Lectures on quantum aspects of black holes

Lecture 3: Long times

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The results obtained using the semiclassical black hole geometry suggest that the spectrum of states of the black hole is continuous.

But the finiteness of the entropy suggests it should be discrete.

What is the spacing between different energy levels ?

$$\dim(H) \propto e^{S} \quad \rightarrow \quad \Delta E \sim e^{-S}$$

- We expect that these energy levels to be like those of a chaotic system.
- It is believed that they should have spacings similar to those of a random matrix.
- Should include eigenvalue repulsion.

Why eigenvalue repulsion? → to produce a degeneracy we would need to finetune three real numbers. Imagine a general 2x2 Hermitian matrix. $\rho(\varepsilon)d\varepsilon= \varepsilon^2 d\varepsilon$, $\varepsilon= E-E'$.

Integrable systems → Poisson distributed. Energy is sum of single particle energies: $E = \sum_i E_i$

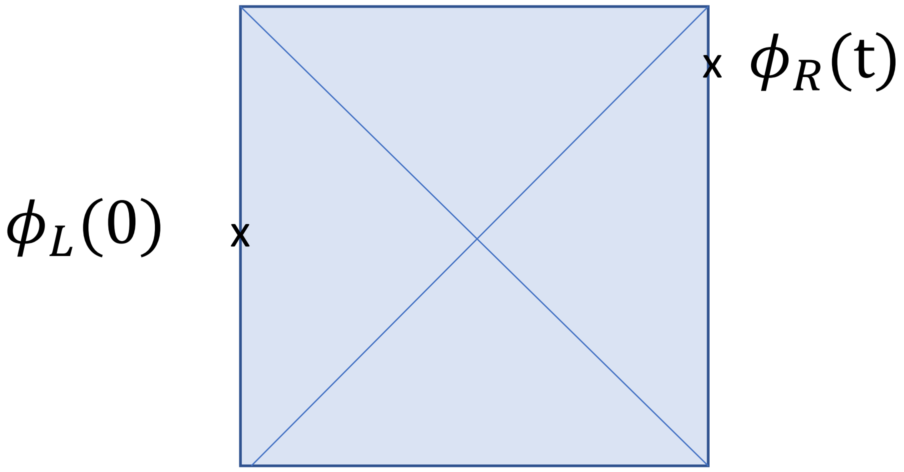
Can we see effects of this discreteness?

- Two point function at long times.

$$\langle \phi_L(0)\phi_R(t)\rangle = Tr[e^{-\beta H/2}\phi(0)e^{-\beta H/2}\phi(t)] = \sum_{ij} e^{-\beta(E_i+E_j)/2} \left|\langle E_i|\phi|E_j\rangle\right|^2 e^{it(E_j-E_i)}$$

JM

(Assume $\langle \phi \rangle = 0$ for all temperatures $\rightarrow \langle E_i|\phi|E_i\rangle = 0$)



At t=0, all phases cancel → has its maximum value.

Decays away from t=0 as e $e^{-\Gamma t}$, $\Gamma =$ imaginary part of quasi-normal mode frequencies.

Decays because the perturbation falls into the black hole.

If there is a finite number of frequencies → cannot decay for ever.

At very long times → it should have a non-zero average.

$$F(t) = \sum_{ij} c_{ij} e^{-i(E_i - E_j)t}$$

$$\lim_{T\to\infty} \left[\frac{1}{T} \int_0^T dt |F(t)|^2 \right] = \sum_{ij} c_{ij}^2 \sim e^{-2S}$$

Dyson, Lindesay, Susskind

Assumed: $E_i - E_j = E_k - E_l \quad \longrightarrow \quad i = k \quad \& \quad j = l$

No degeneracies, or relations between the energies.

(We also have the solution $E_i = E_j$ & $E_k = E_l$ this is zero due to the $\langle \phi \rangle = 0$ assumption)

In principle, we could have a random jitter with some amplitude given by the above value.

e^{-2S} is very small → non perturbative effect. We could get a contribution from a different topology.

We need to consider contributions with a different topology.

A simpler observable

$$\langle \phi_L(0)\phi_R(t)\rangle = \sum_{ij} e^{-\beta(E_i+E_j)/2} \, |\langle E_i|\phi|E_j\rangle|^2 \, e^{it(E_j-E_i)} \quad \longrightarrow$$

$$Z(\beta+iT)Z(\beta-iT) = \sum_{ij} e^{-\beta(E_i+E_j)+iT(E_j-E_i)}$$

Papadodimas, Raju

- The spectral form factor

- We can look at this simpler quantity that involves only the energy spectrum of the theory.
- It is obviously positive, since it is $|Z(\beta + i\,T)|^2$

At long times

$$\lim_{T' \to \infty} \int_0^{T'} dT |Z(\beta + iT)|^2 = Z(2\beta) \ll Z(\beta)^2$$

Assumed no degeneracies

The SYK model

Sachdev-Ye-Kitaev

N majorana fermions ψ^i

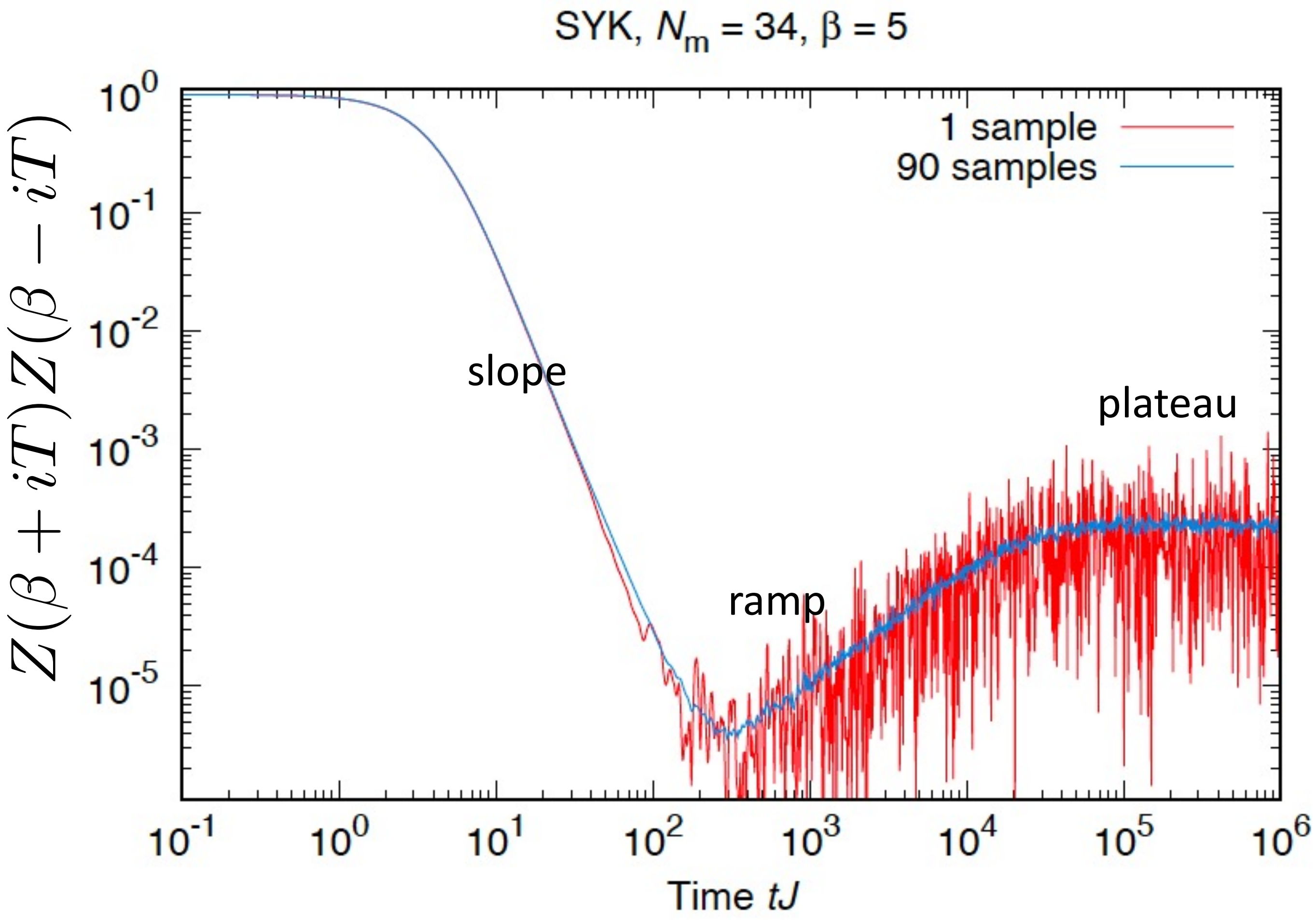
Hamiltonian

$$H = \sum_{ijkl} J_{ijkl} \psi^i \psi^j \psi^k \psi^l$$

J_{ijkl} are picked randomly from a gaussian distribution, independent for each coupling.

This model is not integrable and it is chaotic at low energies.

Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka

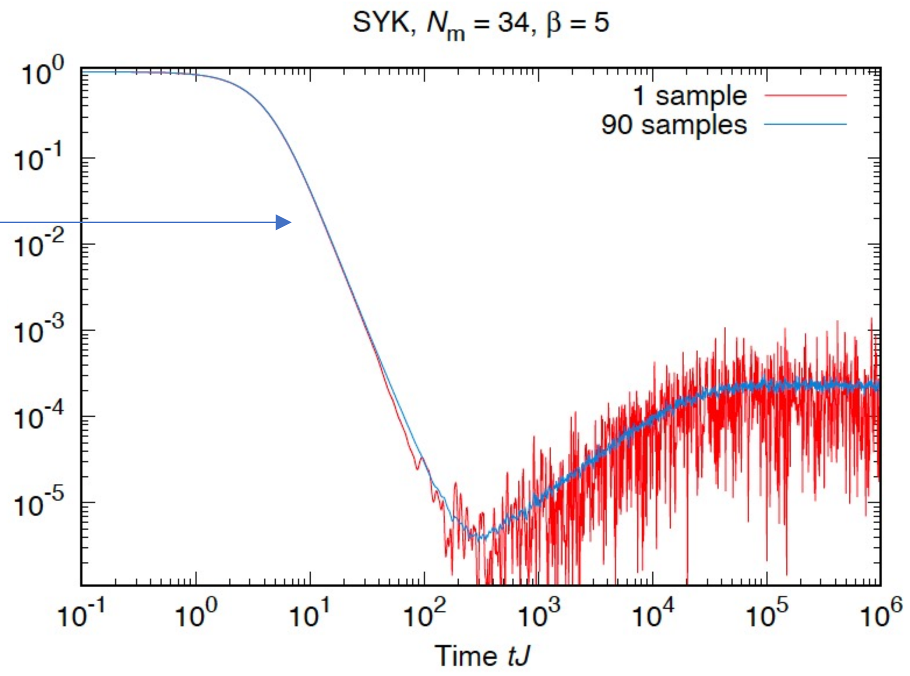


There is a new feature, which is the ramp. It will turn out that this is the one that has the simplest explanation in terms of a new topology.

But fist a few comments…

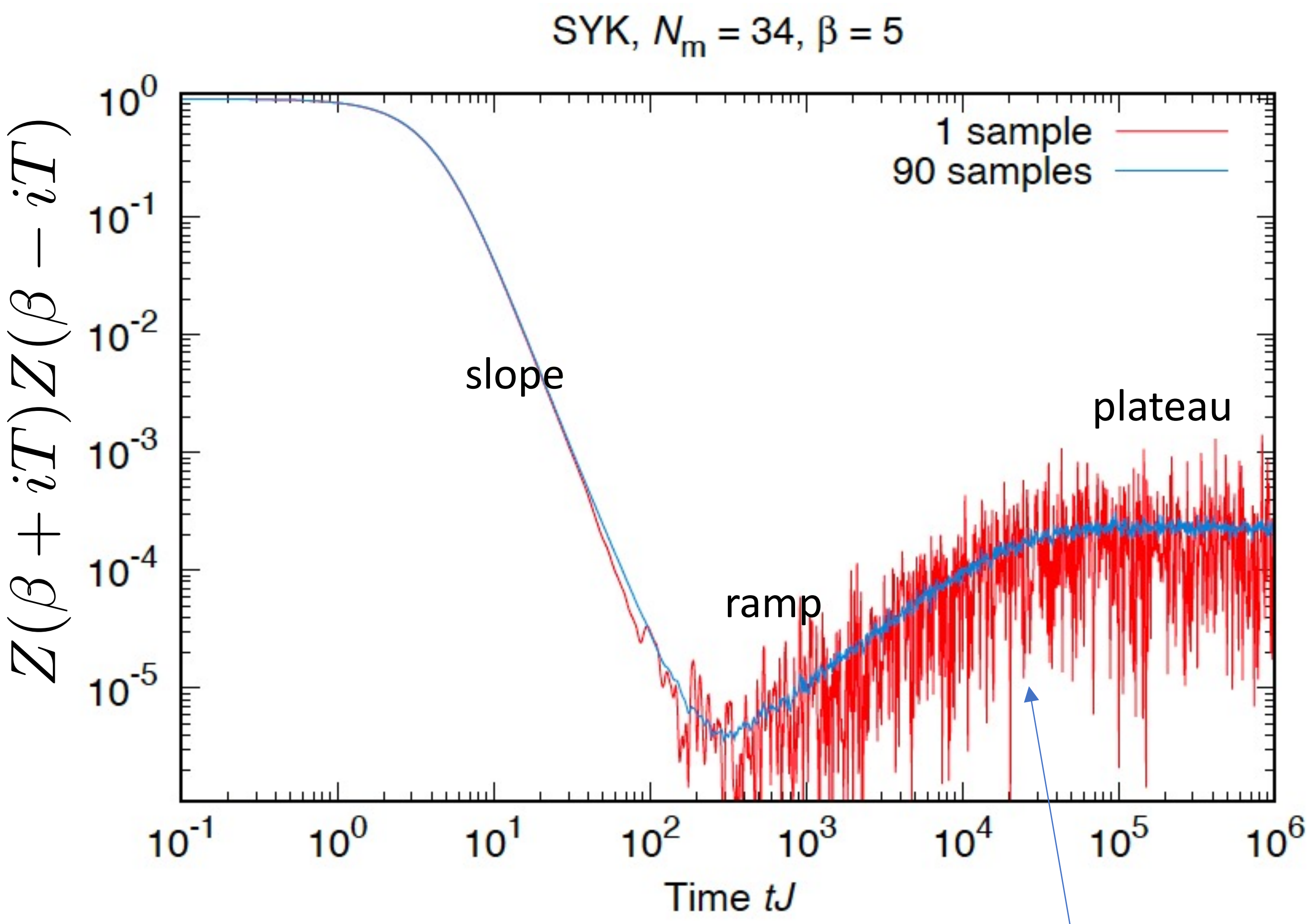
- Consider a theory with random couplings
- There can be quantities that are the same for all generic choices of the couplings. We call these ``self averaging'', each realization gives the same answer as the average over realizations.
- Other quantities might be quite different from realization from realization. Call these ``Non self averaging"

Self averaging



Non-self averaging

In this case, fluctuations are of the same order as the average.



$t_{Jitter} \sim 1/(\Delta E)$

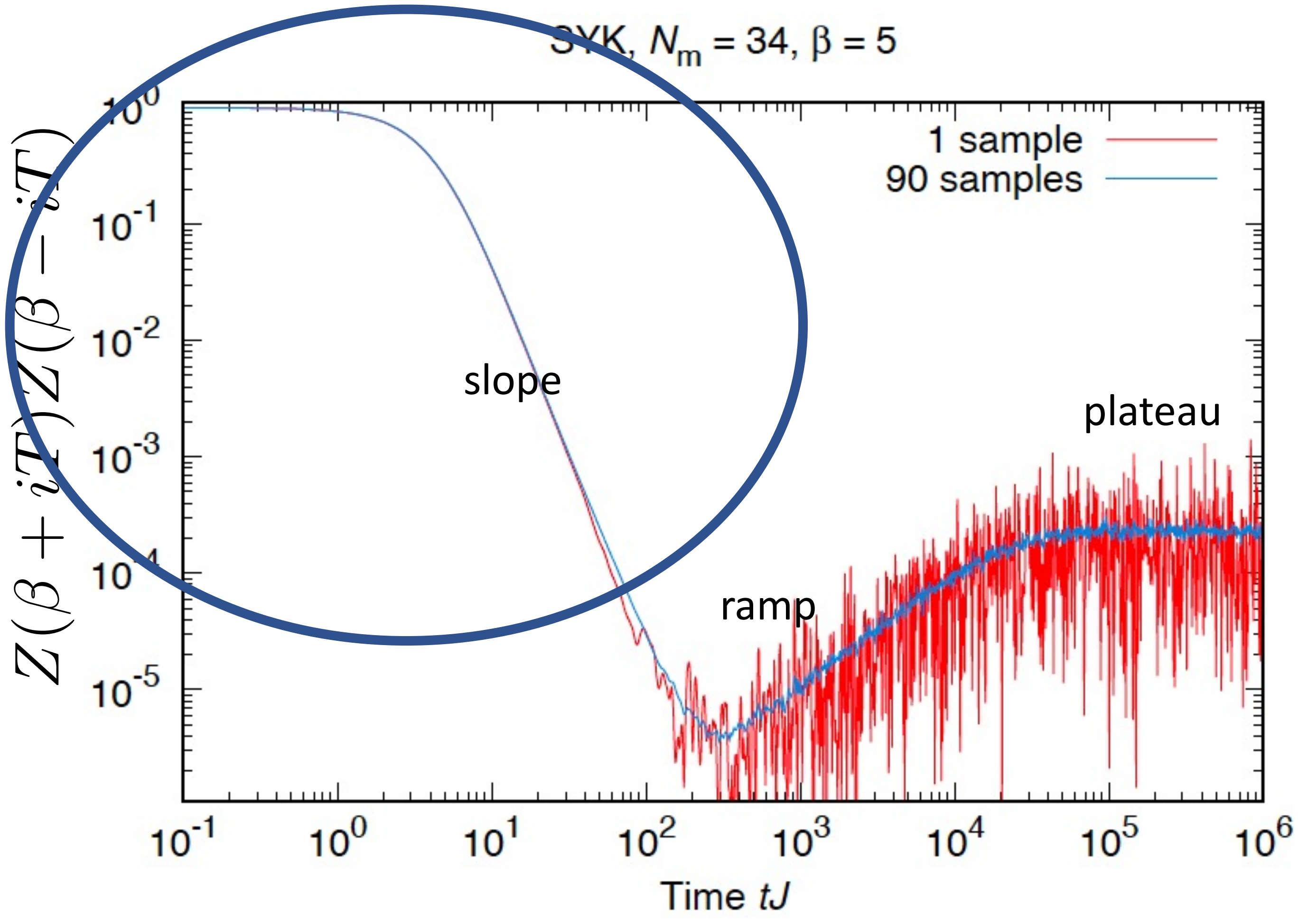
Typical spread in the energies E_i - E_j

We will now discuss various features of this plot.

This plot was for SYK, but we expect a somewhat similar plot in other chaotic theories that have an Einstein gravity dual (say strongly coupled N=4 SYM)

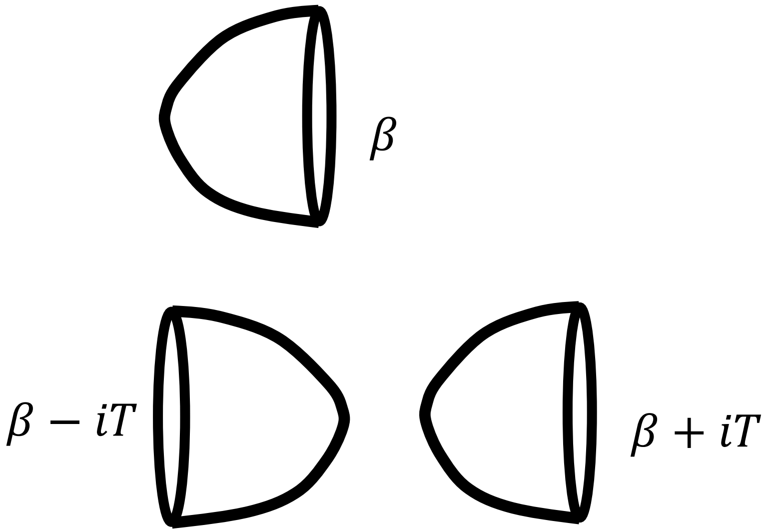
So we want to reproduce its qualitative features from a gravity computation.

The Slope



SYK, N_m = 34, β = 5
1 sample
90 samples
slope
ramp
plateau
$Z(\beta+iT)Z(\beta-iT)$
Time tJ

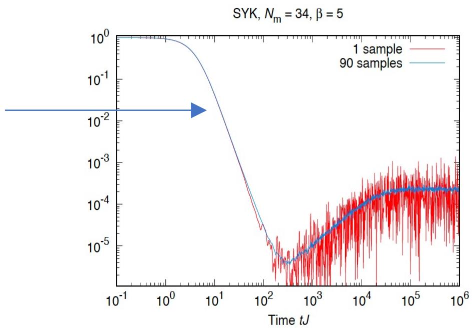
The slope



Consider the usual Euclidean black hole contribution and analytically continue $\beta \to \beta + i\,T$

This is the dominant solution when T=0, but as we increase T it might cease to be the dominant contribution. More precisely, two disconnected disks.

In the particular case of 2d JT gravity we have



$$Z(\beta)_{\text{disk}} = e^{S_0} \frac{1}{(J\beta)^{3/2}} \exp\left(\frac{1}{J\beta}\right)$$

Stanford, Witten

For large T, this decays due to the prefactor. Linear in a log-log plot

In general, the initial decay → $-\partial_\beta^2 \log Z(\beta) T^2 = -T^2 \langle (\Delta E)^2 \rangle$ set by the range of energies present in the thermal state

The ramp

$$|Z(\beta+iT)|^2 = \int dE dE' \rho(E,E') e^{-\beta(E+E')} e^{i(E-E')T}$$

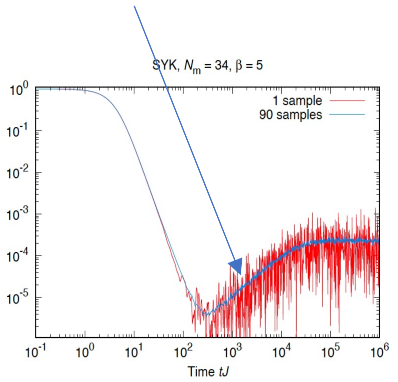
$$\rho(E,E') = \rho(E)\rho(E') + \rho_c(E,E')$$

$$|Z(\beta+iT)|^2 = \left| \int dE \rho(E) e^{-\beta E + iTE} \right|^2 + \int dE dE' \rho_c(E,E') e^{-\beta(E+E')} e^{i(E-E')T}$$

Suppose that the two point function goes as $$\rho_c(E,E') \propto -\frac{1}{(E-E')^2}$$

Then we find that the connected term would give us something linear in T

Linear in time.



We will now discuss some results for random matrices.

We will assume that the Hamiltonian is a random matrix

Results for Random Matrices

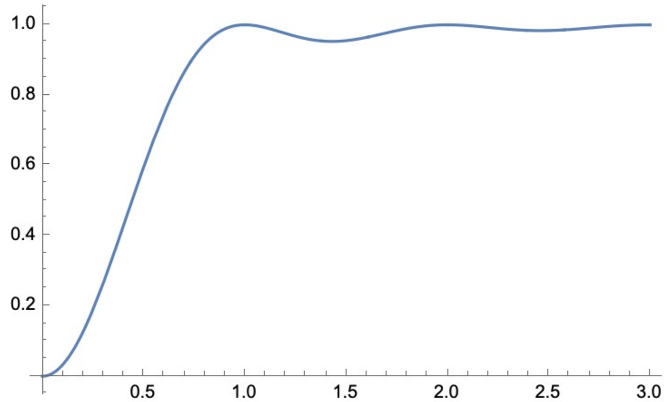
- We could see what we would expect if we had random matrix statistics, for a random hermitian matrix.

- In that case we have the famous sine kernel:

Dyson

$$\rho_c = -\frac{\sin^2[\pi(E-E')\bar{\rho}]}{[\pi(E-E')]^2} + \bar{\rho}\delta(E-E')$$

$$\rho = (\bar{\rho})^2 + \rho_c$$

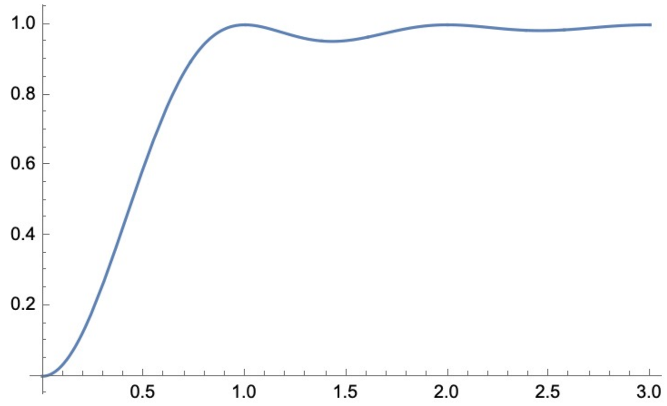


Results for Random Matrices

$$-\frac{\sin^2[\pi(E-E')\bar{\rho}]}{[\pi(E-E')]^2} \to -\frac{1}{2\pi^2(E-E')^2} \quad \text{for} \quad E - E' \gg \frac{1}{\bar{\rho}}$$

- So this eigenvalue distribution is what we expect from a random matrix.
- It gives a ramp with a specific coefficient.

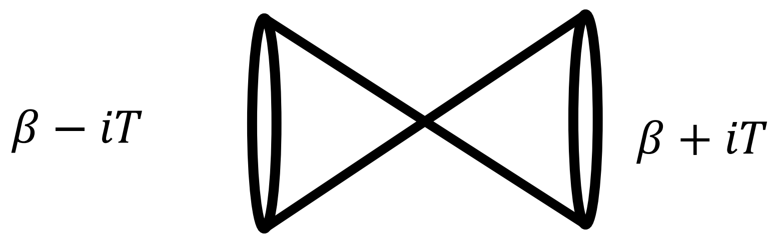
$$\bar{\rho} \sim e^S \propto L$$ = size of the matrix = dimension of the Hilbert space



This is a $1/L^2$ correction relative to the disconnected piece.

Saad, Shenker, Stanford

Ramp from gravity



Wormhole connecting the two circles.

Topology of the cylinder.

Factor of T comes from a zero mode = possibility of translating one circle relative to the other.

No entropy contribution. (Not topological contribution in 2d).

Solution is close to the eternal black hole with an identification under time translations.

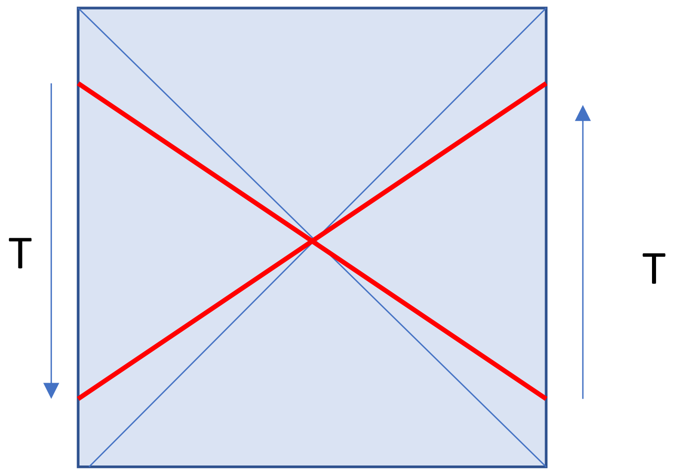
- It turns out that there is no gravity solution for

$$Z(\beta + iT)Z(\beta - iT)$$

- There are near solutions but one parameter is not stabilized. (We are driven to small energies)

- As a first practice problem, we could set $\beta = 0$ → $Z(iT)Z(-iT)$

- Lorentzian eternal black hole. Identify by the time translation isometry, by a shift by T.



The solution looks singular at the horizon.

Displace in a complex direction.

$$ds^2 = -r^2 dt^2 + dr^2$$

$$ds^2 = -(\rho + i\epsilon)^2 dt^2 + d\rho^2 = (i\rho + \epsilon)^2 dt^2 + dr^2 \ , \qquad r = \rho + i\epsilon$$

Obeys the criterion for allowable complex metrics by Kontsevich, Segal, Witten

The solution has zero action, the left and right sides cancel out. Good, no contribution going like exp($1/G_N$)

The solution has a zero mode corresponding to a relative time translation → factor of T

- The solution has an additional non-compact zero mode corresponding to the temperature of this black hole.
- Not too surprising since this would be an infinite temperature situation. (Driven to very high energies).
- The reason we run into this trouble is that the pair correlation function goes like 1/(E-E')^2 , with no large entropy factors. So, if we have a large $\exp(-\beta E)$ we are driven to very low energies. Or to very high if there is no such factor.
- By directly constraining the average energy we get a well defined solution.

Constraining the energy

$$Y_{E,\Delta E}(T) = \int_{\beta_0+iR} d\beta e^{\beta E+\beta^2\Delta E} Z(\beta+iT)$$

$$e^{\beta E+\beta^2\Delta E} \propto \int dE' \exp(E'\beta - \frac{(E-E')^2}{(\Delta E)^2})$$

Is the same as concentrating on states with energies within a region centered on E, $E \pm \Delta E$

$$|Y_{E,\Delta E}(T)|^2 = \int_{\beta_0+iR} d\beta_L e^{\beta_L E+\beta_L^2\Delta E} Z(\beta_L+iT) \int_{\beta_0+iR} d\beta_R e^{\beta_R E+\beta_R^2\Delta E} Z(\beta_R-iT)$$

Do the integrals over β_L , β_R by saddle point.

$$0 = E + 2\beta_L \Delta E + \partial_{\beta_L} \log Z(\beta_L + iT) Z(\beta_R - iT)$$
$$0 = E + 2\beta_R \Delta E + \partial_{\beta_R} \log Z(\beta_L + iT) Z(\beta_R - iT)$$

Guess a solution: $$\beta_L = \beta_R = 0$$

Then we expect that E is conjugate to β_L in the L region, and similarly for the right region.

Now we fix the mass of the black hole to E.

The action is the same as what we had before, which is classically zero.

We then get a factor of T from the relative time translation zero mode. With a prefactor that is independent of G_N

Quantum fields in this background

- Since the background contains closed timelike curves, we might worry that the matter partition function is ill defined.
- It turns out that it is fine and finite.
- This can be understood by the deformation into complex r that we discussed previously.

Kontsevich, Segal, Witten

- In that case, the time circle does not shrink and it is spacelike at the waist.
- In the case of AdS_2 it can be computed more explicitly and the result is the same as what we would get in Euclidean AdS in global coordinates (the strip coordinates), with Euclidean time identified with period of around T.

Success!

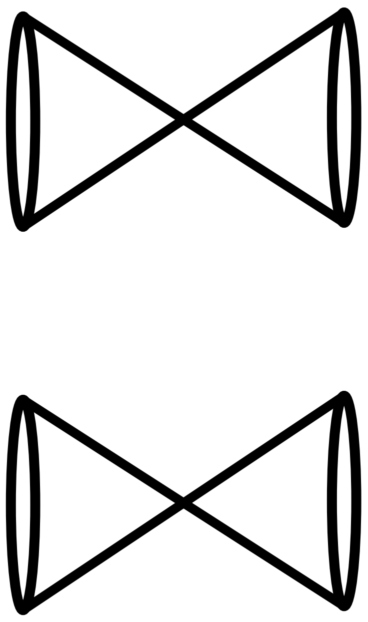
We have found the gravity solution for the ramp!

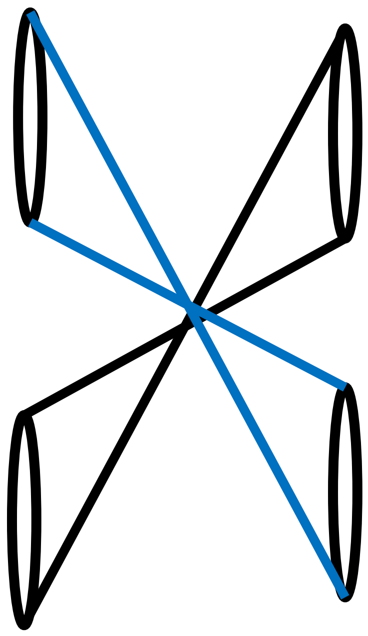
Questions

- The original observable is just a square. Why should we get a connected solution?
- We are getting the <u>average</u> ramp, not the one corresponding to a theory with definite couplings.
- This could be fine for SYK, but what if we consider the gravity dual of a definite theory (like N=4 SYM)?
- Now we have an additional problem, explaining the ``jitter'' associated to definite energies.

Statistical properties of the noise.

$$\langle |Y_{E,\Delta E}(T)|^4 \rangle$$





$$\langle |Y_{E,\Delta E}(T)|^4 \rangle \sim 2\langle |Y_{E,\Delta E}(T)|^2 \rangle \langle |Y_{E,\Delta E}(T)|^2 \rangle$$

If we have k copies → k! So we have a gaussian distribution: $p(Y) \propto e^{-a|Y|^2}$

Gravity lets us compute the statistical properties of the noise.

But so far, not the noise itself.

How should we input into the computation that we have definite couplings as opposed to random couplings ?

Some work in toy models, but no clear answer in general.

Maybe it has to do with the singularity, and we need some information beyond gravity, or a more complete theory of gravity…

Plateaux

- Is related to the eigenvalue spacing as E-E’ → 0
- For a random matrix we expect

$$-\frac{\sin^2[\pi(E-E')\bar{\rho}]}{[\pi(E-E')]^2} = -\frac{1}{2\pi^2(E-E')^2}(1 - \cos[2\pi(E - E')\bar{\rho}])$$

Term that gives the ramp

Term connected to the Plateaux

It is an exponential correction in L , $\bar{\rho} \sim L \sim e^S$, then $\cos(\bar{\rho}\,..) \sim e^{i\bar{\rho}} \sim e^{iL} \sim e^{ie^S}$

Non perturbative in 1/L

Doubly non-perturbative in gravity

Plateaux

- Not a single new geometry but new ``boundaries'' where geometries can end…
- Somewhat clearer picture in the case of pure JT gravity.
- Pure JT gravity can be viewed as arising from a random matrix, where the matrix would be the Hamiltonian of the system.

Connection with replica wormholes

- You have seen how to calculate $Tr[\rho^2] = \sum_{ij} \rho_{ij}\rho_{ji}$ by using wormholes.
- How do we calculate ρ_{ij} itself ?
- We have learnt how to compute entropies, but not individual matrix elements.
- Here we have discussed a similar situation where average properties can be computed, but it is unclear how to compute the actual function.
- A surprising aspect is that these average quantities <u>can</u> be computed via semiclassical gravity.

The end