Geodesic random line processes and the roots of quadratic congruences

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joint work with Matthew Welsh (Maryland)

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Happy 75th birthday, Professor Dani!

Uniform distribution of roots

- Consider the roots μ of the quadratic congruence

$$\mu^2 \equiv D \pmod{m}$$

with $m = 1, 2, 3, \ldots$ and $D > 0$ square-free (all will work also for $D < 0$; it's easier)

- Define sequence $\xi_1, \xi_2, \ldots \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ by normalised roots $\frac{\mu}{m}$, ordered by increasing denominator m (choose arbitrary order for terms with same m)

- **Hooley (1963):** We have uniform distribution mod 1

$$\lim_{N\to\infty} \frac{1}{N} \#\Big\{ j \leq N : \xi_j \in [a, b) + \mathbb{Z} \Big\} = b - a$$

- Extension to higher-order polynomial congruences (Hooley 1964); use of modular forms, Poincaré series (Bykovskii 1984; see also Good 1983); u.d. still holds for m restricted to primes (Duke, Friedlander, Iwaniec 1995); joint distribution (Zahavi 2020); fits in more general CRT framework (Kowalski and Soundararajan 2020).

Randomness mod 1

Given distinct points $\xi_1, \ldots, \xi_N \in \mathbb{T}$, denote by $s_1, \ldots, s_N$ the corresponding "gaps" between those points. Gap distribution:

$$P_N f := \frac{1}{N} \sum_{n=1}^{N} f(N s_n), \qquad f \in \mathrm{C}_b(\mathbb{R}_{\geq 0}) \quad \text{(bounded continuous)}$$

Two point correlation:

$$R_N f := \frac{1}{N} \sum_{\substack{m,n=1 \\ m \neq n}}^{N} \sum_{\ell \in \mathbb{Z}} f(N(\xi_m - \xi_n + \ell)), \qquad f \in \mathrm{C}_c(\mathbb{R}) \quad \text{(cmpct supp)}$$

Theorem A. Let $\xi_1, \xi_2, \ldots$ be iid in $\mathbb{T}$ (uniformly distributed). Then almost surely

$$\lim_{N\to\infty} P_N f = \int_0^\infty f(s) \mathrm{e}^{-s} ds, \qquad \forall f \in \mathrm{C}_b(\mathbb{R}_{\geq 0})$$

$$\lim_{N\to\infty} R_N f = \int_{\mathbb{R}} f(s) ds, \qquad \forall f \in \mathrm{C}_c(\mathbb{R})$$

"Gap and two-point statistics are Poisson"

Random matrices

Theorem B. (Wigner 1950s, Gaudin 1961)
Let $(\xi_{Nn})_{n\leq N}$ the ev's of $A \in U(N)$. Then Haar-a.s.

$$\lim_{N\to\infty} P_N f = \int_0^\infty f(s)\, p_{\mathsf{Gaudin}}(s) ds, \qquad \forall f \in \mathsf{C}_b(\mathbb{R}_{\geq 0})$$

$$\lim_{N\to\infty} R_N f = \int_{\mathbb{R}} f(s)\left(1 - \left(\frac{\sin(\pi s)}{\pi s}\right)^2\right) ds, \qquad \forall f \in \mathsf{C}_c(\mathbb{R})$$

- The probability density $p_{\mathsf{Gaudin}}(s)$ is given by a Painlevé transcendent
- Wigner surmise: $p_{\mathsf{Gaudin}}(s) \approx \frac{32}{\pi^2} s^2 \mathrm{e}^{-4s^2/\pi} ds$

Riemann zeros

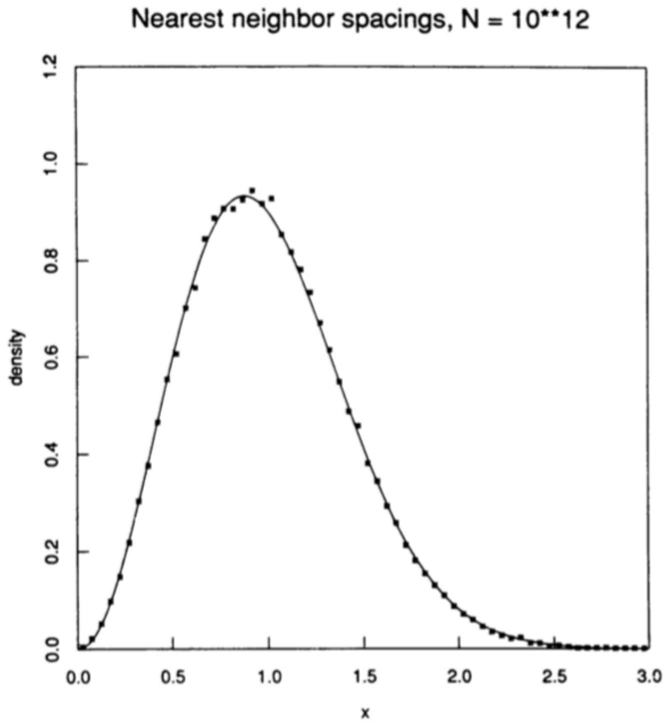


FIGURE 4

Probability density of the normalized spacings δ_n. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12} + 1 \leqslant n \leqslant 10^{12} + 10^5$.

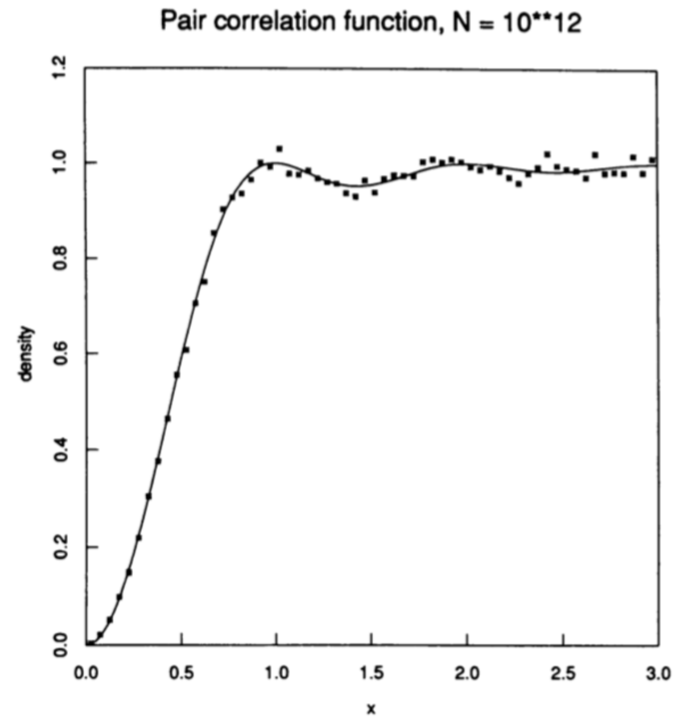


FIGURE 2

Pair correlation of zeros of the zeta function. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $10^{12} + 1 \leqslant n \leqslant 10^{12} + 10^5$.

A. M. Odlyzko, Math. Comp., 48 (1987), pp. 273-308

- Montgomery (1973) ”The pair correlation of zeros of the zeta function”
- Hejhal (1994, 3-point)
- Rudnick and Sarnak (1996, n-point)

Polynomials mod 1

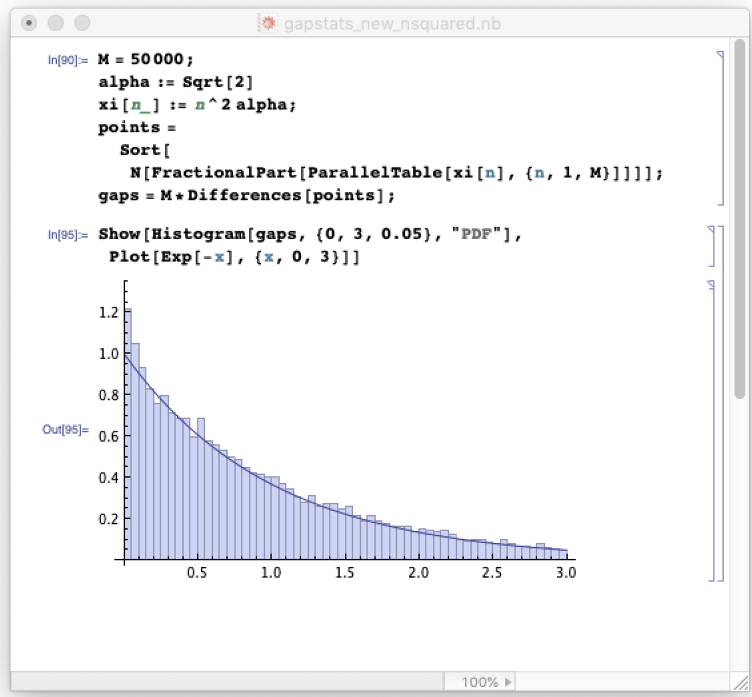
Theorem C. (Rudnick & Sarnak, 1998)
Let $(\xi_n) = (n^d\alpha \text{ mod } 1)$, $d \geq 2$. Then for Lebesgue a.e. α,

$$\lim_{N\to\infty} R_N f = \int_{\mathbb{R}} f(s)ds, \qquad \forall f \in \mathrm{C}_c(\mathbb{R})$$

- Proof uses averages over Weyl sums and estimating solutions to polynomial Diophantine equations

- Rudnick, Sarnak & Zaharescu (2001): for α that are well-approximable by rationals, proof of convergence of gap distribution P_N for $n^2\alpha$ to exponential distribution **along subsequence of** N; for these however convergence not expected along full sequence

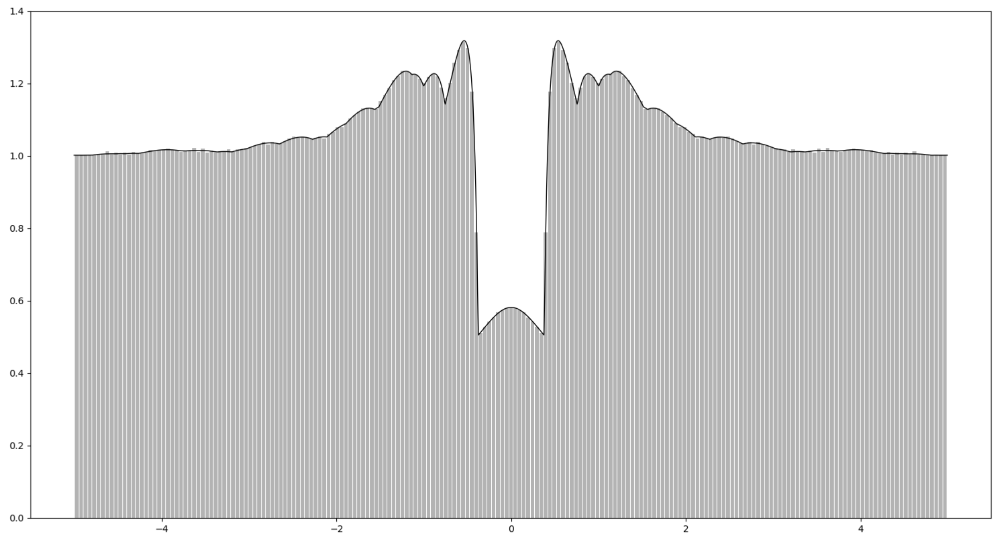
- **No proofs for P_N, nor for R_N for explicit examples of α** e.g. for $\alpha = \sqrt{2}$; cf. algorithmic characterization by Heath-Brown (2010).



Pair correlation for roots

Theorem D. (JM & Welsh, 2021)
Assume $D > 0$ is square-free and $D \not\equiv 1 \pmod 4$. Then there is an even and continuous function $w_D : \mathbb{R} \to \mathbb{R}_{\geq 0}$, such that

$$\lim_{N\to\infty} R_N f = \int_{\mathbb{R}} f(s)\, w_D(s)\, ds, \qquad \forall f \in C_c(\mathbb{R})$$



$D = 2 \qquad N = 10^6$

Pair correlation for roots

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Assume $D > 0$ is square-free and $D \not\equiv 1 \pmod 4$. Then there is an even and continuous function $w_D : \mathbb{R} \to \mathbb{R}_{\geq 0}$, such that

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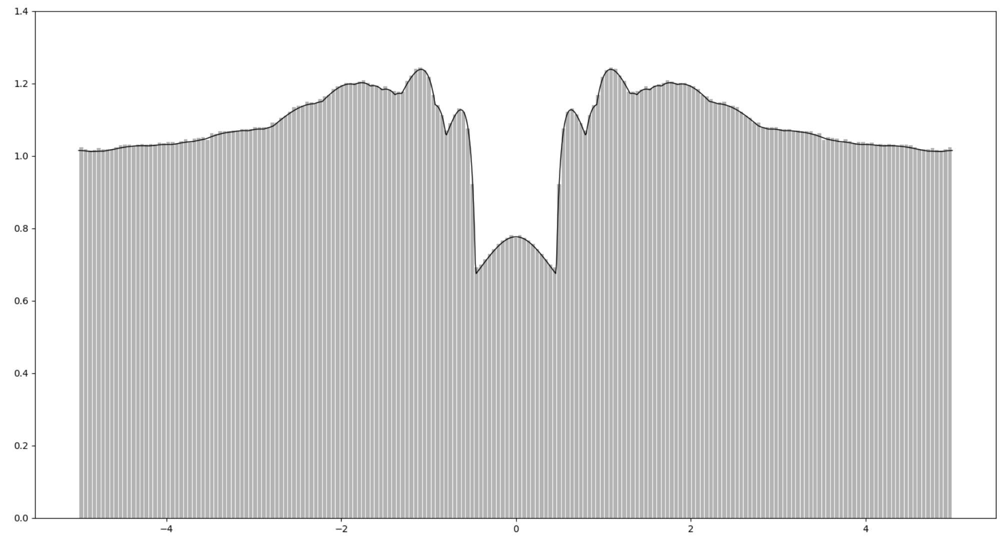


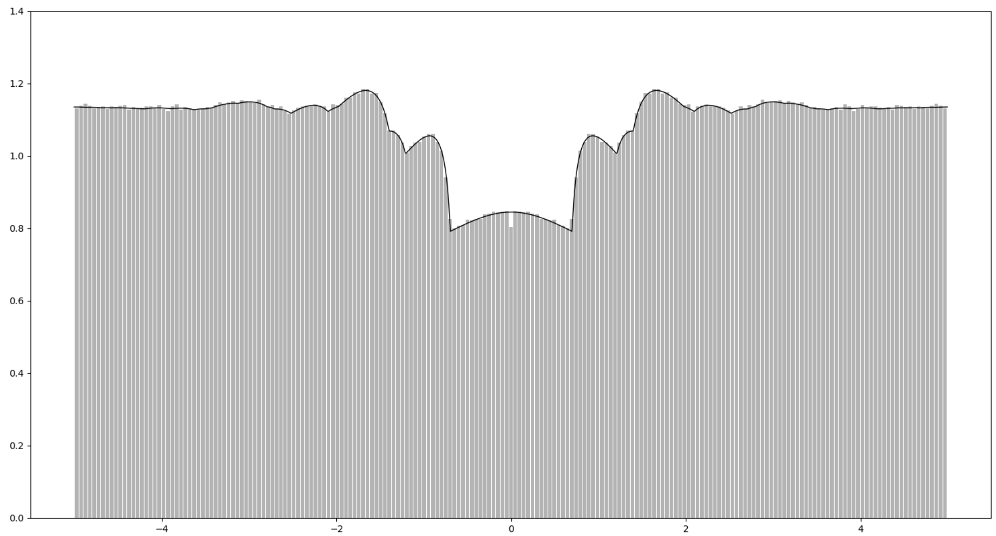
$D = 3 \qquad N = 10^6$

Pair correlation for roots

Theorem D. (JM & Welsh, 2021)

Assume $D > 0$ is square-free and $D \not\equiv 1 \pmod 4$. Then there is an even and continuous function $w_D : \mathbb{R} \to \mathbb{R}_{\geq 0}$, such that

$$\lim_{N\to\infty} R_N f = \int_{\mathbb{R}} f(s)\, w_D(s)\, ds, \qquad \forall f \in \mathrm{C}_c(\mathbb{R})$$



$D = 10 \qquad N = 10^6$

Higher-order statistics

- Define the random counting measure (random point process) on $\mathbb{R}$

$$\Xi_{N,\lambda} = \sum_{j=1}^{N} \sum_{k\in\mathbb{Z}} \delta_{N(\xi_j - \xi + k)}$$

- Here ξ random variable in $\mathbb{T}$ distributed according to Borel prob. measure λ

- **Example:** For any interval $I \subset \mathbb{R}$ and integer k

$$\mathbb{P}(\Xi_{N,\lambda}(I) = k) = \lambda(\{x \in \mathbb{T} : \mathcal{N}_I(x, N) = k\})$$

with

$$\mathcal{N}_I(x, N) = \#\{j \leq N : \xi_j \in x + N^{-1}I + \mathbb{Z}\}.$$

Higher-order statistics

Theorem E. (JM & Welsh, 2021)

For D as above, there exists a random point process Ξ depending only on D so that, for every Borel probability measure λ on $\mathbb{T}$ that is absolutely continuous with respect to the Lebesgue measure, we have convergence $\Xi_{N,\lambda} \to \Xi$ in distribution as $N \to \infty$.

Specifically, for all $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ and finite intervals $I_1, \ldots, I_r \subset \mathbb{R}$, we have that

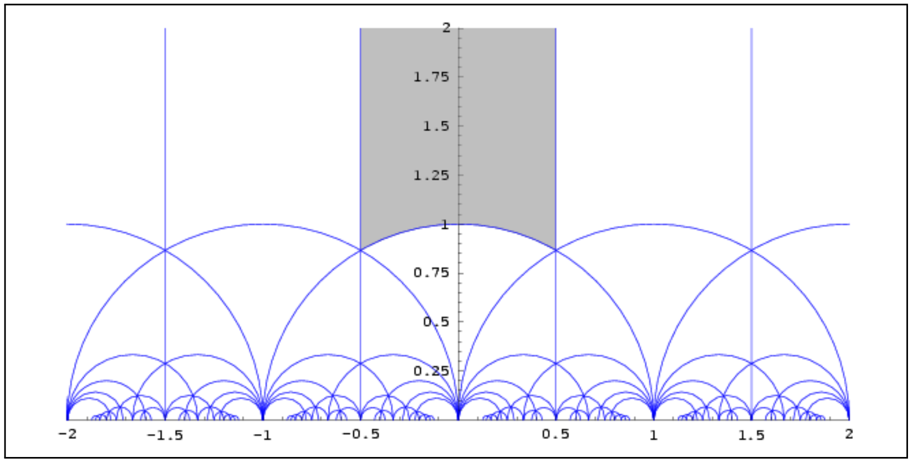
$$\lim_{N\to\infty} \lambda\Big(\Big\{x \in \mathbb{T} : \mathcal{N}_{I_i}(x, N) = k_i \ \forall i\Big\}\Big) = \mathbb{P}\Big(\Xi(I_i) = k_i \ \forall i\Big)$$

and the limit is a continuous function of the endpoints of I_i.

- Implies convergence of (joint) gap distributions
- We also prove convergence of all moments
- Can impose further congruence restrictions on m and μ (leads to different limit process)

Basic hyperbolic geometry

- $\mathbb{H}$ complex upper half plane, $ds^2 = \frac{dx^2+dy^2}{y^2}$
- boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$
- $\mathrm{SL}(2,\mathbb{R})$ acts by Möbius transformations
- geodesics, horocycles
- stabiliser $\Gamma_{\boldsymbol{c}} = \{g \in \mathrm{SL}(2,\mathbb{R}) : g\boldsymbol{c} = \boldsymbol{c}\}$
- $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}$
- $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ modular surface
- $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R}) \simeq$ unit tangent bundle



- $\boldsymbol{c}$ closed geodesic (resp. horocycle) if $\Gamma_{\boldsymbol{c}} \cap \mathrm{SL}(2,\mathbb{Z}) < \mathrm{SL}(2,\mathbb{Z})$ non-trivial

- Denote by $z_{\boldsymbol{c}}$ the "top" of the geodesic $\boldsymbol{c} \in \mathbb{H}$ (i.e. the point on $\boldsymbol{c}$ closest to ∞; in general $z_{\gamma \boldsymbol{c}_l} \neq \gamma z_{\boldsymbol{c}_l}$)

Key insight: the geometry of roots

Theorem F. (JM & Welsh, 2021)
For D as above, there exists a finite set of geodesics $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_h\}$ such that:

(i) For any $m > 0$ and $\mu \pmod{m}$ satisfying $\mu^2 \equiv D \pmod{m}$, there is a unique l and double coset $\Gamma_\infty \gamma \Gamma_{\boldsymbol{c}_l} \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})/\Gamma_{\boldsymbol{c}_l}$ such that

$$z_{\gamma \boldsymbol{c}_l} \equiv \frac{\mu}{m} + \mathrm{i}\frac{\sqrt{D}}{m} \pmod{\Gamma_\infty}. \qquad (*)$$

(ii) Conversely, given l and double coset $\Gamma_\infty \gamma \Gamma_{\boldsymbol{c}_l} \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})/\Gamma_{\boldsymbol{c}_l}$ with $\gamma \boldsymbol{c}_l$ positively oriented, there exist unique $m > 0$ and $\mu \pmod{m}$ satisfying $\mu^2 \equiv D \pmod{m}$ such that (*) holds.

- The geodesics $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_h\}$ project to closed geodesics of equal length in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$
- Extends to setting with additional congruence conditions $m \equiv 0 \pmod{n}$, and $\mu \equiv \nu \pmod{n}$, need to replace $\mathrm{SL}(2,\mathbb{Z})$ by $\Gamma_0(n)$
- See Welsh (Algebra & Number Theory, 2022) for parametrization of roots of higher-degree polynomial congruences
- Theorems D-F recently extended to $D \equiv 1 \bmod 4$ by Li and Welsh (preprint 2022)

Indefinite quadratic forms and closed geodesics

- $F(X,Y) = aX^2 + bXY + cY^2$, $(a,b,c) = 1$, discriminant $d = b^2 - 4ac > 0$

- $F \leftrightarrow \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$

- F, F' equivalent if $\exists \gamma \in \mathrm{SL}(2,\mathbb{Z})$ s.t. $\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = {}^t\gamma \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma$

- finite number h_d of equivalence classes $\{F\}_d$ of F with discriminant d

- Solutions of $F(X,1) = 0$ define end points of geodesic: $x_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

- $F \leftrightarrow$ geodesics in $\mathbb{H}$ that are closed in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$
  $\{F\}_d \leftrightarrow$ closed geodesics of same length ℓ_d in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$

- If $F(X,Y) = mX^2 - 2\mu XY + cY^2$ has discriminant $d = 4D$, then $\mu^2 \equiv D \pmod{m}$ and $x_\pm = \frac{\mu}{m} \pm \frac{\sqrt{D}}{m}$, so top of geodesic is $\frac{\mu}{m} + \mathrm{i}\frac{\sqrt{D}}{m}$

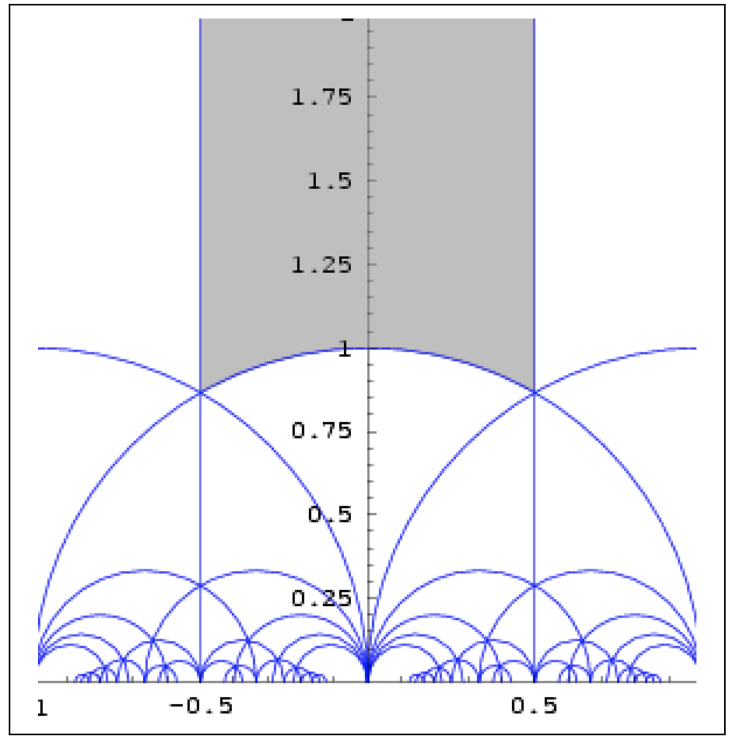
More general geometric setting

- $\Gamma < \mathrm{SL}(2,\mathbb{R})$ discrete subgroup so that $\Gamma\backslash\mathbb{H}$ has finite area with standard cusp at ∞

- $\boldsymbol{c}_1,\ldots,\boldsymbol{c}_h$ collection of geodesics in $\mathbb{H}$ so that each $\boldsymbol{c}_l$ projects to a closed geodesic in $\Gamma\backslash\mathbb{H}$ ($\Leftrightarrow \Gamma_{\boldsymbol{c}_l} \cap \Gamma < \Gamma$ non-trivial)
  assume w.l.o.g. no two $\boldsymbol{c}_l$ are Γ-equivalent

- study distribution of the geodesics

$$\bigcup_{l=1}^{h} \bigcup_{\gamma\in\Gamma/\Gamma_{\boldsymbol{c}_l}} \gamma\boldsymbol{c}_l$$



- and specifically the real parts of geodesic tops with imaginary part larger than y:

$$X(y) = \biguplus_{l=1}^{h} X^l(y)$$

with

$$X^l(y) = \Big\{ \mathrm{Re}(z_{\gamma\boldsymbol{c}_l}) \bmod 1 : \gamma \in \Gamma_\infty\backslash\Gamma/\Gamma_{\boldsymbol{c}_l},\ \mathrm{Im}(z_{\gamma\boldsymbol{c}_l}) \geq y \Big\} \subset \mathbb{T}$$

Distribution in small intervals

- counting in small intervals: $\mathcal{N}_B(x,y) = \#\Big(X(y) \cap (x + yI + \mathbb{Z})\Big)$

- set $B_I = \{u + \mathrm{i}v \in \mathbb{H} : u \in I, v \geq 1\}$

- then for y sufficiently small (so that $y|I| < 1$)

$$\begin{aligned}\mathcal{N}_I(x,y) &= \sum_{l=1}^{h} \#\Big\{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_{\boldsymbol{c}_l} : z_{\gamma \boldsymbol{c}_l} \in x + yB_I + \mathbb{Z}\Big\} \\ &= \sum_{l=1}^{h} \#\Big\{\gamma \in \Gamma / \Gamma_{\boldsymbol{c}_l} : z_{\gamma \boldsymbol{c}_l} \in x + yB_I\Big\} \\ &= \sum_{l=1}^{h} \#\Big\{\gamma \in \Gamma / \Gamma_{\boldsymbol{c}_l} : z_{\gamma \boldsymbol{c}_l} \in n(x)a(y)B_I\Big\}\end{aligned}$$

with $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$

Geodesic line processes

- for $B \in \mathbb{H}$ define $\mathcal{N}_B(g) = \sum_{l=1}^{h} \#\{\gamma \in \Gamma/\Gamma_{\boldsymbol{c}_l} : z_{g^{-1}\gamma \boldsymbol{c}_l} \in B\}$

- then $\mathcal{N}_I(x,y) = \mathcal{N}_B(n(x)a(y))$ for $B = B_I$

- note: $\mathcal{N}_B(\gamma g) = \mathcal{N}_B(g)$ for all $\gamma \in \Gamma$

- this motivates definition of the geodesic random "line" processes

$$\Theta_{y,\lambda} = \sum_{l=1}^{h} \sum_{\gamma \in \Gamma/\Gamma_{\boldsymbol{c}_l}} \delta_{z_{(n(\xi)a(y))^{-1}\gamma \boldsymbol{c}_l}}, \qquad \Theta = \sum_{l=1}^{h} \sum_{\gamma \in \Gamma/\Gamma_{\boldsymbol{c}_l}} \delta_{z_{g^{-1}\gamma \boldsymbol{c}_l}}$$

  - random variable ξ distributed according to a Borel probability measure λ on $\mathbb{T}$
  - random element g distributed with respect to Haar probability measure μ_Γ on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$

- intensity measure: $\mathbb{E}\Theta(B) = \int_{\Gamma\backslash \mathrm{SL}(2,\mathbb{R})} \mathcal{N}_B(g) d\mu_\Gamma(g) = \kappa_\Gamma \,\mathrm{vol}_{\mathbb{H}}(B)$

Convergence in distribution

Theorem G. (JM & Welsh, 2021)

For every a.c. Borel probability measure λ on $\mathbb{T}$ we have convergence $\Theta_{y,\lambda} \to \Theta$ in distribution as $y \to 0$.

In particular, for all $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$, finite intervals I_i, we have that

$$\lim_{y \to 0} \lambda\Big(\Big\{x \in \mathbb{T} : \mathcal{N}_{I_i}(x, y) = k_i \ \forall i\Big\}\Big) = \mathbb{P}\Big(\Theta(B_{I_i}) = k_i \ \forall i\Big)$$

and the limit is a continuous function of the endpoints of I_i.

- Follows from equidistribution of long closed horocycles on $\Gamma\backslash \mathrm{SL}(2, \mathbb{R})$, i.e. for any bounded continuous $f : \Gamma\backslash \mathrm{SL}(2, \mathbb{R}) \to \mathbb{C}$

$$\lim_{y \to 0} \int f(n(x)a(y)) d\lambda(x) = \int f(g) d\mu(g)$$

- Similar results for angles of hyperbolic lattice points: Boca, Paşol, Popa, Zaharescu (2014), Kelmer & Kontorovich (2015), Risager & Södergren (2017), Marklof & Vinogradov (2018), Lutsko (2020)

Moments

- We alse prove convergence of *all* moments

- In particular, for the first moment

$$\lim_{y \to 0} \int_{\mathbb{T}} \mathcal{N}_I(x,y) d\lambda(x) = \mathbb{E}\Theta(B_I) = \kappa_{\Gamma} |I|$$

- This implies uniform distribution of

$$X^l(y) = \left\{ \mathrm{Re}(z_{\gamma \boldsymbol{c}_l}) \bmod 1 : \gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_{\boldsymbol{c}_l},\ \mathrm{Im}(z_{\gamma \boldsymbol{c}_l}) \geq y \right\} \subset \mathbb{T}$$

  as $y \to 0$ and hence (via our geometric interpretation) Hooley's uniform distribution of the roots

Towards spacing statistics: A Poincaré section for the horocycle flow

- Let $\widehat{\boldsymbol{c}}_l$ be the lift of the (oriented) geodesic $\boldsymbol{c}_l$ to $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{T}^1(\mathbb{H})$
- Define two-dimensional Poincaré section for the horocycle flow

$$S^l = \Gamma\backslash\Gamma\widehat{\boldsymbol{c}}_l\{k(-\tfrac{\pi}{2})a(v^{-1}) : v \geq 1\}$$

$$k(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \quad a(y) = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$$

- Natural invariant measure ν for return map on S_l is arc-length measure on $\widehat{\boldsymbol{c}}_l$ times $v^{-2}dv$.

□

- Compare with Athreya-Cheung section (IMRN 2014) for horocycle flow where the (closed) geodesic $\Gamma\backslash\Gamma\widehat{\boldsymbol{c}}_l$ is replaced by a closed horocycle $\Rightarrow$ return map is Boca-Cobeli-Zaharescu map (statistics of Farey fractions)

An equidistribution theorem

Theorem H. (JM & Welsh, 2021)
For $f : \mathbb{T} \times \Gamma\backslash \mathrm{SL}(2,\mathbb{R}) \to \mathbb{C}$ bounded continuous, we have

$$\lim_{y\to 0} y \sum_{\xi \in X(y)} f(\xi, \Gamma n(\xi) a(y)) = \int_{\mathbb{T}} \int_{\Gamma\backslash \mathrm{SL}(2,\mathbb{R})} f(x,g)\, d\nu(g)\, dx.$$

- Key observation is that return times for the periodic orbit $\{\Gamma a(y) n(t) : t \in \mathbb{T}\}$ have the form $y^{-1}\mathrm{Re}(z_{\gamma \boldsymbol{c}_l})$
- Use this equidistribution theorem (in place of the previous horocycle equidistribution) to obtain spacing statistics

Conditioned geodesic line processes

- Earlier we discussed geodesic random line processes

$$\Theta_{y,\lambda} = \sum_{l=1}^{h} \sum_{\gamma \in \Gamma / \Gamma_{\boldsymbol{c}_l}} \delta_{z_{(n(\xi)a(y))^{-1}\gamma \boldsymbol{c}_l}}, \qquad \Theta = \sum_{l=1}^{h} \sum_{\gamma \in \Gamma / \Gamma_{\boldsymbol{c}_l}} \delta_{z_{g^{-1}\gamma \boldsymbol{c}_l}}$$

  - random variable ξ distributed according to a Borel probability measure λ on $\mathbb{T}$
  - random element g distributed with respect to Haar probability measure μ_Γ on $\Gamma \backslash \mathrm{SL}(2,\mathbb{R})$

- Consider now the "conditioned" processes

$$\Theta_y^0 = \sum_{l=1}^{h} \sum_{\gamma \in \Gamma / \Gamma_{\boldsymbol{c}_l}} \delta_{z_{(n(\xi)a(y))^{-1}\gamma \boldsymbol{c}_l}}, \qquad \Theta^0 = \sum_{l=1}^{h} \sum_{\gamma \in \Gamma / \Gamma_{\boldsymbol{c}_l}} \delta_{z_{g^{-1}\gamma \boldsymbol{c}_l}}$$

  - random variable ξ distributed uniformly in $X(y)$
  - random element g distributed with respect to ν on $\Gamma \backslash \mathrm{SL}(2,\mathbb{R})$
  - Θ^0 is related to the Palm distribution of Θ

- Using the previous equidistribution theorem, we can prove $\Theta_y^0 \to \Theta^0$ in distribution and for all moments

Moments

- In particular the intensity measure $\mathbb{E}\Theta^0_y \to \mathbb{E}\Theta^0$ is nothing but the pair correlation measure!
- The limit is $\mathbb{E}\Theta^0(B_I) = \delta_0(I) + \int_I W(v)dv$

where $W : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is the even and continuous function given by

$$W(v) = \frac{1}{\ell v^2} \sum_{l_1,l_2=1}^{h} \sum_{\substack{\gamma\in\Gamma_{c_{l_1}}\backslash\Gamma/\Gamma_{c_{l_2}} \\ \gamma c_{l_2}\neq c_{l_1},\overline{c_{l_1}}}} H_{\mathrm{sign}(g_{l_1}^{-1}\gamma g_{l_2}(0))}(q(\gamma,l_1,l_2),v,v), \tag{1}$$

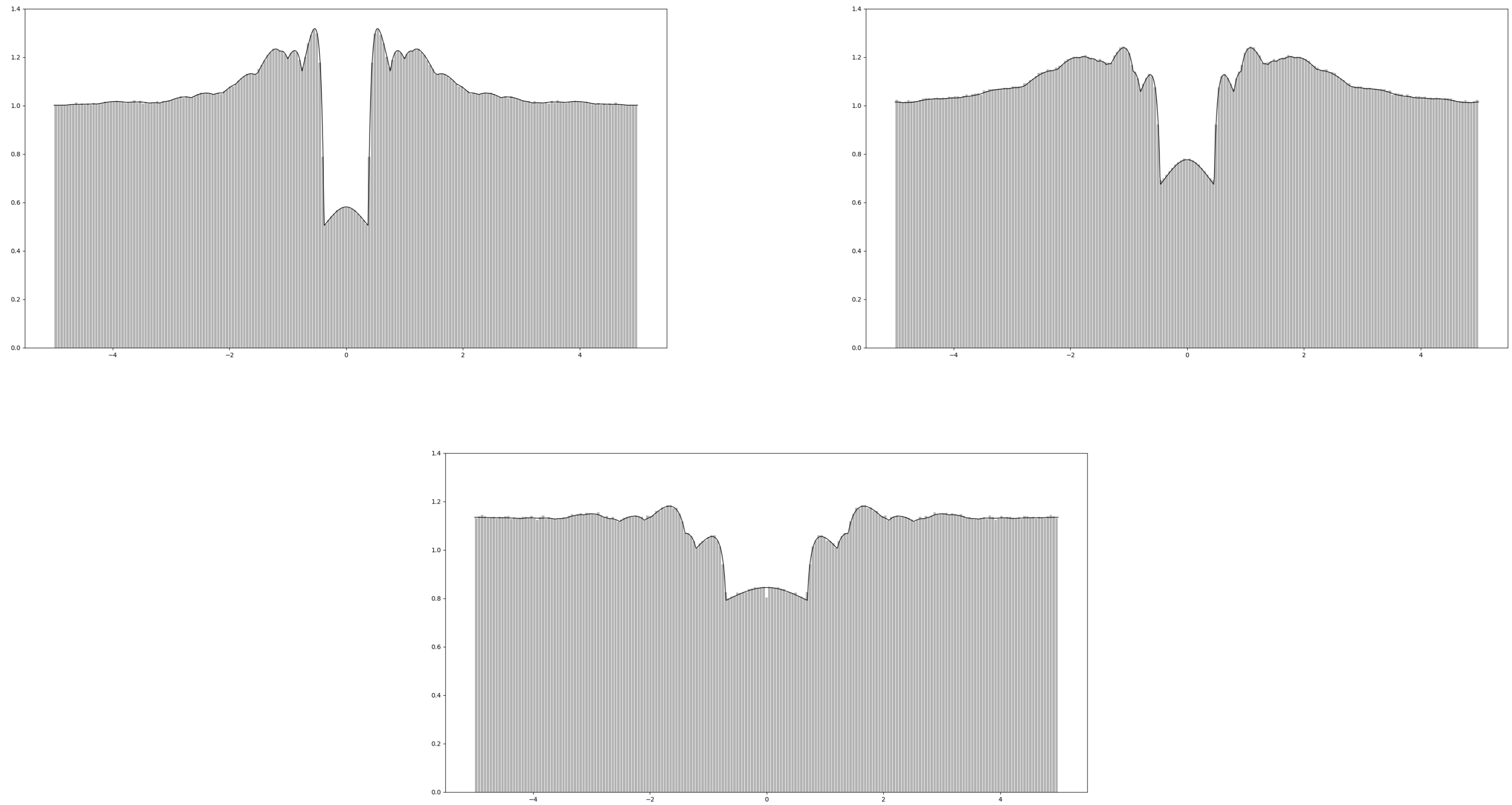
where $q(\gamma,l_1,l_2) = \frac{r+1}{r-1}$ with r the cross-ratio $r = \frac{((\gamma c_{l_2})^+ - c_{l_1}^-)((\gamma c_{l_2})^- - c_{l_1}^+)}{((\gamma c_{l_2})^+ - c_{l_1}^+)((\gamma c_{l_2})^- - c_{l_1}^-)}$, and

$$H_+(q,v,v) = \begin{cases} 0 & \text{if } q < -1 \\ 0 & \text{if } -1 < q < 1 \text{ and } v < \sqrt{2-2q} \\ h_q(s_1(q,v)) - h_q(s_2(q,v)) & \text{if } -1 < q < 1 \text{ and } v > \sqrt{2-2q} \\ h_q(s_1(q,v)) - h_q(-q+\sqrt{q^2-1}) & \text{if } q > 1, \end{cases} \tag{2}$$

$$H_-(q,v,v) = \begin{cases} 0 & \text{if } q < -1 \text{ and } |v| < \sqrt{2-2q} \\ h_q(s_1(q,v)) - h_q(s_2(q,v)) & \text{if } q < -1 \text{ and } |v| > \sqrt{2-2q} \\ h_q(s_1(q,v)) - h_q(s_2(q,v)) & \text{if } -1 < q < 1 \text{ and } v < -\sqrt{2-2q} \\ 0 & \text{if } -1 < q < 1 \text{ and } v > -\sqrt{2-2q} \\ h_q(-q-\sqrt{q^2-1}) - h_q(s_2(q,v)) & \text{if } q > 1, \end{cases} \tag{3}$$

$$h_q(s) = \log\frac{s+q}{1-s^2}, \quad s_1(q,v) = \frac{-q+\sqrt{v^2+q^2-1}}{v+1}, \quad s_2(q,v) = v - q - \sqrt{v^2+q^2-1}.$$

Pair correlation densities ($\rightarrow$ Theorem D)



$D = 2, 3, 10 \qquad N = 10^6$