Quantitative unique continuation for real-valued solutions to second order elliptic equations in the plane

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References

The talk is based on:

- S. Ervedoza, K. Le Balc'h, **Cost of observability inequalities for elliptic equations in 2-d with potentials and applications to control theory**, 2023, *Comm. Partial Differential Equations*
- K. Le Balc'h, D. Souza, **Quantitative unique continuation for real-valued solutions to second order elliptic equations in the plane**, 2023, *arXiv:2401.00441*

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The Landis conjecture on exponential decay

Conjecture (Landis, 1960's)

$$V \in L^\infty(\mathbb{R}^N), \begin{cases} -\Delta u + V(x)u = 0 \textit{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}), \ \varepsilon > 0, \end{cases} \implies u \equiv 0.$$

Если $u(x)$ — решение уравнения $-\Delta u + k^2 u = 0$, определенное в $\mathbf{R}^n$ вне некоторого компакта K, то, как известно, если $u(x)$ убывает при $|x| \to \infty$ со скоростью $e^{-(k+\varepsilon)|x|}$, $\varepsilon > 0$, то $u \equiv 0$.

Kondratev, Landis. *Qualitative theory of second order linear partial differential equations*, 1988

Example: $u(x) = \exp(-\sqrt{|x|^2+1})$ satisfies $-\Delta u + V(x)u = 0$ in $\mathbb{R}^N$.

Proof in 1D (M. Pierre): $-u'' + V(x)u = 0$ in $\mathbb{R}$, $|u(x)| \leq \exp(-|x|^{1+\varepsilon})$.
By integrating, we easily get $|u'(x)| \leq C\exp(-|x|^{1+\varepsilon})$.
Duality argument: Let ϕ s.t. $-\phi'' + V\phi = \mathrm{sign}(u)$, $\phi(0) = \phi'(0) = 0$.
Gronwall's argument: $|\phi(x)| + |\phi'(x)| \leq C\exp(C|x|)$.
$\int_{-R}^{R} |u| = \int_{-R}^{R} u \cdot \mathrm{sign}(u) = \int_{-R}^{R} u(-\phi'' + V\phi) = [-\phi' u + \phi u']_{-R}^{R} \leq e^R e^{-R^{1+\varepsilon}} \to 0.$

The Landis conjecture for complex-valued functions

Theorem (Meshkov's counterxample (1991))

$$\exists V \in L^\infty(\mathbb{R}^2;\mathbb{C}) \text{ and } u \neq 0 \in C^\infty(\mathbb{R}^2;\mathbb{C}), \quad \begin{cases} -\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^2, \\ |u(x)| \leq \exp(-|x|^{4/3}). \end{cases}$$

Extension to $\mathbb{R}^d$ for d even: $v(x_1, x_2, x_3, x_4) = u(x_1, x_2)u(x_3, x_4)$.

Theorem (Meshkov's optimality (1991))

$$V \in L^\infty(\mathbb{R}^N;\mathbb{C}), \quad \begin{cases} -\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{4/3+\varepsilon}), \ \varepsilon > 0, \end{cases} \quad \Longrightarrow u \equiv 0.$$

Theorem (Bourgain, Kenig's quantitative form (2005))

$-\Delta u + Vu = 0$ *in* $\mathbb{R}^N$, *with* $\|V\|_\infty \leq 1$ *and* $\|u\|_\infty = |u(0)| = 1$, *then*

$$\sup_{B(x_0,1)} |u(x)| \geq \exp(-CR^{4/3}\log(R)) \qquad \forall R \geq 1, \ \forall x_0 \in \mathbb{R}^N \text{ with } |x_0| = R.$$

$\Rightarrow$ Meshkov's optimality: $\exp(-R^{4/3+\varepsilon}) \geq \sup_{B(x_0,1)} |u(x)| \geq \exp(-R^{4/3}\log(R))$.

The Landis conjecture for real-valued functions

Bourgain, Kenig' **qualitative** Landis conjecture (2005)

$$V \in L^{\infty}(\mathbb{R}^N; \mathbb{R}), \begin{cases} -\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}), \ \varepsilon > 0, \end{cases} \implies u \equiv 0.$$

Bourgain, Kenig' **quantitative** Landis conjecture (2005)

$-\Delta u + Vu = 0$ in $\mathbb{R}^N$, with $\|V\|_{L^{\infty}(\mathbb{R}^N;\mathbb{R})} \leq 1$ and $\|u\|_{\infty} = |u(0)| = 1$, then

$$\sup_{B(x_0,1)} |u(x)| \geq \exp(-CR^{1+\varepsilon}) \qquad \forall R \geq 1, \ \forall x_0 \in \mathbb{R}^N \text{ with } |x_0| = R. \qquad \text{(QUC)}$$

Quantitative unique continuation results in dimension $N = 2$

1. Kenig, Silvestre, Wang (2015): (QUC) holds for $-\Delta u - \nabla \cdot (Wu) + Vu = 0$ or $-\Delta u + W \cdot \nabla u + Vu = 0$, with $\|W\|_{\infty} \leq 1$ and $0 \leq V \leq 1$.
2. Logunov, Malinnikova, Nadirashvili, Nazarov (2020): (QUC) holds for $-\Delta u + Vu = 0$ with $|V| \leq 1$.

See also Kenig, Davey, Wang, Zhu for (QUC) in dimension $N = 2$.
See also Rossi, Sirakov, Souplet for qualitative results when $N \geq 1$.

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Second order elliptic equations in the plane

Theorem (Le Balc'h, Souza (2023))

For $W_1, W_2 \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$ *and* $u \in H^1_{loc}(\mathbb{R}^2)$ *s.t.*

$$\begin{cases} -\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0 \text{ in } \mathbb{R}^2, \\ |u(x)| \leq \exp(-|x|^{1+\delta}),\ \delta > 0. \end{cases}$$

Then $u \equiv 0$.

Theorem (Le Balc'h, Souza (2023))

For $\|W_1\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} \leq 1$, $\|W_2\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} \leq 1$, $\|V\|_{L^\infty(\mathbb{R}^2;\mathbb{R})} \leq 1$ *and* $u \in H^1_{loc}(\mathbb{R}^2)$ *s.t.* $\|u\|_\infty = |u(0)| = 1$ *and*

$$-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0 \text{ in } \mathbb{R}^2,$$

then

$$\sup_{B(x_0,1)} |u(x)| \geq \exp(-CR^{1+\delta}) \qquad \forall R \geq 1,\ \forall x_0 \in \mathbb{R}^2 \text{ with } |x_0| = R. \qquad (QUC)$$

Extension of [LMNN20] that have treated $-\Delta u + Vu = 0$ in $\mathbb{R}^2$.

Order of vanishing estimate/observability in the plane

Theorem (Le Balc'h, Souza (2023))

For $W_1, W_2 \in L^\infty(B_2; \mathbb{R}^2)$, $V \in L^\infty(B_2; \mathbb{R})$ *and* $u \in H^1_{loc}(B_2)$ *s.t.*

$$-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0 \text{ in } B_2.$$

Assume that for $K \geq 2$,

$$\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}.$$

Then, for every $\delta > 0$, *there exists a positive constant* $C \geq 1$ *s.t.* $\forall r \in (0, 1/2)$,

$$\|u\|_{L^\infty(B_r)} \geq r^{C\left(\|W_1\|_\infty^{1+\delta} + \|W_2\|_\infty^{1+\delta} + \|V\|_\infty^{1/2+\delta}\right)CK} \|u\|_{L^\infty(B_2)} \qquad \text{(Obs)}.$$

Remark 1: Theorem $\Rightarrow$ qualitative and quantitative Landis conjecture by scaling.
Remark 2: For $r = 1/4$, $K = 2$, (Obs) becomes

$$\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|W_1\|_\infty^{1+\delta} + \|W_2\|_\infty^{1+\delta} + \|V\|_\infty^{1/2+\delta}\right)\right) \|u\|_{L^\infty(B_{1/4})}.$$

$$\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|W_1\|_\infty^{2} + \|W_2\|_\infty^{2} + \|V\|_\infty^{2/3}\right)\right) \|u\|_{L^\infty(B_{1/4})} \quad \text{(Carleman)}$$

Remark 3: One can improve $\|V\|_\infty^{1/2+\delta}$ to $\|V\|_\infty^{1/2} \log_+^{3/2}(\|V\|_\infty)$.

Optimal observability inequality in 2-d

Theorem (Ervedoza, Le Balc'h (2023))

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and $\omega \subset \Omega$. For $V \in L^\infty(\Omega; \mathbb{R})$ and $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|u\|_{H^2(\Omega)} \leq C \left(\|-\Delta u + Vu\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)} \right), \qquad \text{(Observability)}$$

where $C > 0$ is given by $\boxed{C = \exp\left(C(\Omega,\omega)\left(1 + \|V\|_\infty^{1/2} \log^{1/2}(\|V\|_\infty)\right)\right).}$

- V has to be real-valued (Meshkov's counterexample).
- $V \in L^\infty(\Omega; \mathbb{R}) \Rightarrow$ one can assume that <u>u is real-valued</u>.
- (Observability) proved by Logunov and al for Ω a 2-d manifold without boundary and $-\Delta u + Vu = 0$.

Applications to control of semi-linear elliptic equations

Take $f \in C^1(\mathbb{R};\mathbb{R})$ such that $f(0) = 0$ and consider the elliptic control problem

$$\begin{cases} -\Delta y + f(y) = F + h1_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{LaplaceNL}$$

where $F \in L^\infty(\Omega)$.
Goal: Find a pair $(y,h) \in [H_0^1(\Omega) \cap L^\infty(\Omega)] \times L^\infty(\omega)$ satisfying (LaplaceNL).

Theorem (Ervedoza, Le Balc'h (2023))

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and $\omega \subset \Omega$.

- *(Positive result) Assume that $f(s) = o_{+\infty}(|s| \log^p(1+|s|))$, $p < 2$, then*

$$\forall F \in L^\infty(\Omega),\ \exists (y,h) \in [H_0^1(\Omega) \cap L^\infty(\Omega)] \times L^\infty(\omega) \textit{ satisfying } \text{(LaplaceNL)}.$$

- *(Negative result) Take $f(s) = |s| \log^p(1+|s|)$, $p > 2$. Then,*

$$\exists F \in L^\infty(\Omega),\ \forall h \in L^\infty(\omega),\ \text{(LaplaceNL)} \textit{ has no solution } y \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

- Negative result is based on the localized eigenfunction method (OK in N-d).
- Positive result is true in 1-d, with $p = 2$.
- Positive result is true in N-d, with $p = 3/2$.

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General scheme of the proof

Hypotheses:

1. For $V \in L^\infty(B_2;\mathbb{R})$, $-\Delta u + Vu = 0$ in B_2.
2. For $K \geq 2$, $\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}$ ("boundary condition").

Goal: $\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|V\|_\infty^{1/2+\delta} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)}$, $\forall r \in (0,1/2)$.

Remark: For $V \in L^\infty(B_2;\mathbb{C})$, a standard Carleman estimate directly gives

$$\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|V\|_\infty^{2/3} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)}, \ \forall r \in (0,1/2). \ ☹$$

3 steps in the proof:

1. Construction of a multiplier $\varphi > 0$ such that $-\Delta\varphi + V\varphi = 0$.
2. Reduction to $-\nabla\cdot(\varphi^2\nabla v) = 0$, $v = u/\varphi$ then simplification to $\Delta\hat{v} = 0$.
3. Unique continuation applied to $\Delta\hat{v} = 0$.

Kenig, Silvestre, Wang's proof (2015)

Hypotheses:

1. For $V \in L^\infty(B_2; \mathbb{R}^+)$, $-\Delta u + Vu = 0$ in B_2.
2. For $K \geq 2$, $\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}$ ("boundary condition").

Goal: $\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|V\|_\infty^{1/2} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)}$, $\forall r \in (0, 1/2)$.

3 steps in the proof:

1. Construction of a multiplier $\varphi > 0$ such that $-\Delta\varphi + V\varphi = 0$ in B_2.
   This is an easy consequence of the maximum principle because $V \geq 0$.
2. Reduction to $-\nabla \cdot (\varphi^2 \nabla v) = 0$, $v = u/\varphi$ then *simplification*:
   2.1 Poincare lemma: $\varphi^2 \nabla v = \mathrm{rot}(\tilde{v})$.
   2.2 $\gamma = \varphi^2 v + i\tilde{v}$ satisfies $\partial_{\overline{z}}\gamma = \alpha\gamma$, $\|\alpha\|_\infty \leq \|\nabla \log(\varphi)\|_\infty \leq \|V\|_\infty^{1/2}$.
   2.3 Cauchy transform: $\partial_{\overline{z}}\beta = \alpha$, , $\|\beta\|_\infty \leq \|\alpha\|_\infty \leq \|\nabla \log(\varphi)\|_\infty \leq \|V\|_\infty^{1/2}$.
   2.4 $\zeta = \exp(-\beta)\gamma$ satisfies $\partial_{\overline{z}}\zeta = 0$.
3. Unique continuation applied to $\partial_{\overline{z}}\zeta = 0$ by Hadamard three-circle theorem.

Logunov, Malinnikova, Nadirashvili, Nazarov's proof (2020)

Hypotheses:

1. For $V \in L^\infty(B_2;\mathbb{R})$, $-\Delta u + Vu = 0$ in B_2.
2. For $K \geq 2$, $\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}$ ("boundary condition").

Goal: $\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|V\|_\infty^{1/2+\delta} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)}, \forall r \in (0,1/2)$.

3 steps in the proof:

1. Construction of a multiplier $\varphi > 0$ such that $-\Delta\varphi + V\varphi = 0$ in Ω_ε.
   - $Z = \{x \in B_2\ ;\ u(x) = 0\}$ the *nodal set* of u,
   - F_ε a union of disks of size ε, that are ε-separated from each other and from Z,
   - $\Omega_\varepsilon = B_2 \setminus (Z \cup F_\varepsilon)$ has Poincaré constant $C_P(\Omega_\varepsilon)^2 \leq C\varepsilon^2$ s.t. $\varepsilon^2\|V\|_\infty \leq c$.
2. Reduction to $-\nabla\cdot(\varphi^2\nabla v) = 0$ in Ω_ε', $v = u/\varphi$ then *simplification*:

   2.1 (Local) Poincaré's lemma: $\varphi^2\nabla v = \mathrm{rot}(\tilde{v})$.

   2.2 $w := \hat{v} + i\tilde{v}$ satisfies $\frac{\partial w}{\partial \bar{z}} = \mu\frac{\partial w}{\partial z}$ with $\mu = \frac{1-\varphi^2}{1+\varphi^2}\frac{\hat{v}_x + i\hat{v}_y}{\hat{v}_x - i\hat{v}_y}$.

   2.3 Estimate on φ: $|\mu| \leq \frac{1-\varphi^2}{1+\varphi^2} \leq C\varepsilon^2\|V\|_\infty$.

   2.4 Beltrami: $\exists\psi$ K-quasiconformal, $\psi(0) = 0$, $\frac{\partial\psi}{\partial\bar{z}} = \mu\frac{\partial\psi}{\partial z}$ in $\mathbb{C}$.

   2.5 Stoilow factorization theorem: $\exists W$ hol. s.t. $w = W\circ\psi$.

   2.6 $\hat{v} = v\circ\psi^{-1}$ satisfies $\Delta\hat{v} = 0$ in $\psi(\Omega_\varepsilon')$.
3. Unique continuation applied to $\Delta\hat{v} = 0$ in $\psi(\Omega_\varepsilon')$ by Carleman estimate.
   - Harnack's inequality to absorb the cut-off terms near $\psi(F_\varepsilon)$.

A brief outline of our proof

Hypotheses:

1. For $W_1, W_2 \in L^\infty(B_2; \mathbb{R}^2)$, $V \in L^\infty(B_2; \mathbb{R})$,
$$-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0 \text{ in } B_2.$$
2. For $K \geq 2$, $\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}$ ("boundary condition").

Goal:

$$\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|W_1\|_\infty^{1+\delta} + \|W_2\|_\infty^{1+\delta} + \|V\|_\infty^{1/2+\delta} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)},$$
$$\forall r \in (0, 1/2).$$

Remark: For $W_1, W_2 \in L^\infty(B_2; \mathbb{C}^2)$, $V \in L^\infty(B_2; \mathbb{C})$, Carleman gives

$$\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|W_1\|_\infty^{2} + \|W_2\|_\infty^{2} + \|V\|_\infty^{2/3} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)},$$
$$\forall r \in (0, 1/2). \text{ ☹}$$

3 steps in the proof:

1. Construction of $\varphi > 0$ such that $-\Delta\varphi + W_1 \cdot \nabla\varphi - \nabla \cdot (W_2\varphi) + V\varphi = 0$.
2. Reduction to $-\nabla \cdot (\varphi^2(\nabla v + Wv)) = 0$, $W = W_1 - W_2$ then simplification to $\partial_{\bar{z}}\zeta = E$.
3. Unique continuation applied to $\partial_{\bar{z}}\zeta = E$.

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Plan of Step 1

- Choice of ε: It is chosen as

$$\varepsilon \leq c + c\|W_1\|_\infty^{-1-\delta/2} + c\|W_2\|_\infty^{-1-\delta/2} + c\|V\|_\infty^{-1/2}. \qquad \text{(Condition)}$$

- Property of the nodal set: $Z := \{x \in \Omega\ ;\ u(x) = 0\}$ satisfies

$$\forall x_0 \in Z,\ \forall \rho \in (0, \varepsilon),\ \partial B(x_0, \rho) \cap (Z \cup \partial B(0,2)) \neq \emptyset, \qquad \text{(P-}\varepsilon\text{)}$$

- Perforation of the domain: $\Omega_\varepsilon = B_2 \setminus (Z \cup F_\varepsilon)$ where $F_\varepsilon = \cup_{j\in J} B(x_j, \varepsilon)$ and

$$C_P(\Omega_\varepsilon) \leq C\varepsilon. \qquad \text{(Small-Poincaré)}$$

- Construction of the multiplier: $\varphi \in H^1(\Omega_\varepsilon)$ satisfies

$$-\Delta\varphi - \nabla \cdot (W_2\varphi) + W_1 \cdot \nabla\varphi + V\varphi = 0 \text{ in } \Omega_\varepsilon, \qquad \text{(Solvability)}$$

and $\tilde{\varphi} := \varphi - 1 \in H_0^1(\Omega_\varepsilon)$ satisfies

$$\|\tilde{\varphi}\|_\infty \leq C\left(\varepsilon^{2/(2+\delta)}\|W_2\|_\infty + \varepsilon^2\|V\|_\infty\right). \qquad \text{(Estimate)}$$

Weak quantitative maximum principles

Lemma (Le Balc'h, Souza (2023))

Let $\varepsilon > 0$, Ω *be a bounded open set contained in* $\mathbb{R}^2$ *such that* $C_P(\Omega)^2 \leq (C')^2\varepsilon^2$, $W \in L^\infty(\Omega;\mathbb{R}^2)$, $f \in L^\infty(\Omega;\mathbb{R})$. *Assume that* $\varepsilon + \varepsilon \|W\|_\infty \leq c$. *Then,*

$$\exists!\Phi \in H_0^1(\Omega),\ -\Delta\Phi + W \cdot \nabla\Phi = f \text{ in } \Omega, \qquad \text{(Solvability)}$$

$$\|\Phi\|_\infty \leq C\varepsilon^2 \|f\|_\infty . \qquad \text{(Estimate)}$$

Lemma (Le Balc'h, Souza (2023))

Let $\varepsilon > 0$, Ω *be a bounded open set contained in* $\mathbb{R}^2$ *such that* $C_P(\Omega)^2 \leq (C')^2\varepsilon^2$, $|\Omega| \leq C'$, $W \in L^\infty(\Omega;\mathbb{R}^2)$, $g \in L^\infty(\Omega;\mathbb{R}^2)$, $p > 2$. *Assume that* $\varepsilon + \varepsilon \|W\|_\infty \leq c$. *Then,*

$$\exists!\Phi \in H_0^1(\Omega),\ -\Delta\Phi + W \cdot \nabla\Phi = \nabla \cdot g \text{ in } \Omega, \qquad \text{(Solvability)}$$

$$\|\Phi\|_\infty \leq C\varepsilon^{2/p} \|g\|_\infty . \qquad \text{(Estimate)}$$

Can we extend to $p = 2$?

Weak quantitative maximum principles and nodal set

Proposition (Le Balc'h, Souza (2023))

Let $\varepsilon > 0$*,* Ω *be a bounded open set contained in* $\mathbb{R}^2$ *such that* $C_P(\Omega)^2 \leq (C')^2\varepsilon^2$*,* $|\Omega| \leq C'$*,* $W_1, W_2 \in L^\infty(\Omega; \mathbb{R}^2)$*,* $V \in L^\infty(\Omega; \mathbb{R})$ *and* $p > 2$*. Assume that*

$$\varepsilon + \varepsilon^{2/p} \|W_1\|_{L^\infty(\Omega)} + \varepsilon \|W_2\|_{L^\infty(\Omega)} + \varepsilon^2 \|V\|_{L^\infty(\Omega)} \leq c. \qquad \text{(Condition 1)}$$

Then there exists a unique $\varphi \in H^1(\Omega)$ *such that*

$$-\Delta\varphi - \nabla \cdot (W_1\varphi) + W_2 \cdot \nabla\varphi + V\varphi = 0 \text{ in } \Omega, \qquad \text{(Solvability)}$$

and $\tilde{\varphi} = \varphi - 1$ *satisfies*

$$\tilde{\varphi} \in H_0^1(\Omega) \text{ and } \|\tilde{\varphi}\|_\infty \leq C\left(\varepsilon^{2/p} \|W_1\|_{L^\infty(\Omega)} + \varepsilon^2 \|V\|_{L^\infty(\Omega)}\right). \qquad \text{(Estimate)}$$

Consequence on the nodal set:
Let u be a real-valued solution to $-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0$ in a ball $B(x, \varepsilon)$ with $\varepsilon > 0$ satisfying (Condition 1) and $u \in H^1(B(x, \varepsilon)) \cap C^0(\overline{B(x, \varepsilon)})$. Then, if $u > 0$ on $\partial B(x, \varepsilon)$ then $u > 0$ in $B(x, \varepsilon)$. $\Rightarrow$ (P-ε) holds for Z.

Weak quantitative maximum principle and multiplier

Proposition (Le Balc'h, Souza (2023))

Let $\varepsilon > 0$, Ω *be a bounded open set contained in* $\mathbb{R}^2$ *such that* $C_P(\Omega)^2 \leq (C')^2\varepsilon^2$, $|\Omega| \leq C'$, $W_1, W_2 \in L^\infty(\Omega;\mathbb{R}^2)$, $V \in L^\infty(\Omega;\mathbb{R})$ *and* $p > 2$. *Assume that*

$$\varepsilon + \varepsilon \|W_1\|_{L^\infty(\Omega)} + \varepsilon^{2/p} \|W_2\|_{L^\infty(\Omega)} + \varepsilon^2 \|V\|_{L^\infty(\Omega)} \leq c. \qquad \text{(Condition 2)}$$

Then there exists a unique $\varphi \in H^1(\Omega)$ *such that*

$$-\Delta\varphi - \nabla\cdot(W_2\varphi) + W_1\cdot\nabla\varphi + V\varphi = 0 \text{ in } \Omega, \qquad \text{(Solvability)}$$

and $\tilde{\varphi} = \varphi - 1$ *satisfies*

$$\tilde{\varphi} \in H_0^1(\Omega) \text{ and } \|\tilde{\varphi}\|_\infty \leq C\left(\varepsilon^{2/p}\|W_2\|_{L^\infty(\Omega)} + \varepsilon^2\|V\|_{L^\infty(\Omega)}\right). \qquad \text{(Estimate)}$$

Consequence: Construction of the multiplier φ in $\Omega_\varepsilon = B_2 \setminus (Z \cup F_\varepsilon)$.
We take $p = 2 + \delta > 2$ and ε satisfying

$$\varepsilon + \varepsilon^{2/(2+\delta)}\|W_1\|_{L^\infty(B_2)} + \varepsilon^{2/(2+\delta)}\|W_2\|_{L^\infty(B_2)} + \varepsilon^2\|V\|_{L^\infty(B_2)} \leq c. \qquad \text{(Condition)}$$

Summary of Step 1

For $W_1, W_2 \in L^\infty(B_2; \mathbb{R}^2)$, $V \in L^\infty(B_2; \mathbb{R})$,

$$-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0 \text{ in } B_2.$$

- Choice of ε: It is chosen as

$$\varepsilon \leq c + c\|W_1\|_\infty^{-1-\delta/2} + c\|W_2\|_\infty^{-1-\delta/2} + c\|V\|_\infty^{-1/2}. \qquad \text{(Condition)}$$

- Construction of the multiplier: $\varphi \in H^1(\Omega_\varepsilon)$ satisfies

$$-\Delta \varphi - \nabla \cdot (W_2 \varphi) + W_1 \cdot \nabla \varphi + V\varphi = 0 \text{ in } \Omega_\varepsilon = B_2 \setminus (Z \cup F_\varepsilon), \qquad \text{(Solvability)}$$

and $\tilde{\varphi} := \varphi - 1 \in H_0^1(\Omega_\varepsilon)$ satisfies

$$\|\tilde{\varphi}\|_\infty \leq C \left(\varepsilon^{2/(2+\delta)} \|W_2\|_\infty + \varepsilon^2 \|V\|_\infty\right). \qquad \text{(Estimate)}$$

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Plan of Step 2

- Reduction to a divergence elliptic equation: Setting $v = u/\varphi$, we have

$$-\nabla \cdot (\varphi^2(\nabla v + Wv)) = 0 \text{ in } \Omega'_\varepsilon = B_2 \setminus F_\varepsilon,\ W = W_1 - W_2. \quad \text{(Divergence)}$$

- Quasiconformal change of variable: $\exists L : B_2 \to B_2$ s.t. $h = v \circ L^{-1}$, satisfies

$$-\Delta h - \nabla \cdot (\tilde{W} h) = 0 \text{ in } L(\Omega'_\varepsilon),$$
$$\text{with } \tilde{W} = \overline{\partial_z L^{-1}} \cdot \hat{W} \circ L^{-1},\ \|\hat{W}\|_\infty \leq \|W_1\|_\infty + \|W_2\|_\infty. \quad \text{(h)}$$

- Quasiconformal estimates: $L(\Omega'_\varepsilon) \approx \Omega'_\varepsilon$, and $\|\tilde{W}\|_\infty \leq \|\hat{W}\|_\infty$, with

$$\varepsilon \leq c\|W_2\|_\infty^{-1-\delta/2} \log^{-1-\delta/2}(\|W_2\|_\infty) + c\|V\|_\infty^{-1/2} \log^{-1/2}(\|V\|_\infty). \quad \text{(Condition)}$$

  Carleman is still not sufficient.

- Simplification to a reduced Beltrami equation:

$$\partial_{\overline{z}} \zeta = F \text{ in } B_2, \quad \text{(Beltrami)}$$

  by approximate type Poincaré lemma and Cauchy transform.

Quasiconformal change of variable

Starting point: $-\nabla \cdot (\varphi^2(\nabla v + Wv)) = 0$ in $\Omega'_\varepsilon = B_2 \setminus F_\varepsilon$, $W = W_1 - W_2$.

Lemma (Le Balc'h, Souza (2023))

There exists an homeomorphic mapping L of $\overline{B(0,2)}$ into itself, $L(0) = 0$ such that

- *$L \in H^1_{loc}(B_2)$ satisfies the following Beltrami equation*

$$\partial_{\overline{z}} L = \mu \partial_z L \text{ in } B_2,\ \mu = \frac{1-\varphi^2}{1+\varphi^2} \cdot \frac{v_x + iv_y}{v_x - iv_y}, \qquad \text{(Equation } L\text{)}$$

- *L is a K-quasiconformal mapping of B_2 into itself, with K satisfying*

$$1 \leq K \leq 1 + C\left(\varepsilon^{2/(2+\delta)} \|W_2\|_{L^\infty(B_2)} + \varepsilon^2 \|V\|_{L^\infty(B_2)}\right), \qquad \text{(Estimate } K\text{)}$$

- *$h = v \circ L^{-1} \in H^1_{loc}(L(\Omega'_\varepsilon))$ satisfies in the weak sense*

$$-\Delta h - \nabla \cdot (\tilde{W} h) = 0 \text{ in } L(\Omega'_\varepsilon) \approx \Omega'_\varepsilon, \qquad (h)$$

with $\tilde{W} = \overline{\partial_z L^{-1}} \cdot \hat{W} \circ L^{-1}$, $\|\hat{W}\|_\infty \leq \|W_1\|_\infty + \|W_2\|_\infty$.

Approximate type Poincare lemma in polar coordinates

Starting from $-\Delta h - \nabla \cdot (\tilde{W} h) = 0$ in $L(\Omega'_\varepsilon) \approx \Omega'_\varepsilon$, then

$$-\nabla \cdot (\chi(\nabla h + \tilde{W} h)) = -\nabla \chi \cdot (\nabla h + \tilde{W} h) \text{ in } \mathbb{R}^2. \qquad \text{(Cut-off equation)}$$

Lemma (Le Balc'h, Souza (2023))

Let us define for $(\rho, \theta) \in (0,2) \times (0, 2\pi)$,

$$\tilde{h}(\rho, \theta) = -\int_0^\rho \chi(s, \theta) \left[\frac{1}{s} \partial_\theta h(s, \theta) - [\tilde{W}_1 h](s, \theta) \sin(\theta) + [\tilde{W}_2 h](s, \theta) \cos(\theta) \right] ds.$$

Then, $\tilde{h} \in H^1(B_2)$ *and satisfies for* $(\rho, \theta) \in (0,2) \times (0, 2\pi)$,

$$\partial_\rho \tilde{h}(\rho, \theta) = -\chi(\rho, \theta) \left[\frac{1}{\rho} \partial_\theta h(\rho, \theta) - [\tilde{W}_1 h](\rho, \theta) \sin(\theta) + [\tilde{W}_2 h](\rho, \theta) \cos(\theta) \right],$$

$$\partial_\theta \tilde{h}(\rho, \theta) = \rho \chi(\rho, \theta) \left[\partial_\rho h(\rho, \theta) + [\tilde{W}_1 h](\rho, \theta) \cos(\theta) + [\tilde{W}_2 h](\rho, \theta) \sin(\theta) \right] + E_h,$$

$E_h(\rho, \theta) =$ *Error term non-local in the* ***radial variable*** ρ. (Error term)

Equivalently, $\chi(\nabla h + \tilde{W} h) = \mathrm{curl}(\tilde{h}) + E_h$ *in* B_2.

Simplification to a Beltrami equation

Starting from

$$\chi(\nabla h + \tilde{W} h) = \mathrm{curl}(\tilde{h}) + E_h \text{ in } B_2, \qquad \text{(Poincare Lemma)}$$

the simplification is done in three steps:

1. $\gamma = \chi h + i\tilde{h}$ in B_2 satisfies

$$\partial_{\overline{z}}\gamma = \alpha\gamma + (\partial_{\overline{z}}\chi)h + \tilde{E}_h \text{ in } B_2, \qquad \text{(Equation } \gamma\text{)}$$

   and

$$\|\alpha\|_{L^{\kappa}(B_2)} \leq C\,\|W_1\|_{\infty} + C\,\|W_2\|_{\infty}. \qquad \text{(Estimate } \alpha\text{)}$$

2. There exists $\beta \in L^{\infty}(B_2)$ such that

$$\partial_{\overline{z}}\beta = \alpha. \qquad \text{(Cauchy transform)}$$

   Moreover,

$$\|\beta\|_{L^{\infty}(B_2)} \leq C\,\|W\|_{\infty},\ |\beta(z_1) - \beta(z_2)| \leq C\,\|W\|_{\infty}\,|z_1 - z_2|^{2/(2+\delta)} \qquad \text{(Estimate } \beta\text{)}$$

3. $\zeta = \exp(-\beta)\gamma$ in B_2 satisfies the following Beltrami equation

$$\partial_{\overline{z}}\zeta = \exp(-\beta)[(\partial_{\overline{z}}\chi)h + \tilde{E}_h] =: F \text{ in } B_2. \qquad \text{(Beltrami)}$$

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Plan of Step 3

We start from:

$$\partial_{\overline{z}}\zeta = \exp(-\beta)[(\partial_{\overline{z}}\chi)h + \tilde{E}_h] \text{ in } B_2. \qquad \text{(Beltrami)}$$

- L^2-Carleman estimate for $\partial_{\overline{z}}$ applied to y, a cut-off version of ζ near 0,

$$\int_{B_2} |y|^2 e^{-2s\log(|z|)+2|z|^2} dz \leq C \int_{B_2} |\partial_{\overline{z}} y|^2 e^{-2s\log(|z|)+2|z|^2} dz. \qquad \text{(Carleman)}$$

- Elimination of cut-off terms involving F_ε by Harnack's inequality,

$$s \geq C\varepsilon^{-1}\log(C\varepsilon^{-1}). \qquad \text{(Choice of } s\text{)}$$

- Elimination of the boundary cut-off terms by using $\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}$

$$s \geq CK. \qquad \text{(Choice of } s\text{)}$$

- Maximal regularity estimates and Sobolev embeddings to get $\forall r \in (0, 1/2)$,

$$\|u\|_{L^\infty(B_2)} \leq \exp\left(C\left(\|W_1\|_\infty^{1+\delta} + \|W_2\|_\infty^{1+\delta} + \|V\|_\infty^{1/2+\delta} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)}.$$

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Results and perspectives

Theorem (Le Balc'h, Souza (2023)):

- For $W_1, W_2 \in L^\infty(B_2; \mathbb{R}^2)$, $V \in L^\infty(B_2; \mathbb{R})$,
$$-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0 \text{ in } B_2.$$
- For $K \geq 2$, $\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}$ ("boundary condition").

$$\text{Then, } \|u\|_{L^\infty(B_2)} \leq$$
$$\exp\left(C\left(\|W_1\|_\infty^{1+\delta} + \|W_2\|_\infty^{1+\delta} + \|V\|_\infty^{1/2+\delta} + K\right)|\log(r)|\right)\|u\|_{L^\infty(B_r)},$$
$$\forall r \in (0, 1/2).$$

$\Rightarrow$ Landis conjecture for $-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = 0$ in $\mathbb{R}^2$.

Perspectives:

- Optimality of $\|W_1\|_\infty^{1+\delta} + \|W_2\|_\infty^{1+\delta} + \|V\|_\infty^{1/2+\delta}$?
- Source terms: $-\Delta u - \nabla \cdot (W_1 u) + W_2 \cdot \nabla u + Vu = f$.
- Control of elliptic equations, see Ervedoza, Le Balc'h (2023).
- Homogeneous Dirichlet boundary conditions.
- $W_1, W_2 \in L^p$ $(p > 2)$, L^q $(q > 1)$ potentials?
- Landis conjecture for $N \geq 3$ in the real-valued setting?