VARIANCE ESTIMATES for GEOMETRIC COUNTING PROBLEMS

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CAST of CHARACTERS:

$\mathcal{M}$ = moduli space / param. space

of geometric objects.

a map $\quad \mathcal{M} \longrightarrow \Sigma(\mathbb{R}^n)$

$\Sigma(\mathbb{R}^n)$ = locally finite atomic meas. on $\mathbb{R}^n$. [loc. fin collections of pts.]

M = moduli space / param. space

of geometric objects.

a map $M \longrightarrow \Sigma(\mathbb{R}^n)$

$m \in M \longmapsto \Lambda_m \subset \mathbb{R}^n$

(point set in $\mathbb{R}^n$).

Counting problems: As m varies in M

$N(m, R) = \# \Lambda_m \cap B(0, R)$

Character #3: a "natural" prob. measure μ on M, giving us a notion of a "random" object in M.

- Think of picking m from M at random using μ.

What can we say about the random (sequence of) variables $N(m, R) = \#(\Lambda_m \cap B(0,R))$?

what can we say about the random variables $N(\mu, R) = \#(\Lambda_\mu \cap B(0,R))$?

Q: • Can we understand expectation?

• Can we understand VARIANCE?

• Can we compare the push-forward of μ to $\Sigma(\mathbb{R}^n)$ to "natural" point processes in $\mathbb{R}^n$? (like Poisson processes)

[Veech ; Marklof]

PERFORMANCE 1 THE SPACE of LATTICES.

$M = X_n = SL_n\mathbb{R}/SL_n\mathbb{Z}$ the space of unimodular lattices in $\mathbb{R}^n$:

$m = g\mathbb{Z}^n \subset \mathbb{R}^n$ $\Lambda_m = g\mathbb{Z}^n_{prim}$

μ Haar prob. meas. on X_n.

$R > 0$, $N(m, R) = \# g\mathbb{Z}^n_{prim} \cap B(0, R)$

Siegel:

$$SL_n\mathbb{R}/N \cap \Gamma$$

$$\mathbb{R}^n \setminus \{0\} \qquad \cancel{X_n}$$

$$\int_{X_n} N(m, R)\, d\mu(m)$$

$$\boxed{\begin{aligned}\hat{f}(m) &= \int f \, d\Lambda_m \\ &= \sum_{v \in \Lambda_m} f(v)\end{aligned}}$$

$$= \frac{1}{\zeta(n)} \operatorname{vol} B(0, R)$$

[EXPECTATION for a random lattice]

$$\left[\int \hat{f} \, d\mu_n = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f \right]$$

Functionals: $f \mapsto \int \hat{f}$ is a $SL_n\mathbb{R}$-inv. linear functional on $C_c(\mathbb{R}^n)$

$\longrightarrow$ comes from integration against a $SL_n\mathbb{R}$-inv. meas. on $\mathbb{R}^n$

PACKING & COVERING.

VARIANCE?

$$\int_{X_n} N(\Lambda, R)^2 \, d\mu(\Lambda).$$

$$\shortparallel$$

Rogers $(n \geq 3)$: $\left(\frac{1}{\zeta(n)} \operatorname{vol} B(0,R)\right)^2$

(Fairchild, $n=2$) + lower order terms.

__APPLICATION__ motivated by a problem in dynamics of unipotent flows on X_n,

__A. - Margulis__ $\mu\left(\Lambda \in X_n : \Lambda \cap A = \emptyset\right)$

$A \subset \mathbb{R}^n$ mble $\leq C_n / \mathrm{vol}(A)$.

("concentration inequality").

... values of polynomials at lattice pts.

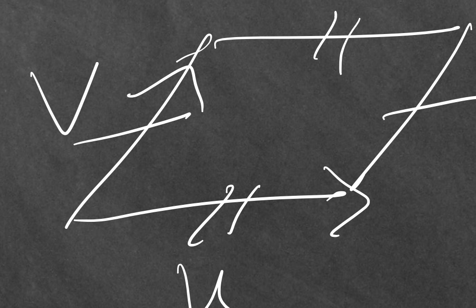
... Oppenheim ...

PRODUCTION #2 :

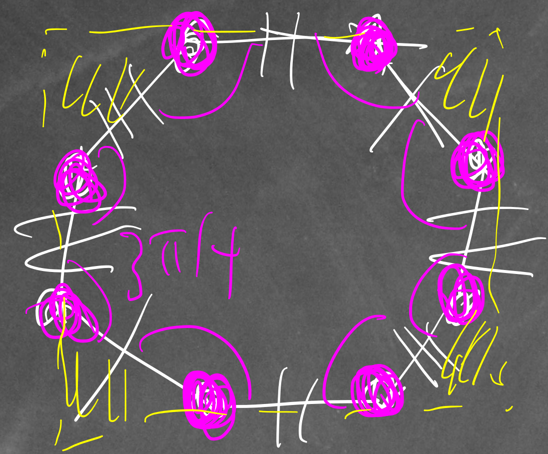
$\mathcal{M}$ = moduli space of TRANSLATION SURFACES

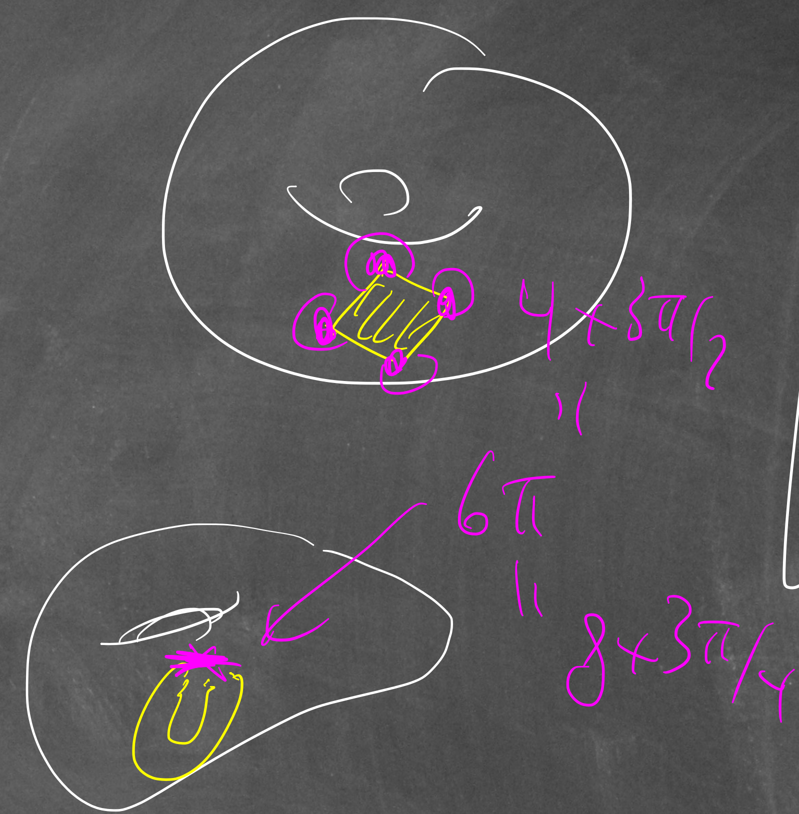
(another generalization of $X_2 = SL_2\mathbb{R}/SL_2\mathbb{Z}$)

$$g\, SL_2\mathbb{Z} \longleftrightarrow g\mathbb{Z}^2 \longleftrightarrow \mathbb{R}^2/g\mathbb{Z}^2 \qquad g = (u\ v)$$



→ take a POLYGON w/ parallel sides id'd by translation to form a cpct Riem surface.





exercise:

hexagon $\to$ torus.

$\mathbb{C}^4/\mathbb{R}^\times \simeq M.$

ω canon. form

genus g RS + hol. 1-form dz

$[d(z+c) = dz].$

SADDLE CONNECTIONS. γ.

traj. connecting pts w/ too much angle.

$$Z_\gamma = \int_\gamma \omega \in \mathbb{C}$$

$4g$-gon $\longrightarrow$ g surface
one pt angle $2\pi(2g-1)$

$4g+2$-gon $\longrightarrow$ genus g surface
2 pts w/ angle $2\pi g$ each.

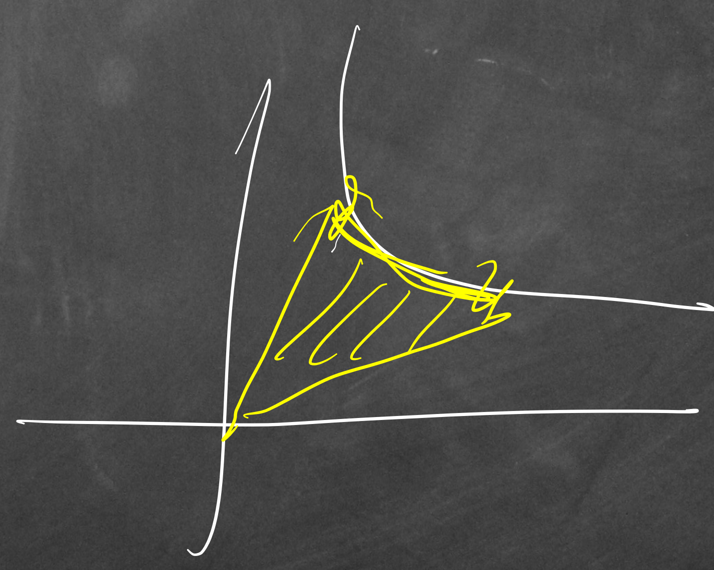
$\mathcal{M}$ = space of TS w/ fixed orders of excess angle (of zeros of 1-form)

$$\Lambda_{\mathcal{M}} = \{ Z_\gamma : \gamma \text{ s.c. in } \mathcal{M} \}.$$

μ ?

Masur-Smillie-Veech measure.

$SL_2\mathbb{R}$-invariant [Masur, Veech: finite, ergodic].



area 1 surface

area $\to 0$ surface

"$n = \sqrt{2}$"

$m \longmapsto \Lambda_m$ $\qquad$ $N(m,R) = \# \Lambda_m \cap B(0,R)$

EXPECTATION: $\displaystyle \int_M N(m,R)\, d\mu \stackrel{?}{=}$

Thm (Veech)
$$\int_M N(m,R)\, d\mu = c_M \pi R^2.$$

VERY INTERESTING

$$\left[f \in C_c(\mathbb{C}),\ \hat{f}(m) = \sum_{z \in \Lambda_m} f(z),\ \int_M \hat{f}\, d\mu = c_M \int_{\mathbb{C}} f. \right]$$

VARIANCE?

A. – CHEUNG – MASUR.

$N(m, R)$ has finite variance

$\left(\hat{f} \in L^{2+\kappa_M}(M, \mu_{MSV}) \text{ for } f \in C_c(\mathbb{C}) \right]$.

[Bonnafoux to all $SL_2\mathbb{R}$-inv. meas on M, Dozier].

APPLICATION: A. – FAIRCHILD – MASUR used ACM + ergodic theory (Nevo, Eskin-Masur) COUNT PAIRS.

Nevo-Rohr-Weiss can be combined w/
our finite variance ideas

EFFECTIVE COUNTING OF PAIRS
(BONNAFOUX).

PERFORMANCE #3

$g \geq 2$

$M = M_g$ the moduli space of HYPERBOLIC genus g surfaces.

[also a generalization of $SO(2) \backslash X_2 = \mathbb{H}^2 / SL_2\mathbb{Z}$]

space of const-curv. metrics on a genus 1 surface]

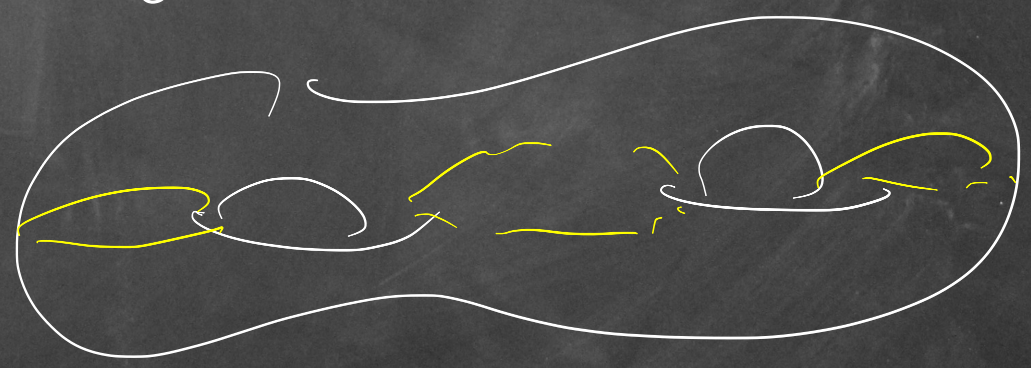
$\mathcal{M} = \mathcal{M}_g$ = moduli space of hyp. surfaces.

$$m \in \mathcal{M} \longmapsto \Lambda_m \subset \mathbb{R}^+$$

$$= \left\{ \ell_\gamma(m) : \begin{array}{l} \gamma \text{ s.c.g.} \\ \text{on } m \end{array} \right\}.$$

$$N(m, R) = \#\{\gamma \text{ s.c.g.} : \ell_\gamma(m) \le R\}.$$

μ on $\mathcal{M}_g$ is WP volume on $\mathcal{M}_g$.



WOLPERT: $\prod d\ell_i \wedge d\tau_i$

MIRZAKHANI: $N(m, R)$ has

FINITE expectation & the expectation is a POLYNOMIAL of degree $6g-6$ in R, and the coeff can be computed in terms of intersection #'s on M_g.

MIRZAKHANI: $N(m, R) \in L^{2-\varepsilon}(M, \mu)$

$\notin L^{2+\varepsilon}(M, \mu)$

$\forall \varepsilon > 0$.

A. — ARANA-HERRERA:

$$N(m, R) \in L^2(M, \mu)$$

We USED this to get COUNTING RESULTS for PAIRS of curves on surfaces.

A — AH — Bell:

$$\left\{ \frac{i(\gamma_1, \gamma_2)}{\ell(\gamma_1)\ell(\gamma_2)} : \begin{array}{c} \gamma_1, \gamma_2 \\ \text{s.c.c.} \\ \text{on } \text{surf.} \\ \ell \le R \end{array} \right\}.$$