of cocompact discontinuous groups may happen for arbitrary higher dimensions!

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* A. Weil, On discrete subgroups of Lie group, II, Ann. Math., (1962).

** TK, JGP (1993); Math. Ann. (1998).

Generalities: Deformation of quotients $\Gamma\backslash X = \Gamma\backslash G/H$

Formulation

$$\underset{\text{fix}}{\Gamma} \quad \underset{\text{discrete}}{\overset{\boxed{\varphi}}{\hookrightarrow}} \quad \underbrace{G \curvearrowright X = G/H}_{\text{fix}}$$

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Vary a homomorphism φ

$\rightsquigarrow$ Can we say that $\varphi(\Gamma)\backslash X$ is a deformation of $\Gamma\backslash X$?

Generalities: Deformation of quotients $\Gamma\backslash X = \Gamma\backslash G/H$

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$$\underset{\text{fix}}{\Gamma} \quad \underset{\text{discrete}}{\overset{\varphi}{\hookrightarrow}} \quad \underbrace{G \curvearrowright X = G/H}_{\text{fix}}$$

Vary a homomorphism φ

$\rightsquigarrow$ Can we say that $\varphi(\Gamma)\backslash X$ is a deformation of $\Gamma\backslash X$?

We need to consider homomorphisms φ through which Γ acts properly discontinuously on X.

Deformation of $\Gamma\backslash G/H$ (formulation)

Assume G acts faithfully on $X = G/H$. Fix a discrete subgroup Γ.
Formulate* "deformation" of $\Gamma\backslash G/H$ by varying $\varphi\colon \Gamma \to G$.

$$R(\Gamma, G; X) := \{\varphi\colon \Gamma \overset{\text{injective}}{\longrightarrow} G \mid \varphi(\Gamma) \curvearrowright X \ \underline{\text{properly discontinuous}}\}.$$

$$\mathrm{Aut}(\Gamma) \curvearrowright R(\Gamma, G; X) \curvearrowleft \mathrm{Int}(G).$$

<u>Definition</u> (Higher Teichmüller space and moduli space)*

$$\mathcal{T}(\Gamma, G; X) = R(\Gamma, G; X)/\,\mathrm{Int}(G),$$
$$\mathcal{M}(\Gamma, G; X) = \mathrm{Aut}(\Gamma)\backslash R(\Gamma, G; X)/\,\mathrm{Int}(G).$$

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* T. Kobayashi, JGP (1993), Math. Ann., (1998); see also TK, Discontinuous groups for non-Riemannian homogeneous spaces. Mathematics Unlimited — 2001 and Beyond, pages 723–747. Springer-Verlag, 2001.

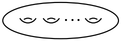
Classical example: $\Sigma_g \simeq$  $(g \geq 2)$

$$R(\Gamma, G; X) := \{\varphi\colon \Gamma \overset{\text{injective}}{\to} G \mid \varphi(\Gamma) \curvearrowright X \text{ properly discontinuous}\}.$$

<u>Definition</u> (previous slide)

$$\mathcal{T}(\Gamma, G; X) = \quad R(\Gamma, G; X)/\operatorname{Int}(G),$$
$$\mathcal{M}(\Gamma, G; X) = \quad \operatorname{Aut}(\Gamma)\backslash R(\Gamma, G; X)/\operatorname{Int}(G).$$

<u>Example</u> Let $G := PSL(2, \mathbb{R}) \supset K := PSO(2)$, and

$\Gamma := \pi_1($  $) \subset G \quad (g \geq 2).$

Then

$\mathcal{T}(\Gamma, G; X) =$ Teichmüler space of the Riemann surface Σ_g,

$\mathcal{M}(\Gamma, G; X) =$ Moduli space of the Riemann surface Σ_g.

Deformation of quotients $\Gamma\backslash X = \Gamma\backslash G/H$

Formulation

$$\underset{\text{fix}}{\Gamma} \quad \overset{\varphi}{\underset{\text{discrete}}{\hookrightarrow}} \quad \underbrace{G \curvearrowright X = G/H}_{\text{fix}}$$

Vary an injective homomorphism φ

⇝ Can we say that $\varphi(\Gamma)\backslash X$ is a deformation of $\Gamma\backslash X$?

- Two problems
  - — existence of nontrivial deformation φ;
  - — stability of proper actions under deformation.

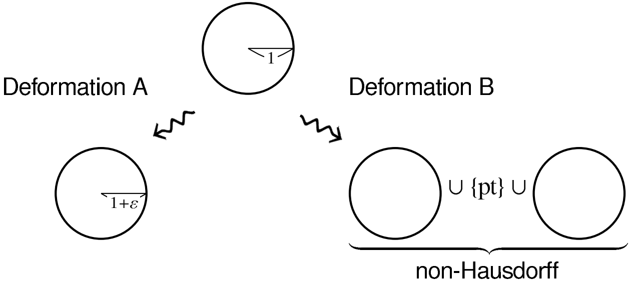
Small deformation of $\mathbb{Z} \curvearrowright \mathbb{R}$

Natural action of $\mathbb{Z}$ on $\mathbb{R}$ generated by

$$x \mapsto x + 1$$

Deformation A	$x \mapsto x + (1 + \varepsilon)$
Deformation B	$x \mapsto (1 + \varepsilon)x + 1$

Quotient space $\mathbb{Z}\backslash\mathbb{R}$



Deformation A

Deformation B

non-Hausdorff

Proper action: preserved! destroyed!

Small deformation of $\mathbb{Z} \curvearrowright \mathbb{R}$: Group theoretic interpretation

$$G := \mathrm{Aff}(1, \mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) : a \in \mathbb{R}^\times, b \in \mathbb{R}\}$$

$$H := \{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times\}$$

$\Gamma := \mathbb{Z}$ acts on $X := G/H \simeq \mathbb{R}$ via

$$\varphi \colon \mathbb{Z} \to G, \quad n \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^n .$$

Deformation A $x \mapsto x + (1 + \varepsilon) \quad \leftrightarrow \quad \varphi_\varepsilon^A(1) = \begin{pmatrix} 1 & 1+\varepsilon \\ 0 & 1 \end{pmatrix}$

Deformation B $x \mapsto (1+\varepsilon)x + 1 \leftrightarrow \varphi_\varepsilon^B(1) = \begin{pmatrix} 1+\varepsilon & 1 \\ 0 & 1 \end{pmatrix}$

Both φ_ε^A and φ_ε^B $\in \mathrm{Hom}(\Gamma, G)$ are “small deformations” of φ.

Γ does not act on X properly discontinuously via φ_ε^B $(\varepsilon \neq 0)$.

3-dimensional anti-de Sitter manifold AdS^3

$$SL(2,\mathbb{R}) = \{g = \begin{pmatrix} x_1 + x_4 & -x_2 + x_3 \\ x_2 + x_3 & x_1 - x_4 \end{pmatrix} : \det g = 1\}$$
$$= \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}$$

has a Lorentzian structure induced from

$$\mathbb{R}^{2,2} = (\mathbb{R}^4, dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2),$$

which has a constant sectional curvature -1.
Thus one may identify

$$G := SL(2,\mathbb{R}) \simeq \mathrm{AdS}^3 \qquad \text{(anti-de Sitter space)}.$$

The direct group $G \times G$ acts on AdS^3 as isometries on

$$(G \times G)/\operatorname{diag} G \simeq \mathrm{AdS}^3$$

Any discrete subgroup of G acts properly discontinuously on G from the left, yielding an anti-de Sitter manifold

$$\Gamma \backslash \mathrm{AdS}^3 = \Gamma \backslash G \simeq (\Gamma \times \{e\}) \backslash G \times G / \operatorname{diag} G.$$

A conjecture of Goldman

$$\Gamma \underset{\text{discrete}}{\subset} G = SL(2,\mathbb{R}) \simeq \mathrm{AdS}^3 \quad \text{(anti-de Sitter space)}$$

$\rightsquigarrow$ $(\Gamma \times \{e\})\backslash(G \times G)/\operatorname{diag} G = \Gamma\backslash \mathrm{AdS}^3$ is an anti-de Sitter mfd.

Deform $\Gamma \times \{e\}$ by considering a ‘graph’

$$\Gamma_\rho := \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\} \subset G \times G,$$

where $\rho\colon \Gamma \to G$ is a homomorphism.
Note that $\Gamma_{\mathbf{1}} \simeq \Gamma \times \{e\}$ where **1** denotes the trivial homomorphism.

Conjecture (Goldman 1985) Suppose $\Gamma\backslash G$ is compact.
If ρ is sufficiently ‘close to’ $\mathbf{1}$, then Γ_ρ acts
properly discontinuously on $(G \times G)/\operatorname{diag} G \simeq G$.

$\ldots$ Different situation from the deformation of $\mathbb{Z} \curvearrowright \mathbb{R}$.

* W. Goldman, Nonstandard Lorentz space forms, J. Differential Geometry **21** (1985), pp. 301–308.

Stability of properly discontinuous action

Theorem I (K- 1998)* Let $\Gamma \underset{\text{cocompact}}{\subset} G' \underset{\text{reductive}}{\subset} G \underset{\text{reductive}}{\supset} H$ such that $G' \curvearrowright G/H$ properly. If $\rho_1 \times \rho_2 : \Gamma \to Z_G(G') \times Z_G(H)$ is 'close to' **1**, then $\Gamma_{(\rho_1,\rho_2)} \curvearrowright X$ properly discontinuously.

$$\Gamma_{(\rho_1,\rho_2)} := \{(\gamma\rho_1(\gamma), \rho_2(\gamma)) : \gamma \in \Gamma\} \subset G \times G.$$

Applying Theorem E to $H = \{e\}$ and $G' = G = PSL(2, \mathbb{R})$, one sees

Corollary* Goldman's conjecture (1985) is true.

Since then, rapid developments include

- Solvable case (*e.g.*, $\mathbb{Z} \curvearrowright \mathbb{R}$)

  TK–Nasrin (2006), Baklouti and his collaborators, Yoshino, ...
- Reductive case

  Kassel (2012), Guéritard–Guichard–Kassel–Wienhard, Kannaka (2023), ...

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* T. Kobayashi, "Deformation of compact Clifford–Klein forms of indefinite $\cdots$", Math. Ann., **310** (1998), pp. 395–409.

Higher dimensional deformation ("Teichmüller theory")

(1) $\Gamma \curvearrowright G/K \iff (\Gamma \times 1) \curvearrowright (G \times G)/\triangle G$ (2)

$\Gamma \subset G$ simple Lie gp

Theorem H (Selberg–Weil's local rigidity, 1962)
$^{\exists}$ uniform lattice Γ admitting continuous deformations for (1)
$\iff G \approx SL(2, \mathbb{R})$ (loc. isom).

Theorem J (local rigidity in the non-Riemannian setting '98) **
$^{\exists}$ uniform lattice Γ admitting continuous deformations for (2)
$\iff G \approx SO(n+1, 1)$ or $SU(n, 1)$ $(n = 1, 2, 3, \dots,)$.

$\iff$ trivial representation is not isolated in the unitary dual
(not having Kazhdan's property (T))

** T. Kobayashi, "Deformation of $\cdots$" Math. Ann. (1998), 305–409.

Sketch of proof for Theorems I and J

Theorem J (local rigidity in the non-Riemannian setting)**
$^{\exists}$ uniform lattice Γ admitting continuous deformations for (2)
$\iff$ $G \approx SO(n+1,1)$ or $SU(n,1)$ $(n = 1, 2, 3, \ldots,)$.

Local rigidity of $\Gamma \curvearrowright G/H$ in the non-Riemannian setting
follows from infinitesimal rigidity $H^1(\Gamma, \mathfrak{g}) = 0$.

Deformation of $\Gamma\backslash G/H$

Need to prove that proper discontinuity is preserved under small deformation in the reductive case.

Idea: Use properness criterion (Theorem G).

Further examples for $O(8,8)/O(8,7)$

Construct compact standard quotients from Lecture 1.

Example (TK '96)* $^\exists$ 15-dimensional compact manifold having pseudo-Riemannian signature (8,7) with sectional curvature -1.

Take $L := Spin(8,1) \hookrightarrow G = O(8,8) \curvearrowright X = O(8,8)/O(8,7)$.

Testing deformations (bending constructions) (Kannaka et al. 2023)**

1) $\Gamma \underset{\text{cocompact}}{\subset} L \rightsquigarrow$ No continuous deformation of $\Gamma\backslash X$.

2) $^\exists\Gamma \underset{\text{cocompact}}{\subset} Spin(6,1)\ (\subset L) \rightsquigarrow {}^\exists$ continuous deformation φ such that $\varphi(\Gamma)$ is a discont subgroup for X and is Zariski dense in G.

3) $^\exists\Gamma = \pi_1($  $) \overset{\exists\varphi}{\hookrightarrow} G$ such that $\varphi(\Gamma)$ is a discontinuous group for X and is Zariski dense in G.

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* T. Kobayashi, Discontinuous groups and Clifford–Klein forms . . . , Academic Press 1996, pp.99–165.

** Johnson–Milson (1987); K. Kannaka–T. Okuda–K. Tojo, arXiv:2309.0833.

Plan of Lecture 2

Properness Criterion and its Quantification

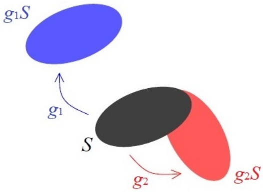
- Proper Actions
- Properness Criterion
- Deformation *vs* local rigidity
- Quantifying Properness (“sharp” action)
- Counting of Γ-orbits

Quantify proper actions

Definition A continuous action $G \curvearrowright X$ is called proper if the subset

$$G_S := \{g \in G : S \cap gS \neq \emptyset\}$$

is compact for any compact subset $S \subset X$.



$$\Longrightarrow \begin{array}{l} g_1 \not\in \\ g_2 \in \end{array} G_S$$

Two "quantifications" of properness of the action on $X = G/H$:

"asymptotic" volume	proper	"sharpness condition"
(Benoist–K, '15)	⟶	(Kassel–K, '15)
	"more proper"	
⇣		⇣
spectrum of		deformation theory of
$G \curvearrowright L^2(X)$		$\Gamma\backslash G/H$

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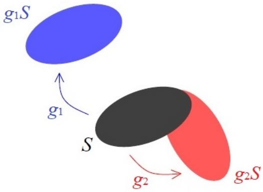


$$\Longrightarrow \quad \begin{array}{l} g_1 \notin \\ g_2 \in \end{array} G_S$$

Two "quantifications" of properness of the action on $X = G/H$:

"asymptotic" volume

(Benoist–K, '15)

↓

spectrum of

$G \curvearrowright L^2(X)$

proper → "sharpness condition"

"more proper"

"sharpness condition"

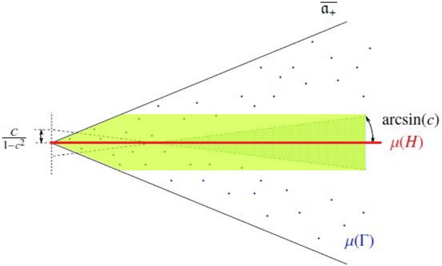
(Kassel–K, '15)

↓

deformation theory of

$\Gamma \backslash G/H$

Reminder: properness criterion (Theorem G)



$\Gamma \pitchfork H$ $\Longleftrightarrow$ $\mu(\Gamma) \pitchfork \mu(H)$ in $\mathfrak{a}$

properness criterion

$\Updownarrow$ $\Updownarrow$

$\Gamma \curvearrowright G/H$ $\#(\mu(\Gamma) \cap \underline{\varepsilon\text{-nbd of } \mu(H)}) < \infty$

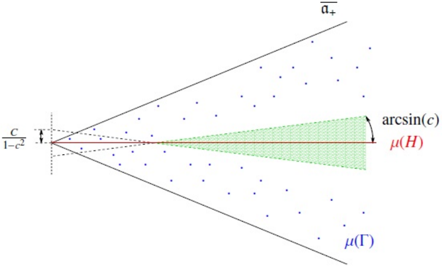
$\underline{\text{properly discontinuously}}$

Sharpness constant (c, C) for $X = G/H$

$G \supset H$ real reductive groups

Definition (Strongly proper action)* We say subgroup Γ of G is sharp for X if ${}^{\exists}c \in (0,1]$ and $C \geq 0$ such that

$$\|\mu(\gamma) - \mu(H)\| \geq c\|\mu(\gamma)\| - C.$$



- Γ is sharp for $X \Longrightarrow \Gamma \curvearrowright X$ properly discontinuously.
- Well-behaved under deformation of Γ.

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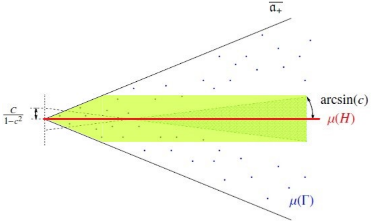
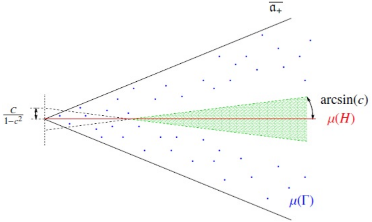
* F. Kassel–T. Kobayashi, Poincaré series for non-Riemannian locally symemtric spaces, Adv. Math., 2016. 123–226.

Sharpness (strongly proper action) vs proper action

- properness criterion uses



- sharpness definition uses



Plan of Lecture 2

Properness Criterion and its Quantification

- Proper Actions
- Properness Criterion
- Deformation *vs* local rigidity
- Quantifying Properness (“sharp” action)
- Counting of Γ-orbits

Counting : $\Gamma \cdot x \cap B_R$ in the Riemannian manifold

Classical Riemannian setting $\Gamma \curvearrowright X$ isometry

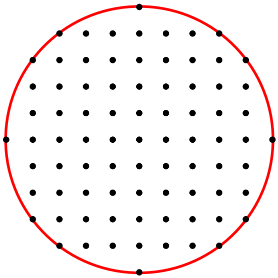
B_R : ball of radius R from a base point $x_0 \in X$

$\Gamma \cdot x$: Γ-orbit through $x \in X$

$$N_\Gamma(x; R) := \#(\Gamma \cdot x \cap B_R)$$

Example $\Gamma = \mathbb{Z}^2,\ X = \mathbb{R}^2,\ x = x_0 = (0, 0)$

$m(x; R) \sim \pi R^2$ (= volume of B_R)



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$$N_\Gamma(x; R) := \#(\Gamma \cdot x \cap B_R)$$

Comparison with volume $\mathrm{vol}(B_R)$ in the Riemannian setting.

Proposition (counting $\lesssim$ volume)

Let Γ be any discrete group of isometries of a complete Riemannian manifold X.

$$\Rightarrow \qquad {}^\forall x \in X, \ {}^\exists c > 0$$

$$\sup_{R>0} \frac{N_\Gamma(x; R)}{\mathrm{vol}(B_{R+c})} < \infty.$$

Counting : $\Gamma \cdot x \cap B_R$ in pseudo-Riemannian manifolds

New setting $X = G/H$ reductive homogeneous space
$\rightsquigarrow$ definition of "ball" $\underline{B_R}$ needs to be modified
(the G-invariant "metric" is indefinite)

$\underline{B_R}$: ball of "pseudo-radius" R from a base point $x_0 \in X$
$\Gamma \cdot x$: Γ-orbit through $x \in X$

$$N_\Gamma(x; R) := \#(\Gamma \cdot x \cap B_R)$$

$\underline{B_R}$

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* F. Kassel and TK, Poincaré series for non-Riemannian locally symmetric spaces, Adv. Math. **287**, (2016), pp.123–236.

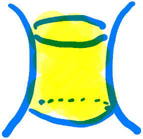
pseudo-ball for AdS^3 (Example)

$$X = \mathrm{AdS}^3 \simeq \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\} \subset \mathbb{R}^{2,2}$$

- pseudo-distance $\|x\|$ of x from the origin $o := (1, 0, 0, 0)$

$$\cosh \|x\| = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

- Volume of the pseudo-ball B_R with pseudo-radius R.



$$B_R = \{x \in X : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq \cosh R\}$$

$$\mathrm{vol}(B_R) = 4\pi^2(\sinh \frac{R}{2})^2 \sim 4\pi^2 e^R.$$

- Counting of Γ-orbits

$$N_\Gamma(x; R) = \#(\Gamma \cdot x \cap B_R).$$

It may grow faster than $\mathrm{vol}(B_R)$ even if Γ is a discont gp!

Counting : $\Gamma \cdot x \cap B_R$ in pseudo-Riemannian manifolds

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$\rightsquigarrow$ definition of "ball" B_R needs to be modified
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$$N_\Gamma(x; R) := \#(\Gamma \cdot x \cap B_R)$$

- Eskin–McMullen, . . . : Γ lattice of G, $x \in X$ special position

  Remark: $N_\Gamma(x; R) = \infty$ for $x \in X$ in generic position
  In fact Γ acts ergodically on X.

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* F. Kassel and TK, Poincaré series for non-Riemannian locally symmetric spaces, Adv. Math. **287**, (2016), pp.123–236.

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  Remark: $N_\Gamma(x; R) = \infty$ for $x \in X$ in generic position
  In fact Γ acts ergodically on X.
- Kassel–TK* : Γ discontinuous group for X, $x \in X$ general
  Upper estimates of $N_\Gamma(x; R)$ uniformly
  with respect to $x \in X$ and deformation of Γ (Lecture 3).

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* F. Kassel and TK, Poincaré series for non-Riemannian locally symmetric spaces, Adv. Math. **287**, (2016), pp.123–236.

Counting $N_\Gamma(x;R) = \#(\Gamma \cdot x \cap B_R)$ for G/H

Theorem K[*] Let $X = G/H$ be a semisimple symmetric space, and B_R a ball with pseudo-distance R from the origin. Let Γ be a sharp discontinuous group for X. Then $^\exists a > 0$, $^\exists A > 0$ such that

$$N_\Gamma(x;R) \leq Ae^{aR} \qquad ({}^\forall R > 0).$$

Remark (1) (non-symmetric case, 2023)[**] Theorem K can be extended for any reductive homogeneous space.

(2) (Kannaka 2023)[***] For any function $F(t)$ (*e.g.*, $\exp(e^t)$), there exists a non-sharp discontinuous group Γ for $X = \mathrm{AdS}^3$ s.t.

$$\sup_{R>0} \frac{N_\Gamma(x;R)}{F(\mathrm{vol}\, B_R)} = \infty.$$

[*] F. Kassel and TK, Poincaré series for non-Riemannian locally symmetric spaces, Adv. Math. **287**, (2016), pp.123–236.

[**] –, Infinite multiplicity for Poincare series $\cdots$, preprint.

[*] K. Kannaka, (Ph. D. thesis, to appear in Selecta math.)

Global Analysis of Locally Symmetric Spaces with Indefinite Metric

Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (Jan 1st)

Lecture 2

Properness Criterion and its Quantification (Jan 2nd)

- Proper Actions and Discontinuous Groups
- Properness Criterion
- Deformation *vs* local rigidity
- Quantifying Properness ("sharp" action)
- Counting of Γ-orbits

Lecture 3

Global Analysis on Locally Symmetric Spaces
Beyond the Riemannian Case (Jan 3rd)

Some references for Lecture 2

- Properness Criterion for Homogeneous Spaces of Reductive Groups

  T. Kobayashi (Math Ann '89, J. Lie Theory 1996)
  Y. Benoist (Ann Math 1996)

- Deformation theory

  Goldman (JDG, 1985); TK, (Math. Ann, 1998); Kassel (Math Ann 2012), Kannaka–Okuda–Tojo (arXiv:2309.08331).

- Quantification of proper action
  - **Weakening** proper actions via volume estimates
    Y. Benoist–TK, Tempered homogeneous spaces I, II, III, IV (2015–2023)
  - **Strengthening** proper action ('sharpness constant')
    F. Kassel and T. Kobayashi,
    Poincaré series for non-Riemannian locally symmetric spaces.
    Adv. Math. 287, (2016), pp.123–236.

Thank you very much for your attention!