Symmetric closed subsets of real affine root systems and regular subalgerbas

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In this talk I shall discuss about the classification of symmetric closed subsets of real affine root systems and their correspondence with the regular subalgebras of affine Lie algebras generated by them. This is a joint work with Dipnit Biswas and Venkatesh R.

Finite and affine Lie algebras and their root systems

- $\mathring{\mathfrak{g}}$(resp. $\mathfrak{g}$)$-$ finite dimensional semi-simple Lie algebra (resp. affine Lie algebra).
- $\mathring{\mathfrak{h}}$(resp.$\mathfrak{h}$)$-$ a Cartan subalgebra of $\mathring{\mathfrak{g}}$(resp. $\mathfrak{g}$).
- $\mathring{\Phi}$(resp. Φ)$-$ the set of all roots (resp. real roots) of $\mathring{\mathfrak{g}}$ (resp. $\mathfrak{g}$).
- $\Delta-$ the set of roots of $\mathfrak{g}$.
- $\mathring{W}$(resp.W)$-$ the Weyl group of $\mathring{\Phi}$ (resp. Φ).
- Irreducible finite root systems are classified in terms of their Dynkin diagrams. They are of type $A, B, C, D, E_{6,7,8}, F_4, G_2$.

Closed subroot systems and regular subalgebras

Definition

A subset $\mathring{\Psi}$ of $\mathring{\Phi}$ is called

- a symmetric subset if $\mathring{\Psi} = -\mathring{\Psi}$.
- a subroot system if $s_\alpha(\beta) \in \mathring{\Psi}$ for all $\alpha, \beta \in \mathring{\Psi}$.
- a closed subset if $\alpha, \beta \in \mathring{\Psi}$ and $\alpha + \beta \in \mathring{\Phi}$ implies $\alpha + \beta \in \mathring{\Psi}$.
- a closed subroot system if $\mathring{\Psi}$ is a closed subset and a subroot system.

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Definition

A subalgebra $\mathring{\mathfrak{s}}$ of $\mathring{\mathfrak{g}}$ is called a regular subalgebra if

$$\mathring{\mathfrak{s}} = (\mathring{\mathfrak{h}} \cap \mathring{\mathfrak{s}}) \oplus \bigoplus (\mathring{\mathfrak{g}}_\alpha \cap \mathring{\mathfrak{s}}).$$

An example

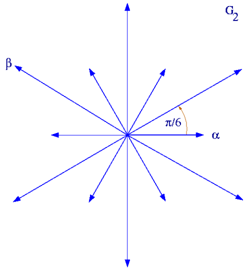

Figure: finite root system of type G_2

The set of short roots $\mathring{\Phi}_s$ is a non-closed subroot system but the set of long roots $\mathring{\Phi}_\ell$ is a closed subroot system of $\mathring{\Phi}$.

Known results in finite dimensional theory

1. There is a one to one correspondence between the subroot systems of $\mathring{\Phi}$ and the subgroups of $\mathring{W}$ which are generated by reflections and the correspondence is given by

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2. Any symmetric closed subsets of $\mathring{\Phi}$ is a closed subroot system of $\mathring{\Phi}$.
3. There is a one to one correspondence between the closed subsets of $\mathring{\Phi}$ and the regular subalgebras of $\mathring{\mathfrak{g}}$ and the correspondence is given by

$$\mathring{\Psi} \longmapsto \langle \mathring{\mathfrak{h}}, \mathring{\mathfrak{g}}_\alpha : \alpha \in \mathring{\Psi} \rangle.$$

Definitions in the affine setting

Definition

A subset $\Psi \subseteq \Phi$ is called a

- (real) subroot system of Δ if $s_\alpha(\beta) \in \Psi$ for $\alpha, \beta \in \Psi$.
- (real) closed subset of Δ if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Psi$.

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Definition

For a symmetric subset Ψ of Φ we define $\mathfrak{g}(\Psi) := \langle \mathfrak{g}_\alpha : \alpha \in \Psi \rangle$.
A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is called regular if $\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{s} \cap \mathfrak{g}_\alpha)$.

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Recently in 2019, Roy and Venkatesh studied the (maximal) closed subroot systems and their correspondence with regular subalgerbas.

Subroot systems of affine root systems

For a subroot system Ψ of Φ define

$$Gr(\Psi) := \{\alpha \in \mathring{\Phi} : \alpha + r\delta \in \Psi \text{ for some } r \in \mathbb{Z}\},$$

$$Z_\alpha(\Psi) := \{r \in \mathbb{Z} : \alpha + r\delta \in \Psi\}, \quad \alpha \in \mathring{\Phi}.$$

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The next Proposition has proved by Dyer and Lehrer in 2011 for untwisted affine root systems and has been generalized by by Roy and Venkatesh in 2019.

Proposition

Let Ψ be a subroot system of an affine root system Φ. Then there exists a function $p^\Psi : Gr(\Psi) \to \mathbb{Z}/2$ and non-negative integers $n_\alpha^\Psi, \alpha \in Gr(\Psi)$ such that

$$Z_\alpha(\Psi) = p^\Psi(\alpha) + n_\alpha^\Psi \mathbb{Z},$$

Moreover p^Ψ is $\mathbb{Z}$-linear if $Gr(\Psi)$ is reduced and we have $n_\alpha^\Psi = n_{w\alpha}^\Psi, \ w \in \mathring{W}$.

Classification of closed subroot system

Roy and Venkatesh classified the closed subroot systems of affine root systems in 2019.

Proposition

Let Ψ be an irreducible closed subroot system of a reduced affine root system Φ. Then there exists a $\mathbb{Z}$-linear function $p : Gr(\Psi) \to \mathbb{Z}$ and a non-negative integer n such that Ψ is one of the following form:

- $m|n$ and
$$\Psi = \{\alpha + (p_\alpha + n\mathbb{Z})\delta : \alpha \in Gr(\Psi)\},$$
- $m \nmid n$ and
$$\Psi = \{\alpha + (p_\alpha + n\mathbb{Z})\delta : \alpha \in Gr(\Psi)_s\} \cup \\ \cup \{\alpha + (p_\alpha + mn\mathbb{Z})\delta : \alpha \in Gr(\Psi)_\ell\}$$

The following has been proved by Roy and Venkatesh in 2019.

Proposition

The map

$$\left\{ \begin{array}{c} \text{Set of all closed subroot} \\ \text{systems of } \Phi \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Set of all regular} \\ \text{subalgebras of } \mathfrak{g} \end{array} \right\}$$

$$\Psi \longmapsto \mathfrak{g}(\Psi)$$

defines an injection from the real closed subroot systems of Δ and the regular subalgebras of $\mathfrak{g}$

Classification for the reduced case

Let Ψ be a symmetric closed subset of a reduced affine root system Φ and let $\Psi = \Psi_1 \sqcup \cdots \sqcup \Psi_r$ be the decomposition of Ψ into irreducible components.

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Result (Dipnit,I-,Venkatesh'23)

There exists a $\mathbb{Z}$-linear function $p : Gr(\Psi) \to \mathbb{Z}$ such that $p_\alpha \in Z_\alpha(\Psi),\ \forall \alpha \in Gr(\Psi)$ and a non-negative integer n such that Ψ is one of the following forms:

1. $\Psi_i = \{\alpha + (p_\alpha + n\mathbb{Z})\delta : \alpha \in Gr(\Psi_i)\}$
2. $\Psi_i = \{\alpha + (p_\alpha + n\mathbb{Z})\delta : \alpha \in Gr(\Psi_i)_s\} \cup \{\alpha + (p_\alpha + mn\mathbb{Z})\delta : \alpha \in Gr(\Psi_i)_\ell\}$
3. $\Psi_i \cap \Phi^+ = \{\epsilon_i + (p_{\epsilon_i} + A_i)\delta : i = 1, 2\} \cup \{\alpha + (p_\alpha + n\mathbb{Z})\delta : \alpha \in \mathring{\Phi}_\ell^+\}$ where $A_i = n_\ell\mathbb{Z} \cup (a_i + n_\ell\mathbb{Z})$, $n_\ell \in 2\mathbb{Z}_+$, $0 \leq a_1, a_2 < n_\ell$ such that $a_1 + a_2 \equiv 0 \pmod{n_\ell}$ and both a_1, a_2 are odd.

First theorem

Theorem (Dipnit,I-,Venkatesh, 2023)

Let Ψ be a symmetric closed subset of an *untwisted* affine root system. Then Ψ is a closed subroot system.

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Remark

If any irreducible component of Ψ is of type (3) in the list above, then Ψ is a closed subroot system if and only if $a_1 = a_2 = n/2$. In particular, any symmetric real closed subsets of real affine root systems need not be a closed subroot system.

Regular subalgebras generated by symmetric closed subsets

$\mathcal{C}_{\text{sym}}(\mathfrak{g})-$ the set of symmetric closed subsets of Φ.
$\mathcal{R}(\mathfrak{g}) = \{\mathfrak{g}(\Psi) : \Psi \in \mathcal{C}_{\text{sym}}(\mathfrak{g})\}-$ the set of all regular subalgebras of $\mathfrak{g}$ generated by the symmetric closed subsets of Φ. Define a map

$$\iota_{\mathfrak{g}} : \mathcal{C}_{\text{sym}}(\mathfrak{g}) \to \mathcal{R}(\mathfrak{g})$$

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As we mentioned earlier, the restriction of $\iota_{\mathfrak{g}}$ to the set of closed subroot systems is injective.

Is this map injective? If not, then what is the pre-image $\iota_{\mathfrak{g}}^{-1}(\mathfrak{g}(\Psi))$?

A counter-example

Unlike in the finite case, the map $\iota_{\mathfrak{g}}$ is not injective for any $\mathfrak{g}$, even if we restrict it to symmetric closed subsets of Φ.

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Example

Let $\mathfrak{g}$ be any affine Lie algebra not of type $A_{2n}^{(2)}$ and α be a short root in $\mathring{\Phi}$. Consider two symmetric, real closed subsets Ψ_1, Ψ_2 of Φ defined by

$$\Psi_1 := \{\alpha+\delta, -\alpha+\delta, -\alpha-\delta, \alpha-\delta\}, \;\; \Psi_2 := \{\alpha+3\delta, \alpha+\delta, -\alpha-3\delta, -\alpha-\delta\}.$$

Then we have

$$\mathfrak{g}(\Psi_1) = \mathfrak{g}(\Psi_2) = \bigoplus_{r\in\mathbb{Z}} \mathfrak{g}_{\pm\alpha+(2r+1)\delta} \oplus \bigoplus_{r\in\mathbb{Z}} \mathbb{C}\alpha^\vee \otimes t^{2r}.$$

The correspondence

Proposition (Dipnit,I-,Venkatesh, 2023)

Suppose $\mathcal{S}$ is a subclass of the set of symmetric closed subsets of Φ containing the closed subroot systems such that the restriction of $\iota_{\mathfrak{g}}$ is injective, then $\mathcal{S}$ must be the set of all closed subroot systems of Φ.

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Corollary

Let Ψ be a symmetric closed subset of Φ. Then $\Delta(\mathfrak{g}(\Psi)) \cap \Phi = \Psi$ if and only if Ψ is a closed subroot system of Φ.

Inverse images

The next Lemma provides the inverse image of $\mathfrak{g}(\Psi)$ when Ψ is given by (2) in the classification list.

Lemma (Dipnit,I-,Venkatesh 2023)

Let Ψ be an irreducible closed subroot system of Φ given by (2) in the classification list.

- For any given positive integer r, there are exactly $\varphi(2r)$ symmetric, real closed subsets Ψ' in $\iota_{\mathfrak{g}}^{-1}(\mathfrak{g}(\Psi))$ such that $n_\ell(\Psi') = 2rn$, where φ is the Euler's totient function.
- For a fixed n and r, all $\Psi' \in \iota_{\mathfrak{g}}^{-1}(\mathfrak{g}(\Psi))$ with $n_\ell(\Psi') = 2rn$ is of the form (2) or (3) and a_1 is a cyclic generator of the group $\langle n \rangle$ in $\mathbb{Z}/(2rn)\mathbb{Z}$.
- $\iota_{\mathfrak{g}}^{-1}(\mathfrak{g}(\Psi))$ is infinite.

Injectivity of the map

Proposition (Dipnit, I-,Venkatesh 2023)

For any irreducible real closed subroot system Ψ of Φ, we have

$$\iota_{\mathfrak{g}}^{-1}(\mathfrak{g}(\Psi)) = \{\Psi\}$$

if one of the following holds

- $Gr(\Psi)$ is not of type A_1, B_2.
- if $Gr(\Psi) = B_2$ and Ψ is given by (1) in the classification list.

And $\iota_{\mathfrak{g}}^{-1}(\mathfrak{g}(\Psi))$ is infinite for all other cases.

A few questions

- Does the statement '$\Delta(\mathfrak{g}(\Psi)) \cap \Delta^{\text{re}} = \Psi$ if and only if Ψ is a real closed subroot system' hold for Kac-Moody algebras or Extended affine Lie algebras? (For rank 2 Kac-Moody Lie algebras it is known to be true).

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- What are all the imaginary roots in $\Delta(\mathfrak{g}(\Psi))$? (we don't know the answer even for rank 2 Kac-Moody algebras).

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- What are all the imaginary roots in $\Delta(\mathfrak{g}(\Psi))$? (we don't know the answer even for rank 2 Kac-Moody algebras).
- Any geometric description of maximal closed subroot systems in affine root systems?

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Thank You for your attention!

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Questions?