TURBULENCE ENERGY SPECTRUM

RANDOMLY FORCED MODELS

HOMOGENEOUS ISOTROPIC TURBULENCE

- Amid all the complexities of the turbulent flow involving vortices etc there are some robust universal features related to the kinetic energy content at a given wave number :

$$K = \frac{1}{2V}\int d^3r \langle u_\alpha u_\alpha \rangle = \int \frac{d^3k}{(2\pi)^3}\langle u_\alpha(k) u_\alpha(-k)\rangle = \frac{1}{4\pi^2}\int C(k) k^2 dk = \int E(k) dk$$

- This defines the correlator $C(k)$ and the spectrum $E(k)$

- In the absence of dissipation and forcing, energy conserved

- In a steady state where forcing rate and dissipation rate are equal one has the Kolmogorov spectrum

Kolmogorov Spectrum

- Kolmogorov (1941)

$$E(k) = K_0 \varepsilon k^{-5/3}$$

- The rate of energy flow $\varepsilon(k) = \varepsilon$ across a particular wave-number (from below it to above it) is independent of 'k' and the pre-factor is an universal constant

- This defines a "mean field theory "

- One needs to define over what range of wave- numbers the above picture is valid

The Universal Range

- The energy of turbulent flow has to come from external forcing. Usually done at large length scales . Stirring coffee
- Consider forced incompressible Navier Stokes flow in D=3:

$$\dot{u}_\alpha + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p + \eta \nabla^2 u_\alpha + f_\alpha$$

$$\partial_\alpha u_\alpha = 0$$

- Energy equation

$$\partial_t \frac{1}{2V} \int u^2 dV = -\frac{\eta}{V} \int \left(\partial_\alpha u_\beta \right)^2 dV + \frac{1}{V} \int f_\alpha u_\alpha dV$$

- Large length scale input, short length scale dissipation, In steady state the energy flux is constant. Mean field theory.

A STOCHASTIC MODEL

DeDominicis and Martin 1979 : Make the external force stochastic with long range correlation . The external force is defined by the correlation function

$$\left\langle f_\alpha(k,\omega) f_\beta(k',\omega') \right\rangle = D_0 P_{\alpha\beta}(k)\delta(\vec{k}+\vec{k}')\delta(\omega+\omega')\frac{1}{k^{D-4+y}}$$

The focus of that work and the subsequent effort of Yakhot and Orszag was the use of the Renormalization Group . This created a lot of adverse reactions because the Kolmogorov spectrum required y=4 where there were infinite number of marginal operators, further there was another kind of divergence at y=3.

The problem with scales

A RG kind of approach immediately asks the question -in which asymptotic limit are we going to find the scaling law

The standard answer would be that in the long wavelength limit -but thats not where the Kolmorgorov universal range is

Use the model but use age old self consistent perturbation theory and see if it works and makes sense

Need to work in Fourier space for convenience

$$\partial_t u_\alpha(k) + M_{\alpha\beta\gamma}\int u_\beta(p)u_\gamma(k-p)\frac{d^3p}{(2\pi)^3} = -\eta k^2 u_\alpha(k) + f_\alpha(k)$$

$$M_{\alpha\beta\gamma}(k) = \frac{i}{2}\left[k_\beta P_{\alpha\gamma}(k) + k_\gamma P_{\alpha\beta}(k)\right]$$

A Different Point of View

- Self -consistent calculation in Fourier space and use the DeDominicis-Martin parameter ‘y’ for setting up a perturbative evaluation of integrals. The form of the stochastic forcing is a convenient tool for that.
- What are the quantities which one should calculate. The rate of transfer of energy across the surface of a sphere of radius ‘k’ is clearly one
- Another is the old Heisenberg-Chandrasekhar picture – the eddy viscosity which is related to the previous point
- The energy in the range 0 to ‘k’

$$\mathrm{E}(k) = \int_0^k \frac{d^3 p}{(2\pi)^3} \langle u_\alpha(p) u_\alpha(-p) \rangle$$

Eddy viscosity

- Rate of change of energy in 0 < p < k

$$\varepsilon(k)=\partial_t \mathrm{E}(k)=\int d^3p\left[\left\langle \partial_t u(p)u(-p)+u(p)\partial_t u(-p)\right\rangle\right]$$

$$=\int d^3p d^3q[\langle \; M_{\alpha\beta\gamma}u(q)u(p-q)u(-p)+\ldots) \; \rangle]-\int d^3p\eta p^2\left\langle u(p)u(-p)\right\rangle$$

$$=T(k)-\eta\int p^2E(p)dp$$

- T(k) is the transfer term due to nonlinearity and in analogy with the viscous term write it as

$$T(k)=\eta_e\int_0^k p^2E(p)dp$$

eddy viscosity and the transfer rate

- The two key players have been identified
- The dynamics is

$$(-i\omega+\eta k^2)u_\alpha(k,\omega)=f(k,\omega)+M_{\alpha\beta\gamma}(k)\int d^3p\,d\omega' u_\beta(p,\omega')u_\gamma(k-p,\omega-\omega')$$

- The effective viscosity is clearly the self energy obtained by studying the propagator
- The correlation function is the energy spectrum
- The transfer rate is related to self energy and correlation function (energy spectrum)
- For Kolmogorov

$$\varepsilon(k)=T(k)$$

Perturbation theory niceties

$$G_0^{-1} = -i\omega + \eta k^2$$

$$G^{-1}(k,\omega) = -i\omega + \eta k^2 + \Sigma(k,\omega)$$

- In the Kolmogorov range dominant $\Sigma(k,0) = \eta_e k^2$
- Arrange perturbation theory by self consistent loops- there is no small parameter really
- This is where the parameter ‘y’ in the external stirring force correlator becomes important.
- It orders the perturbation theory

Perturbation theory

- Assume 'y' is small
- The one loop contribution starts with a leading term of 1/y

$$\Sigma(k,\omega) = \int d^D p\, d\omega' M(k) M(p) G(\omega', p) C(\omega - \omega')$$

- Correlator structure is $G \langle ff \rangle G^*$
- zero frequency self energy $\Sigma(k) \sim k^z$
- Self –consistency ensures $z = 2 - \dfrac{y}{3}$

- The integral has an ultraviolet pole at y=0

Perturbation theory parameter

- One needs the amplitudes as well . The frequency scale is set by

$$\Sigma(k) = \Gamma_0 k^{2-\frac{y}{3}}$$

- The correlator : $\int C(k,\omega)\frac{d\omega}{2\pi} = \frac{C_0}{k^{D-4+y+z}}$
- The ratio

$$\Gamma_0^2 / C_0 = \frac{A}{y} + O(1)$$

- The two loop carries an extra (above the one loop) factor C_0 / Γ_0^2

- That drives the loop ordering in 'y'

The infrared problem is not real

- What is it ? Consider the integral for eddy viscosity

$\Sigma(k)/k^2$

$$\int d^3p\,d\omega \frac{C(p,\omega)}{-i\omega+\Sigma(k-p,-\omega)} M^2$$

- Focus on the contribution from p<<k : M is now like k, the self energy is big and out of the integral, the frequency integral gives the equal time correlator and the momentum integral diverges

Infrared cure

- The zero frequency self-energy has a divergence at small wave-numbers. Dynamic scaling says that the zero wave-vector self energy will have a small frequency divergence. The correlator at low wave number will be screened by the frequency of the response function and remove the divergence.

A recent result on the Batchelor spectrum of passive scalar turbulence

- Bedrossian, Blumenthal and Punshon –Smith
- Mathematically rigorous proof of Batchelor's prediction that passive scalars advected in fluids at finite Reynolds number with a small diffusivity should display a $|k|^{-1}$ power spectrum
- D=2 Navier Stokes and D=3 hyper-viscous Navier Stokes
- Forcing white in time regular in space
- Scalar subjected to white noise

THE STABLY STRATIFIED FLUID- BOLGIANO ,OBUKHOV SCALING

- Density gradient caused by temperature gradient $\Delta T / d$
- Navier Stokes equation has the buoyancy force in a Boussinesque approximation where incompressibility is maintained after the relaxation to include the buoyancy
- Temperature fluctuations $\theta = \delta T / \Delta T$ follow the usual heat diffusion equation

$$\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p - Ri\theta\delta_{\alpha 3} + \eta \nabla^2 u_\alpha + f_\alpha$$

$$\partial_\alpha u_\alpha = 0$$

$$\partial_t \theta + u_\alpha \partial_\alpha \theta = \lambda \nabla^2 \theta + \frac{u_3}{d} + h$$

THE ENERGY BALANCE

$$E = \frac{1}{2}\int d^3r[u_\alpha u_\alpha + Ri\theta^2]$$

$$\partial_t E = -\int d^3r[\eta(\partial u)^2 + Ri\theta^2] + \int d^3r[f_\alpha u_\alpha + h\theta]$$

Stably stratified has all signs always working right. Energy flows from large length scales to short length scales and it is very much like Kolmogorov. But there is a new situation- the kinetic energy spectrum may be dominated by a thermal energy flow.

This is the pure Bolgiano-Obukhov limit.

THE BOLGIANO LIMIT

- The spectrum is always defined by $K = \int E(k)dk$ where K is the kinetic energy
- The total energy in the sphere of radius 'k' is

$$\mathrm{E}(k) = \int_0^k d^3 p \frac{1}{2}\left[u_\alpha(p)u_\alpha(-p) + Ri\theta(p)\theta(-p)\right]$$

The rate of change which is the transfer rate to momenta beyond 'k' is

$$\varepsilon(k) = \partial_t \mathrm{E}(k) = \int d^3 p\left[u_\alpha(-p)\partial_t u_\alpha(p) + Ri\theta(-p)\partial_t\theta(p)\right]$$

The Bolgiano limit is " drop the kinetic energy flux" in the above and Kolmogorov is as before assume the transfer k- independent

Kolmogorov Argument

- The dynamics of the velocity field is slaved to the temperature in the Bolgiano limit
- This is what determines the “dimension “ of the temperature field as L/T^2
- The energy transfer rate now has the dimension L^2/T^5
- Dimensional analysis now yields the spectrum

$$E(k) = B_0 \varepsilon^{2/5} k^{-11/5}$$

- Universal spectrum once again and a universal pre-factor

A Scalar Model for Bolgiano limit ?

$$\partial_t u_\alpha(k,t) + \eta k^2 u_\alpha(k,t) = -Ri\left(\delta_{\alpha 3} - \frac{k_\alpha k_3}{k^2}\right)\theta(k,t)$$

$$(\partial_t + \lambda k^2)\theta(k,t) = h(k,t) - i\int p_j u_j(k-p)\theta(p) d^3 p$$

- The transfer integral

$$T_\theta(p) = \left\langle \theta(-p)\int p_j u_j(p-q)\theta(q) d^3 q \right\rangle$$

THE SCALAR MODEL WAVE NUMBER FREQUENCY SPACE

- We use Fourier space and write the slaving of the velocity field to the temperature field as

$$(-i\omega+\eta k^2)u_\alpha = -Ri\left(\delta_{\alpha 3}-\frac{k_\alpha k_3}{k^2}\right)\theta$$

- The dynamics of the temperature field is

$$h(k,\omega)= (-i\omega+\lambda k^2)\theta(k)+$$

$$+ik_\alpha\int[-i(\omega-\omega')+\eta(k-p)^2]^{-1}\theta(k-p,\omega-\omega')\theta(p,\omega')d^3p d\omega'$$

Transfer Integrals and scaling laws

- The correlation of $h(k,\omega)$ is exactly as in the DeDominicis – Martin set up
- The transfer integral is k-independent for y=4
- The self consistent one-loop answer for the relaxation rate gives the dynamic exponent z=2/5
- The universal amplitude ratio
- The thermal energy spectrum $E_\theta(k) \propto k^{-7/5}$ defined as

$$\int \frac{1}{V} d^3r \left\langle \theta^2(r) \right\rangle = \int E_\theta(k)dk$$

- The Kinetic Energy spectrum gives Bolgiano –Obukhov

The Crossover

- We have looked at Kolmogorov spectrum and Bolgaiano spectrum separately focussing on the two extremes

  $$Ri = 0 \qquad\qquad\qquad Ri >> 1$$

  The passage from one to the other can be accomplished by exploiting a crossover trick found by Heisenberg and Chandrasekhar (1950)

  The result is for any given computation of the energy spectrum $E(k)$

$$E(k) \propto k^{-5/3}, k < k_1 \;\; k_1 \propto Ri^{-3/4} \;\; E(k) \propto Ri^{-11/5}, k >> k_2 \;\; k_2 \propto Ri^{-5/8}$$