Global Analysis of Locally Symmetric Spaces with Indefinite-Metric

Lecture 1
Local to Global in Non-Riemannian Geometry

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Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (1/1)

Lecture 2

Properness criterion and its quantification (1/2)

Lecture 3

Global Analysis on Locally Symmetric Spaces
Beyond the Riemannian Case (1/3)

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— Introduction to pseudo-Riemannian space forms

— Construction/Obstruction of compact quotients $\Gamma\backslash X$

— Digression

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Local to global in Riemannian geometry

The study " from local to global in Riemannian geometry " has been a major trend in geometry since the 20th century with remarkable developments.

As a warming up, one may recall one of classic theorems by Bonnet and Myers:

> Example (Myers 1941) A complete Riemannian manifold with Ricci curvature $\geq c(> 0)$ is compact.

How about " local to global " in pseudo-Riemannian geometry such as Lorentzian geometry?

Reminder : pseudo-Riemannian manifold

Definition A pseudo-Riemannian manifold (X, g) is a manifold equipped with non-degenerate bilinear form

$$g_x : T_x X \times T_x X \to \mathbb{R} \quad (x \in X)$$

depending smoothly on $x \in X$.

The signature (p, q) of g_x is locally constant.

(X, g) is a Riemannian manifold if $q = 0$,
a Lorentzian manifold if $q = 1$.

- The Laplacian Δ is a second-order differential operator on X defined by

$$\Delta = \operatorname{div} \circ \operatorname{grad} .$$

This is not an elliptic differential operator if g is indefinite.

Homogeneous pseudo-Riemannian manifold

Proposition 1 Let $G \supset H$ be a pair of real reductive Lie groups.
$\rightsquigarrow$ $X = G/H$ carries a pseudo-Riemannian structure

such that G acts isometrically on G/H.

Sketch of Proof For simplicity, suppose G is semisimple. We write $\mathfrak{g} \supset \mathfrak{h}$ for the Lie algebras of $G \supset H$, respectively.

- $\rightsquigarrow$ The Killing form B of $\mathfrak{g}$ is non-degenerate. Moreover, its restriction to $\mathfrak{h}$ is also non-degenerate.
- $\rightsquigarrow$ B induces an H-invariant non-degenerate bilinear form on the tangent space $T_o(G/H) \simeq \mathfrak{g}/\mathfrak{h}$.
- $\rightsquigarrow$ Its G-translation gives the desired pseudo-Riemannian structure.

Example Let $X = G/H$ with $(G, H) = (O(p+1, q), O(p, q))$.
The induced pseudo-Riemannian structure is of signature (q, p).

The Calabi–Markus phenomenon (1962)

In contrast to the Bonnet–Myers theorem in Riemannian geometry, global features of pseudo-Riemannian manifolds are quite mysterious:

Theorem A (Calabi–Markus, 1962*)
Any de Sitter manifold is non-compact.

de Sitter mfd = Lorentzian manifold with sectional curvature $\equiv 1$

This is an example of space form (the $q = 1$ case in the next slide) for pseudo-Riemannian geometry.

* E. Calabi–L. Markus, Relativistic space forms, Ann. Math., **75**, (1962), 63–76.

Space forms in pseudo-Riemannian geometry (definition)

(M, g) : pseudo-Riemannian manifold of signature (p, q), geodesically complete.

Definition* (M, g) is a space form
$\Longleftrightarrow$ sectional curvature κ is constant

Model space of space form

$$\{(x_1, \ldots, x_{p+q+1}) : \sum_{i=1}^{p+1} x_i^2 - \sum_{j=1}^{q} y_j^2 = 1\}$$

in $\mathbb{R}^{p+1,q} = (\mathbb{R}^{p+q+1}, dx_1^2 + \cdots + dx_{p+1}^2 - dy_1^2 - \cdots - dy_q^2)$
has a pseudo-Riemannian structure of signature (p, q) with $\kappa \equiv 1$.

The space is identified with $O(p+1, q)/O(p, q)$.

* J. A. Wolf, Spaces of Constant Curvature, 6th ed. AMS, 2011

Curvature in pseudo-Riemannian manifold

Remark The change of signature

$$(M, g) \rightsquigarrow (M, -g)$$

switches

the signature $(p, q) \rightsquigarrow (q, p)$,

the curvature $\kappa \rightsquigarrow -\kappa$.

Example The space form

$$O(p+1, q)/O(p, q) \simeq \{(x_1, \ldots, x_{p+q+1}) : \sum_{i=1}^{p} x_i^2 - \sum_{i=1}^{q} y_i^2 = 1\}$$

has a pseudo-Riemannian structure of signature (p, q) with $\kappa \equiv 1$

$$\Updownarrow$$

has a pseudo-Riemannian structure of signature (q, p) with $\kappa \equiv -1$

Space forms (examples)

Space form $\cdots$

$$\begin{cases} \text{Signature } (n-q, q) \text{ of pseudo-Riemannian metric} \\ \text{Curvature } \kappa \in \{+, 0, -\} \end{cases}$$

Example $q = 0$ (Riemannian manifold)

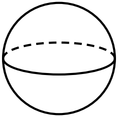
sphere S^n	$\mathbb{R}^n$	hyperbolic sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$O(n+1)/O(n)$		$O(n,1)/O(n)$

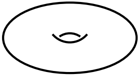
Example $q = 1$ (Lorentzian manifold)

de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$O(n,1)/O(n-1,1)$	$\mathbb{R}^{n-1,1}$	$O(n-1,2)/O(n-1,1)$

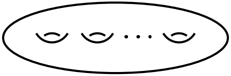
2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2,0)$)





Σ_g $(g \geq 2)$



curvature	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

Lorentzian case ($\Longleftrightarrow$ signature $(1,1)$)

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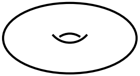

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curvature	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

There does NOT exist a compact form if $\kappa > 0$ and $\kappa < 0$

Local to global problem in pseudo-Riemannian geometry

General Problem 1 (Space form problem* for pseudo-Riemannian manifolds)

Local Assumption

signature (p, q), curvature $\kappa \in \{+, 0, -\}$

$$\Downarrow$$

Global Results

- Do compact forms exist?
- What groups can arise as their fundamental groups?

* T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces. Mathematics Unlimited — 2001 and Beyond, pages 723–747. Springer-Verlag, 2001;

TK, Conjectures on reductive homogeneous spaces, Lect. Notes in Math. **2313**, 217–231, Springer, 2023.

Formulation in group language (special case)

Uniformization theorem*: Any complete pseudo-Riemannian manifold M of signature (q, p) with $\kappa \equiv -1$ and $p \neq 1$ is of the form

$$\Gamma \backslash G/H$$

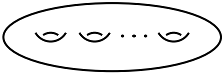
where $G = O(p+1, q)$, $H = O(p, q)$, and Γ is a discrete subgroup of G such that Γ acts properly discontinuously and freely on G/H.

Example (Klein–Poincaré–Koebe)

$(p, q) = (0, 2)$. Set $\Gamma := \pi_1(\Sigma_g)$.

$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \simeq SO(1, 2)/O(2)$

$\downarrow$ covering

$\Sigma_g =$  $(g \geq 2) \simeq \pi_1(\Sigma_g) \backslash \mathbb{H}$

surface group

* J. A. Wolf, Spaces of Constant Curvature, 6th ed. AMS, 2011

Generalities: Proper action & properly discontinuous action

We recall a continuous action $L \curvearrowright X$ is called proper if

$$L \times X \to X \times X. \quad (g, x) \mapsto (x, gx)$$

is a proper map, *i.e.*, the full inverse of a compact set is compact.

A properly discontinuous action is a proper action if L is a discrete goup.

General Problem 2 Find a (useful) properness criterion for the action of a subgroup L on $X = G/H$.

We shall discuss this problem in the second lecture (Jan. 2nd).

Definition: Discontinuous group Γ for $X = G/H$

Setting	Γ	$\subset$	G	$\supset$	H
	discrete		Lie group		closed subgroup

Definition: Discontinuous group Γ for $X = G/H$

Setting	Γ	$\subset$	G	$\supset$	H
	discrete		Lie group		closed subgroup

$$
\begin{array}{ccccc}
 & & G & & \\
 & \text{covering} \swarrow & & \searrow & \\
\Gamma\backslash G & & & & G/H = X \\
 & \searrow & p_\Gamma & \swarrow & \\
 & & \Gamma\backslash G/H = \Gamma\backslash X & &
\end{array}
$$

- $\Gamma\backslash G$ and $X = G/H$ are C^∞ manifolds.

Definition: Discontinuous group Γ for $X = G/H$

Setting	Γ	$\subset$	G	$\supset$	H
	discrete		Lie group		closed subgroup

$$\begin{array}{ccccc} & & G & & \\ & \text{covering} \swarrow & & \searrow & \\ \Gamma\backslash G & & & & G/H = X \\ & \searrow & p_\Gamma & \swarrow & \\ & & \Gamma\backslash G/H = \Gamma\backslash X & & \end{array}$$

- $\Gamma\backslash G$ and $X = G/H$ are C^∞ manifolds.
- The quotient $\Gamma\backslash X = \Gamma\backslash G/H$ is not necessarily Hausdorff.

It becomes a Hausdorff C^∞ manifold with p_Γ being a covering, if Γ acts properly discontinuously and freely on X.

Definition: Discontinuous group Γ for $X = G/H$

Setting $\quad \underset{\text{discrete}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{closed subgroup}}{H}$

$$
\begin{array}{ccc}
 & G & \\
{}^{\text{covering}}\swarrow & & \searrow \\
\Gamma\backslash G & & G/H = X \\
\searrow & {}^{p_\Gamma} & \swarrow \\
 & \Gamma\backslash G/H = \Gamma\backslash X &
\end{array}
$$

- $\Gamma\backslash G$ and $X = G/H$ are C^∞ manifolds.
- The quotient $\Gamma\backslash X = \Gamma\backslash G/H$ is not necessarily Hausdorff.

It becomes a Hausdorff C^∞ manifold with p_Γ being a covering, if Γ acts properly discontinuously and freely on X.

Definition Such Γ is called a discontinuous group for X.

Global properties of space forms $\kappa \equiv -1$, signature (q, p)

Theorem A′[*,**] Let $G/H = O(p+1, q)/O(p, q)$.

(1) (Calabi–Markus phenomenon) G/H admits an infinite discontinuous group if and only if $p < q$.

(2) If G/H admits a cocompact discontinuous group, then either $pq = 0$ or "$p < q$ and q is even".

(3) G/H admits a cocompact discontinuous group for (p, q) below.

p	$\mathbb{N}$	0	1	3	7
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	8

[*] Calabi–Markus (1962), Wolf (1962), Kulkarni (1981), Kobayashi (1996), Tholozan (2015), Morita (2017).

[**] T. Kobayashi, Conjectures on reductive homogeneous spaces, Lect. Notes Math. **2313**, 217–231, Springer, 2023.

Basic question on discontinuous groups for G/H

Let $G \supset H$ be a pair of real reductive (linear) groups
$\rightsquigarrow$ $X = G/H$ has a G-invariant (pseudo-) Riemannian structure.

General Problem 3* Which homogeneous space G/H admits ' large ' discontinuous groups Γ with the following properties?

1) $\#\Gamma = \infty$;

2) co-compact (or co-volume finite);

3) "deformable"/ "rigid".

- The first problem (Calabi–Markus phenomenon 1962) is solved in the reductive case (K- 1989). We shall discuss the general idea together with the third problem in the second lecture.
- The second problem is open, but has fruitful progress over the decades.

* **Problems A, B, C** in T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces. **Mathematics Unlimited — 2001 and Beyond**, pages 723–747. Springer-Verlag, 2001.

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Beyond the Riemannian Case (1/3)

Existence problem for compact quotients $\Gamma\backslash X$ of $X = G/H$

- Arithmetic groups of G (remarks).

- Construction
  — standard quotients.

- Obstruction
  — various methods and examples.

$\Gamma\backslash G/H$ is not Hausdorff if Γ is an arithmetic subgp of G.

- Borel–Harish-Chandra, Mostow–Tamagawa

For a linear semisimple Lie group G, one can define arithmetic subgroups Γ of G (*e.g.*, $\Gamma = SL(n,\mathbb{Z})$ in $G = SL(n,\mathbb{R})$)

- $\mathrm{vol}(\Gamma\backslash G) < \infty$ for any arithmetic subgroup Γ,
- $^{\exists}\Gamma$ such that $\Gamma\backslash G$ is compact (Borel 1962).

- Suppose H is a non-compact subgroup of G and $H \neq G$.

One should remark that the quotient topology of $\Gamma\backslash G/H$ never becomes Hausdorff if Γ is an arithmetic subgroup.

Idea of "standard quotient" $\Gamma\backslash G/H$

Γ	$\subset$	G	$\supset$	H
	discrete	reductive Lie gp		reductive subgp

Difficulties $\Gamma\backslash G/H$ is not always Hausdorff.

Idea of "standard quotient" $\Gamma\backslash G/H$

Γ	$\subset$	G	$\supset$	H
	discrete	reductive Lie gp		reductive subgp

<u>Difficulties</u> $\Gamma\backslash G/H$ is not always Hausdorff.

$$G \overset{\text{isometry}}{\curvearrowright} X = G/H \quad \text{pseudo-Riemannian}$$

Idea of "standard quotient" $\Gamma\backslash G/H$

Γ $\subset$ (discrete) G (reductive Lie gp) $\supset$ H (reductive subgp)

Difficulties $\Gamma\backslash G/H$ is not always Hausdorff.

G' —proper→

$\cap$

G $\curvearrowright$ (isometry) $X = G/H$ (pseudo-Riemannian)

Idea 1 Find a reductive subgp G' acting properly on X.

Such G' can be detected by properness criterion* (2nd lecture).

* T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann. (1989).

Idea of "standard quotient" $\Gamma\backslash G/H$

$$\underset{}{\Gamma} \quad \underset{\text{discrete}}{\subset} \quad \underset{\text{reductive Lie gp}}{G} \quad \supset \quad \underset{\text{reductive subgp}}{H}$$

Difficulties $\Gamma\backslash G/H$ is not always Hausdorff.

$$\begin{array}{ccccc} \Gamma & \subset & G' & \xrightarrow{\text{proper}} & \\ & & \cap & \text{isometry} & \text{pseudo-Riemannian} \\ & & G & \curvearrowright & X = G/H \end{array}$$

Idea 1 Find a reductive subgp G' acting properly on X.

Such G' can be detected by properness criterion* (2nd lecture).

Idea 2 Take a torsion-free discrete subgp Γ inside G'.

Definition The resulting C^∞-manifold $\Gamma\backslash X = \Gamma\backslash G/H$ is called a standard quotient of $X = G/H$.

* T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann. (1989).

Construction: Standard quotients $\Gamma \backslash X = \Gamma \backslash G/H$

Let $G \supset H$ be a pair of real reductive Lie groups, and $X := G/H$.

$\Gamma \subset G'$ — properly → X

$\cap$

G — isometry $\curvearrowright$ — $X = G/H$ pseudo-Riemannian

Theorem B (standard quotient) $X = G/H$ admits a cocompact discontinuous group if there exists a reductive subgp G' acting properly and cocompactly on X.

⇝ List* of standard compact quotients $\Gamma \backslash X$ (1989–2005*). Some exhaustion results ** by Tojo, Bocheński–Tralle, ...

* T. Kobayashi, Math. Ann. (1989); TK, Euro School, 1996; TK–Yoshino, PAMQ (2005) for the list;

** Tojo (PJA 2019; preprint), Bocheński–Tralle, for classification.

Examples of compact standard quotients

- By Theorem B, one finds several families of non-Riemannian homogeneous spaces G/H and finitely many sporadic ones that admit cocompact/co-volume finite discontinuous groups**.

Example* There exists a 15-dim'l compact manifold having pseudo-Riemannian signature (8,7) with sect. curvature -1.

Proof Take $L := Spin(8,1) \hookrightarrow G = O(8,8) \curvearrowright X = O(8,8)/O(8,7)$.
L acts properly on X, and $d(L) + d(H) = 8 + 56 = 64 = d(G)$. □

Example** There exists a 7-dimensional compact complex manifold modelled on $SO(8,\mathbb{C})/SO(7,\mathbb{C})$.

Proof. Take $L := Spin(7,1)$.

* T. Kobayashi, Discontinuous groups and Clifford-Klein forms …, Academic Press 1966, pp. 99–165.

** T. Kobayashi–T. Yoshino, PAMQ (2005) for the list; K. Tojo, PJA (2019), Bocheński–Tralle, for the exhaustion.

Global properties of Space forms $\kappa \equiv -1$, signature (q, p)

Theorem A′* (reminder) Let $G/H = O(p+1, q)/O(p, q)$.

(1) (Calabi–Markus phenomenon, Thm. A) G/H admits an infinite discontinuous group if and only if $p < q$.

(2) If G/H admits a cocompact discontinuous group, then either $pq = 0$ or "$p < q$ and q is even".

(3) G/H admits a cocompact discontinuous group for (p, q) below.

p	$\mathbb{N}$	0	1	3	7
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	8

Conjecture ([K01, Conj. 2.6]**) The converse of (3) holds.

* Calabi–Markus (1962), Wolf (1962), Kulkarni (1981), Kobayashi (1996), Tholozan (2015), Morita (2017).

** T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces, Mathematics Unlimited — 2001 and Beyond, pages 723–747, Springer-Verlag, 2001.

Existence problem for cocompact $\Gamma \curvearrowright G/H$

For more general geometry, we ask which homogeneous space G/H admits a "large" discontinuous group Γ.

$$G/H \to \Gamma \backslash G/H.$$

local global

- Standard construction for compact quotients (Theorem B),
- Obstructions for the existence of compact quotients
  Various approaches including
  - — properness criterion, sharpness (2nd lecture).
  - — cohomological obstructions (*e.g.*, characteristic classes)*,
  - — cohomological dimension (today),
  - — unitary representation, (today),
  - — ergodic theory,
  - — geometric group theory, $G/H \supset K/H \cap K, \ldots,$ (Morita's talk).

* T. Kobayashi–Ono, Hirzebruch's proportionality principle (1990); Morita, (Ph.D. thesis, 2017), Tholozan, ...

Naïve ideas for finding obstructions for compact quotients

- Numerical formulation of “non-compactness”?

— Using cohomologies $\rightsquigarrow$ Theorem C

$\cdots$ *e.g.*, ‘non-compactness dimension’ of $\mathbb{Z}^n$ or $\mathbb{R}^n$.

— Using unitary representations $\rightsquigarrow$ Theorem D

$\cdots$ topology $\rightsquigarrow$ measure theory
e.g., volume estimates rather than compactness, etc.

Cohomological dimension $\mathrm{cd}(\Gamma)$

Consider $\Gamma = \mathbb{Z}^k \curvearrowright X = \mathbb{R}^n$ affine action

Elementary observation

(1) If $\mathbb{Z}^k$ acts properly discontinuously on $\mathbb{R}^n$, then $k \leq n$.

(2) Furthermore, $\mathbb{Z}^k$ acts cocompactly on $\mathbb{R}^n$ if and only if $k = n$.

- What does k mean in $\Gamma = \mathbb{Z}^k$? Use cohomology of groups.
- What does n mean in $X = \mathbb{R}^n$? (next slide)

Definition (cohomological dimension of group)

For an abstract group Γ, one defines

$\mathrm{cd}_{\mathbb{R}}(\Gamma)$:= the projective dimension of $\mathbb{R}$ as the trivial $\mathbb{R}[\Gamma]$-module

$= \sup\{n \in \mathbb{N} : H^n(\Gamma; A) \neq 0$ for some left $\mathbb{R}[\Gamma]$-module $A\}$.

Example $\mathrm{cd}_{\mathbb{R}}(\mathbb{Z}^k) = k$,

$\mathrm{cd}_{\mathbb{R}}(\pi_1(\Sigma_g)) = 2$.

Cocompactness criterion for $\Gamma \curvearrowright G/H$

K : maximal compact subgroup of a connected reductive Lie group G.

$d(G) := \dim G - \dim K$ ("non-compactness dimension").

$d(G) - d(H) \cdots$ "non-compactness dimension"** of G/H

Theorem C** Let H be a reductive subgroup of G. Suppose Γ is a discontinuous group for G/H. Then one has

(1) $\mathrm{cd}_{\mathbb{R}}(\Gamma) \le d(G) - d(H)$.

(2) $\mathrm{cd}_{\mathbb{R}}(\Gamma) = d(G) - d(H) \iff \Gamma \backslash G/H$ is compact.

Use Serre's spectral sequence* for

$$G/H \simeq K \underset{H\cap K}{\times} \mathbb{R}^{d(G)-d(H)} \underset{\text{homotopic}}{\approx} K/H \cap K$$

$$\downarrow$$

$$\Gamma \backslash G/H$$

where K is chosen s.t. $H \cap K$ is a maximal compact subgp in H.

* J. P. Serre, Cohomologie des groupes discrètes, Annals of Math. Studies, 1971;

** Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989), 249–263.

Cocompactness criterion for $L \curvearrowright G/H$ (continuous analog)

$d(G) := \dim G - \dim K$ ("non-compactness dimension" of G).

$d(G) - d(H) \cdots$ "non-compactness dimension" of G/H.

Theorem 2 Let H be a reductive subgroup of G. Suppose Γ is a torsion-free discontinuous gp for G/H.

(1) $\mathrm{cd}_{\mathbb{R}}(\Gamma) \leq d(G) - d(H)$.

(2) $\mathrm{cd}_{\mathbb{R}}(\Gamma) = d(G) - d(H) \iff \Gamma\backslash G/H$ is compact.

$\wr$ discrete $\Gamma \rightsquigarrow$ continuous L

Theorem C′* Let H be a reductive subgroup of G. Suppose L is a reductive subgroup of G acting properly on G/H. Then one has

(1) $d(L) \leq d(G) - d(H)$.

(2) $d(L) = d(G) - d(H) \iff L\backslash G/H$ is compact.

* Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989).

Cohomological obstruction to cocompact discont gp

Let $H \sim H' \overset{\text{def}}{\iff} H \subset SH'S$ and $H' \subset SHS$ ($^\exists$ compact set $S \subset G$)

Theorem D* (non-existence) If there exists $H' \subset G$ such that

$$H \sim H' \text{ and } d(H) < d(H'),$$

then G/H does not admit a cocompact discontinuous group Γ.

Proof If such Γ existed, Γ would be a discontinuous group also for G/H' because $H \sim H'$. This would be a contradiction since

$$\underbrace{\mathrm{cd}_{\mathbb{R}}(\Gamma) = d(G) - d(H)}_{\text{Thm C}'} \underset{\text{Assumption}}{>} d(G) - d(H') \underset{\text{Thm C}}{\geq} \mathrm{cd}_{\mathbb{R}}(\Gamma).$$

□

Example (1990)** $(G, H) = (SL(3,\mathbb{C}), SL(2,\mathbb{C}))$

Take $H' = SU(2,1)$. Then $H \sim H'$ and $d(H') = 4 > d(H) = 3$.

$\Rightarrow$ $SL(3,\mathbb{C})/SL(2,\mathbb{C})$ does not admit a cocompact discontinuous gp.

* T. Kobayashi, Duke Math. J., 1992; ** T. Kobayashi, Proc. ICM-1990 satellite (Kawaguchi Lake).

Conjecture for non-existence of compact $\Gamma\backslash SL(n)/H$

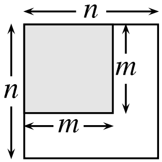
Conjecture (K–)* Let $G = SL(n,\mathbb{F})$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$, and H $(\subsetneq G)$ be any non-compact reductive subgroup. Then G/H does not admit a cocompact discontinuous group.

This is unsolved even when H is simple such as $SL(m,\mathbb{F})$.

Example $H = \text{Image}$ of a group hom $\tau\colon SL(m,\mathbb{F}) \to SL(n,\mathbb{F})$.

Among various choices of τ, special cases include:

1) τ: standard representation (not irreducible).

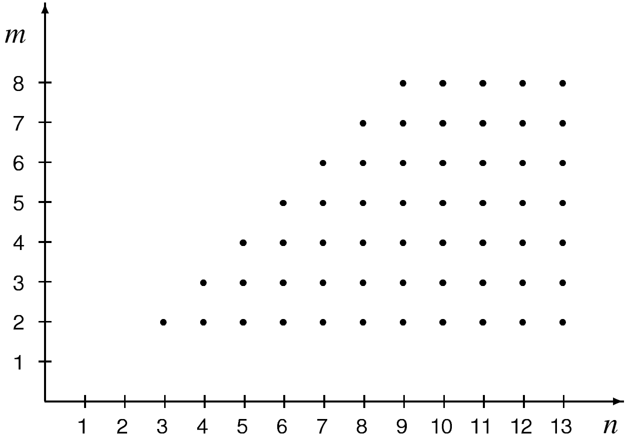


2) τ: n-dimensional irreducible representation.

* Special case of Conjecture 4.3 in T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces. Mathematics Unlimited — 2001 and Beyond, pages 723–747. Springer-Verlag, 2001.

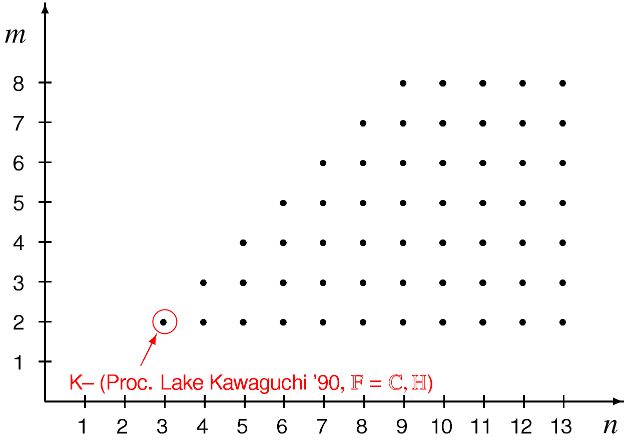
Example 1. Nonexistence of Compact quotients for $SL(n)/SL(m)$

Conjecture has been proved for the standard rep τ:



Example 1. Nonexistence of Compact quotients for $SL(n)/SL(m)$

Conjecture has been proved for the standard rep τ:



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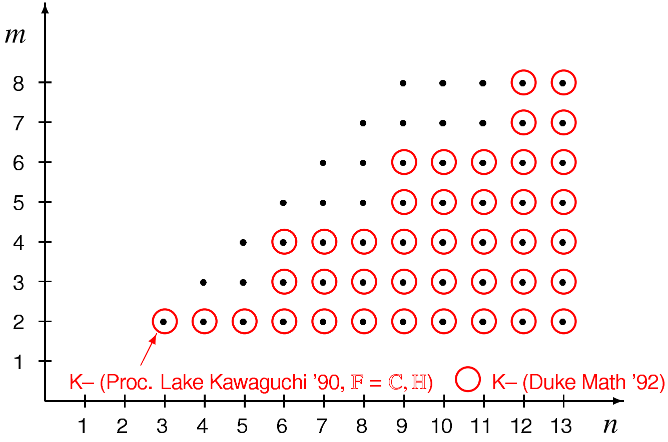

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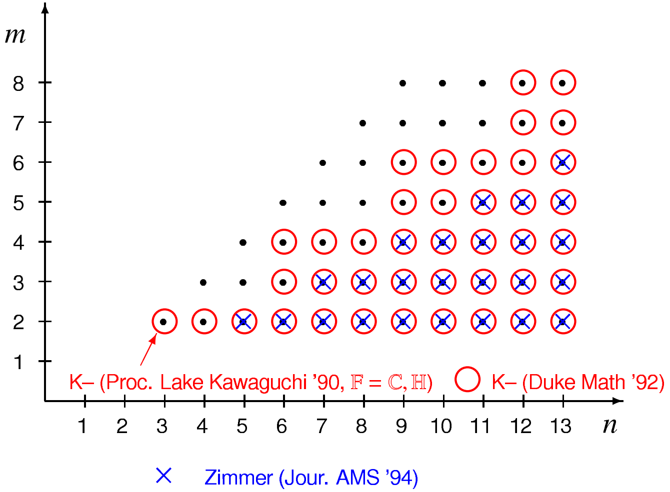

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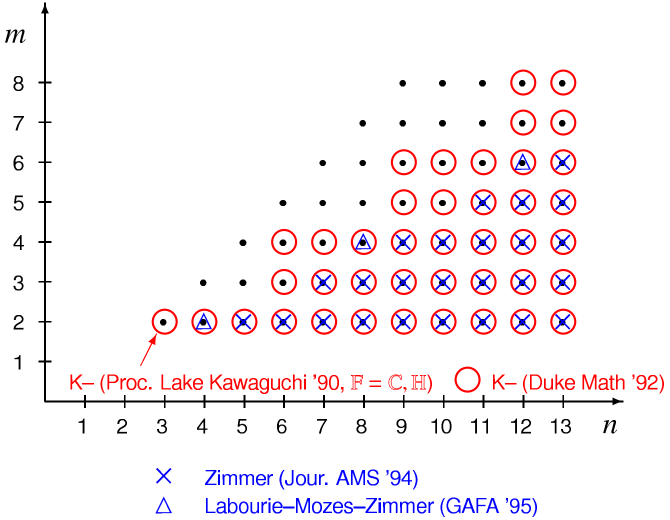

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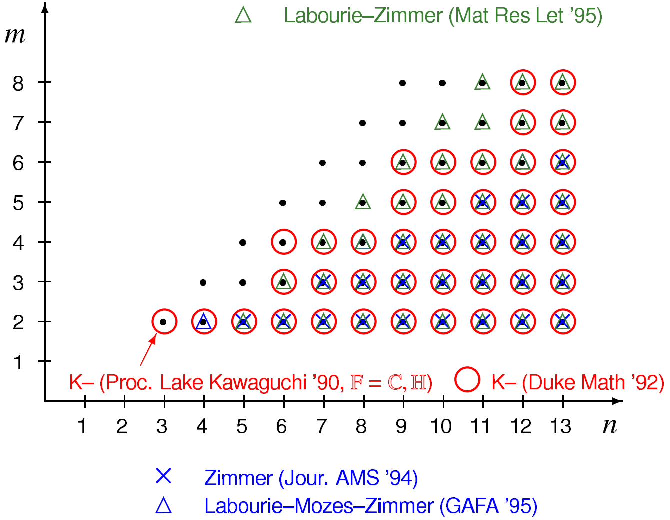

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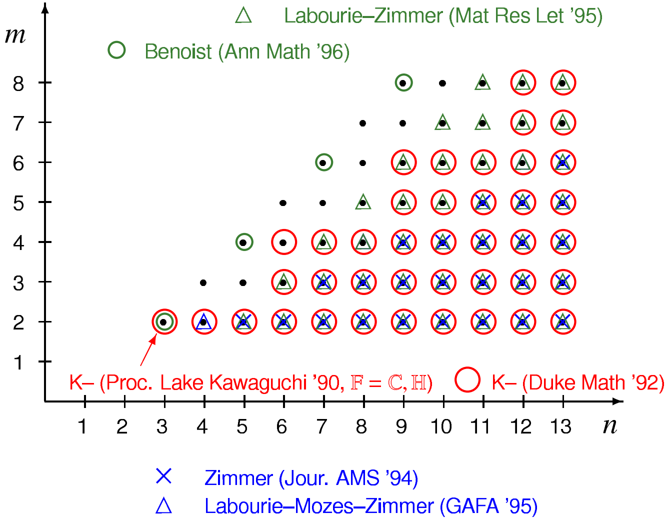

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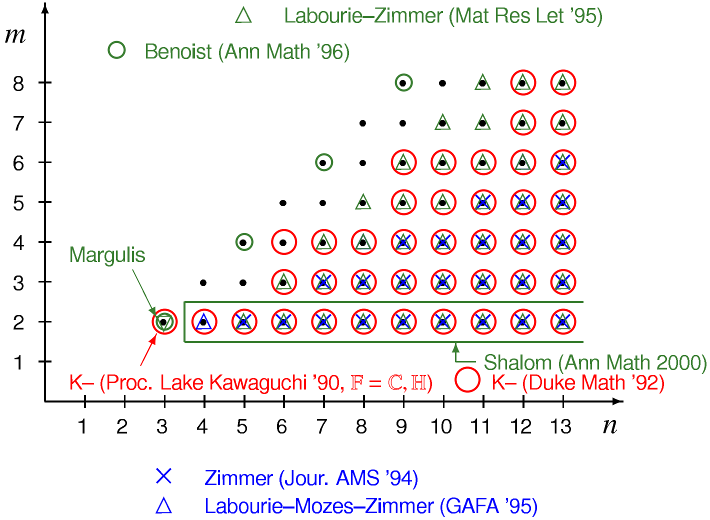

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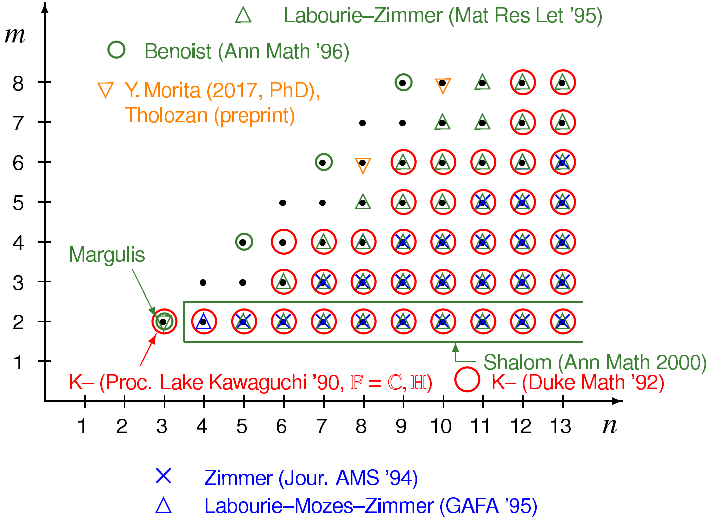

Non-existence of compact quotient $\Gamma \backslash SL(n, \mathbb{F})/H$

Conjecture (K–)* Let $G = SL(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$), and H ($\subsetneq G$) be any non-compact reductive subgroup. Then G/H does not admit a cocompact discontinuous group.

Conjecture is true in the setting where $H = SL(m, \mathbb{F})$ and $\tau \colon H \hookrightarrow G$ is the standard embedding.

K–	criterion of proper actions + cohomological dimension	$n > 3\lceil \frac{m}{2} \rceil$ (Theorem C)
Zimmer	orbit closure thm (Ratner)	$n > 2m$ (alternative proof)
Labourier–Mozes–Zimmer, Labourier–Zimmer		
	ergodic action	$n \geq 2m \rightsquigarrow n \geq m + 3$
Benoist	criterion of proper actions	$n = m + 1,\ m$ even
Shalom	unitary representation	$n \geq 4,\ m = 2$ (alternative)
Kassel–Morita–Tholozan		
Kassel–Tholozan		$n > m$

Example 2: Non-existence of compact quotients for $SL(n)/H$

Conjecture (K–)* Let $G = SL(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$), and H ($\subsetneq G$) be any non-compact reductive subgroup. Then G/H does not admit a cocompact discontinuous group.

In another special setting where H acts irreducibly on $\mathbb{R}^n$, Conjecture is open, but has found some evidence (Theorem D, below).

The first example in this setting is due to Margulis (1997)* for $H = SL(2, \mathbb{R})$. He used a concept "measurably proper action", a weaker notion of "proper action".

* G. A. Margulis, Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients, Bull. Soc. Math. France **125** (1997), 447–456.

Compact-like geometric/linear actions

Geometric actions on topological space M

$L \curvearrowright M$ proper actions
L behaves nicely in $\mathrm{Homeo}(M)$
as if it were a compact group.

Linear actions on Hilbert space $\mathcal{H}$

$L \curvearrowright \mathcal{H}$?
L behaves nicely in $U(\mathcal{H})$ (unitary operators)
as if it were a compact group.

How to capture a candidate for the property "?"

- decay conditions of matrix coefficients (Margulis '97) $\cdots$ today
- discrete decomposability (TK- '94, '98, 2017) $\cdots$ 3rd lecture

* Margulis, Bull. Soc. Math. France '97; TK Invent. '94, Ann. Math., '98; Invent. '98.

Finite-dim rep $\tau\colon \mathfrak{h} \to \mathrm{End}(V) \rightsquigarrow p_V > 0$

An approach to quantify "properness" of the linear action $H \curvearrowright V$ is given by a number $p_V > 0$ as below.

Let $\mathfrak{h}$ be a Lie algebra, and $\mathfrak{a}$ its max split abelian subspace. For a finite-dimensional rep $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduce*

$$p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\sum |\text{eigenvalues of } \mathrm{ad}(Y) \text{ in } \mathrm{End}(\mathfrak{h})|}{\sum |\text{eigenvalues of } \tau(Y) \text{ in } \mathrm{End}(V)|}$$

Some table of p_V can be found in Benoist–TK (2022)*.

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Obstruction from unitary reps and p_V

Theorem D* Let $G := SL(n, \mathbb{R})$ and H a reductive subgroup. Then G/H does not admit a cocompact discontinuous group if $p_V < 1$.

Here p_V is defined for the natural rep of H on $V := \mathbb{R}^n$ via $H \hookrightarrow G$. This gives an evidence for Conjecture in “generic cases.

Sketch of Proof.

$p_V < 1 \overset{*,**}{\iff} H$ is a “G-tempered subgroup”.

$\Longrightarrow$ Matrix coefficients of the unitary reps of G on $L^2(\Gamma\backslash G)$ decay “fast” along the subgroup H

$\underset{\text{Margulis}}{\Longrightarrow}$ $\Gamma\backslash G/H$ cannot be compact.

* (to appear),

** Margulis, Bull. Math. France (1997). H. Oh (1998), Y. Benoist–TK (2015).

Global Analysis of Locally Symmetric Spaces with Indefinite Metric

Plan

Lecture 1

Local to Global in Non-Riemannian Geometry (1/1)

— Introduction to pseudo-Riemannian space forms

— Construction/Obstruction of compact quotients $\Gamma\backslash X$

— Digression: "Tangential homogeneous space X_θ"

Lecture 2

Properness criterion and its quantification (1/2)

Lecture 3

Global Analysis on Locally Symmetric Spaces
Beyond the Riemannian Case (1/3)

Digression: 150, 100, 60 years ago $\cdots$

100 years ago	Radon–Hurwitz number (1922-1923)
– – – ––	
150 years ago	Klein's Erlangen program (1872)
60 years ago	Calabi–Markus (Ann Math 1962)
	Any de Sitter manifold is non-compact.
	Adams (Ann Math 1962)
	Vector fields on spheres.

Multiplication of two quadratic forms

One has the following identities:

$$(a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (ay + bx)^2$$

$$(a^2 + b^2)(x^2 + y^2 + z^2 + w^2) = (ay + bz)^2 + (ax + bw)^2 + (-aw + bx)^2 + (az - by)^2$$

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2 + w^2) = (ay + bz + cw)^2 + (ax + bw - cz)^2 + (-aw + bx + cy)^2 + (az - by + cx)^2$$

However, no such identity exists for

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (\text{—})^2 + (\text{—})^2 + (\text{—})^2$$

bilinear forms on $\{(a, b, c, \cdots)\} \times \{(x, y, z, w, \cdots)\}$

Radon–Hurwitz number (1922-1923)

$$(a^2+b^2)(x^2+y^2) = (ax-by)^2+(ay+bx)^2$$

$$(a^2+b^2)(x^2+y^2+z^2+w^2) = (ay+bz)^2+(ax+bw)^2+(-aw+bx)^2+(az-by)^2$$

$$(a^2+b^2+c^2)(x^2+y^2+z^2+w^2) = (ay+bz+cw)^2+(ax+bw-cz)^2+(-aw+bx+cy)^2+(az-by+cx)^2$$

bilinear forms on $\{(a,b,c,\cdots)\} \times \{(x,y,z,w,\cdots)\}$

Question For which pairs (p,q), does there exist a bilinear map $f\colon \mathbb{R}^{p+1}\times\mathbb{R}^q \to \mathbb{R}^q$ such that

$$||v||\cdot||w|| = ||f(v,w)|| \quad {}^\forall v\in\mathbb{R}^{p+1}, {}^\forall w\in\mathbb{R}^q.$$

Example Yes for $(p,q) = (1,2),(1,4),(2,4),(3,4)$.
No for $(p,q) = (2,3)$.

Example $p+1 = q \in \{1,2,4,8\}$ corresponding to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Observation (p,q) OK $\Rightarrow (p',q)$ OK $\quad {}^\forall p' \le p$

Radon–Hurwitz number (1922-1923) and Adams (1962)

Definition We set

$$\varphi(p) = \left[\frac{p}{2}\right] + \begin{cases} 0 & (p \equiv 0,6,7 \quad \mod 8), \\ 1 & (p \equiv 1,2,3,4,5 \quad \mod 8). \end{cases}$$

Theorem E (Radon* (1922), Hurwitz* (1923), Adams*** (1962))
The following three conditions on (p, q) are equivalent:
(i) $^{\exists}$bilinear map $f: \mathbb{R}^{p+1} \times \mathbb{R}^q \to \mathbb{R}^q$ such that

$$||v|| \cdot ||w|| = ||f(v,w)|| \qquad {}^{\forall}v \in \mathbb{R}^{p+1}, {}^{\forall}w \in \mathbb{R}^q.$$

(ii) $^{\exists}$bilinear map $f: \mathbb{R}^{p+1} \times \mathbb{R}^q \to \mathbb{R}^q$ such that

$$f(v,w) = 0 \text{ only if } v = o \text{ or } w = 0.$$

(iii) q is a multiple of $2^{\varphi(p)}$.

p	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\varphi(p)$	0	1	2	2	3	3	3	3	4	5	6	$\cdots$
q	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	$4\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$16\mathbb{N}$	$32\mathbb{N}$	$64\mathbb{N}$	$\cdots$

(i) ⇔ (iii) by Radon and Hurwitz.
(ii) ⇒ (iii) is a reformulation of Adams (1962).

* J. Radon, Abh. math. Sem. Hamburg **1** (1922); ** A. Hurwitz, Math. Ann. (1923), J. F. Adams, Ann. Math. (1962).

Mackey analogy — from geometric viewpoints

$$G = K \exp \mathfrak{p} \rightsquigarrow G_\theta = K \ltimes \mathfrak{p} \quad \text{(Cartan motion group)}$$

Example Riemannian manifold with constant sectional curvature κ

$\kappa < 0$	$\kappa = 0$	$\kappa > 0$
hyperbolic space H^n	$\mathbb{R}^n$	sphere S^n
$\parallel$	$\parallel$	$\parallel$
G/K	G_θ/K	G_U/K

κ

$$G = \mathrm{Isom}(H^n)_0 = K \exp \mathfrak{p} \simeq SO_0(n,1)$$

change

$$G_\theta = \mathrm{Isom}(\mathbb{R}^n)_0 = K \ltimes \mathfrak{p} \simeq SO(n) \ltimes \mathbb{R}^n$$

change

$$G_U = \mathrm{Isom}(S^n)_0 = K \exp(\sqrt{-1}\mathfrak{p}) \simeq SO(n+1)$$

Tangential homogeneous spaces G_θ/H_θ

$G \supset H$ real reductive Lie groups

$$G/H \rightsquigarrow G_\theta/H_\theta$$

easier

G : real reductive linear Lie group

θ : Cartan involution, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$

$K = \{g \in G : \theta g = g\}$ maximal compact subgroup of G

$G = K \exp \mathfrak{p}$

<u>Definition</u> (Cartan motion group G_θ)

$$G_\theta := K \ltimes \mathfrak{p}$$

$$(k_1, X_1) \cdot (k_2, X_2) := (k_1 k_2, X_1 + \mathrm{Ad}(k_1) X_2)$$

$\mathfrak{g}_\theta = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{p}$ <u>abelian</u> subalgebra.

Existence of cocompact discontinuous group for G_θ/H_θ

$$G/H = O(p,q+1)/O(p,q) \rightsquigarrow G_\theta/H_\theta$$

tangentail space

space form

<u>Theorem F (2005)</u>*

There exists a cocompact discontinuous group for G_θ/H_θ
$\Longleftrightarrow$ q is a multiple of $2^{\varphi(p)}$.

Here, $\varphi(p) = \left[\frac{p}{2}\right] + \begin{cases} 0 & (p \equiv 0,6,7 \mod 8) \\ 1 & (p \equiv 1,2,3,4,5 \mod 8) \end{cases}$

p	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\varphi(p)$	0	1	2	2	3	3	3	3	4	5	6	$\cdots$
q	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	$4\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$8\mathbb{N}$	$16\mathbb{N}$	$32\mathbb{N}$	$64\mathbb{N}$	$\cdots$

* T. Kobayashi, T. Yoshino, Pure and Appl. Math. Quarterly **1**, (2005), 603–684. Special Issue: In Memory of A. Borel.

Global properties of Space forms $\kappa \equiv -1$, signature (q, p)

Theorem A′* (reminder) Let $G/H = O(p+1, q)/O(p, q)$.

(1) (Calabi–Markus phenomenon, Thm. A) G/H admits an infinite discontinuous group if and only if $p < q$.

(2) If G/H admits a cocompact discontinuous group, then either $pq = 0$ or "$p < q$ and q is even".

(3) G/H admits a cocompact discontinuous group for (p, q) below.

p	$\mathbb{N}$	0	1	3	7
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	8

Conjecture ([K01, Conj. 2.6]**) The converse of (3) holds.

* Calabi–Markus (1962), Wolf (1962), Kulkarni (1981), Kobayashi (1996), Tholozan (2015), Morita (2017).

** T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces, Mathematics Unlimited — 2001 and Beyond, pages 723–747, Springer-Verlag, 2001.

Global Analysis of Locally Symmetric Spaces with Indefinite Metric

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Open problem: Existence problem of compact $\Gamma\backslash G/H$

Conjecture* A reductive homogeneous space G/H admits a cocompact discontinuous group iff G/H admits a compact standard quotient.

Example An irreducible complex symmetric space $X_{\mathbb{C}}$ admits a cocompact discontinuous group iff $X_{\mathbb{C}} \simeq S^3_{\mathbb{C}}$ or $S^7_{\mathbb{C}}$.

The general theory for compact standard quotients (*e.g.*, criterion and basic properties) may be found in [K- '89]**.

Remark Even if G/H admits cocompact discontinuous groups, there may exist "non-standard ones", too. Deformation theory initiated by Goldman et al. will be discussed in Lecture 2.

* T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces, Mathematics Unlimited — 2001 and Beyond, 723–747. Springer-Verlag, 2001.

** T. Kobayashi, Proper action on homogeneous space of reductive type, Math. Ann., 1989.

Some evidence of Conjecture for 1990–2024

We have seen that some recent results support the conjecture:

- Results for tangential spaces G_θ/H_θ (*e.g.,* Theorem F, TK–Yoshino, 2005).

- $SL(n,\mathbb{F})/SL(m,\mathbb{F})$ (standard embedding) does not admit compact quotients. This is proved by TK- '90, TK-'92, Zimmer '94, Labourie–Mozes–Zimmer '95, Benoist '96, Shalom '00, Morita '07, and Kassel–Tholozan (in preparation) by various methods.

- $SL(n,\mathbb{F})/\varphi(SL(m,\mathbb{F}))$ via irreducible rep φ. Margulis '97, Oh '98, TK- (Theorem D, in preparation).

- Conjecture applied to $O(p,q+1)/O(p,q)$ implies the converse of Theorem A′ (3) is also true (the space form conjecture). Recent progress: q must be a multiple of $2^{\varphi(p)}$ exactly as in Theorem F for G_θ/H_θ (announced by Kassel, Morita–Tholozan).

- $O(p_1+p_2,q_1+q_2)/O(p_1,q_1)\times O(p_2,q_2)$ with $p_1,p_2,q_1,q_2>0$ (TK, Duke 92)

Expository Papers and Open Problems for Lecture 1

Expository papers

- T. Kobayashi,
Discontinuous groups for non-Riemannian homogeneous spaces,
Mathematics Unlimited — 2001 and Beyond, pp. 723–747.
Springer-Verlag, 2001.

- T. Kobayashi.
On discontinuous group actions on non-Riemannian homo spaces,
Sugaku Expositions, 22, pp.1–19, 2009. Amer. Math. Soc.,
Translation by M. Reid of the original article (Math. Soc. Japan).

Open Problems

- T. Kobayashi, Chapter 4 in
Conjectures on Reductive Homogeneous Spaces, Mathematics
Going Forward, Lecture Notes in Mathematics **2313**, pp. 213–231
(2023), Springer-Nature.

Some references for Lecture 1

For the **first half** of Lecture 1,

• T. Kobayashi, Math Ann (1989), Proc. Lake Kawaguchi (1990), Duke Math (1992), J. Geom Physics (1993), European School Lecture 1994 (Academic Press 1996).

For a **measure theoretic approach**,

• Margulis, (Bull. Math. France, 1997). H. Oh (Duke, 1998), Y. Benoist–TK (J. Euro Math, 2015).

For compact quotients of **tangential spaces** G_θ/H_θ.

• T. Kobayashi, T. Yoshino, Pure and Appl. Math. Quarterly **1**, (2005), 603–684. Special Issue: In Memory of A. Borel.

For **Radon–Hurwitz numbers**,

• J. Radon, Abh. math. Sem. Hamburg **1** (1922); A. Hurwitz, Math. Ann. (1923); J. F. Adams, Ann. Math. (1962).

Thank you very much for your attention!