Maths Exploration Sheet: Power of an Invariant

1. Introduction

This exploration sheet is based on problem-solving techniques from "Power of an Invariant". You will explore a sequence of activities, record your observations, try out examples, and look for persistent patterns or "invariants" that remain unchanged in the process.

2. Activities

Activity 1: Sequences and Invariant Products

Background:

Start with a point (a, b) in the plane with $0 < b < a$. Generate a sequence (x_n, y_n) by:

$$x_0 = a,\ y_0 = b$$
$$x_{n+1} = \frac{x_n + y_n}{2}$$
$$y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

Instructions:

(a) Choose a, b (e.g. $a = 4$, $b = 1$).

(b) Fill out several steps in the table below.

n	x_n	y_n	$x_n y_n$	$x_n - y_n$
0				
1				
2				
3				

Questions:

- What stays invariant in this process?
- Do x_n and y_n approach each other?

Activity 2: Reducing a List by Subtraction

Background:

Write numbers 1 to $2n$ (for odd n) on the board. At each step, pick any two numbers a, b, erase both, and write $|a - b|$ instead.

Instructions:

- Try the process for $n = 3$ $(1, 2, 3, 4, 5, 6)$. Record each step.
- Try for even n. What changes?

Record your process:

Step	Numbers Remaining
0	
1	
2	
3	
...	

Questions:

- What can you say about the parity (odd/even) of the final number?

Activity 3: Equalizing Numbers in a Circle

Background:

A circle is divided into six sectors labeled $1, 0, 1, 0, 0, 0$. At each step, you may increase *two neighboring numbers* by 1.

Instructions:

- List the numbers after each move.
- Try to make all sectors equal.

- Try with other initial arrangements (e.g., $2, 0, 0, 0, 0, 0$).

Step	Arrangement
0	
1	
2	
...	

Questions:

- Is equalization possible? Why or why not?
- Can you identify any invariant quantities?

Activity 4: Grouping with Constraints

Background:

A parliament has members, each with at most three enemies. Can you split the members into two houses so each has at most one enemy in their own house?

Instructions:

- Model the problem for 6 people; draw links for "enemies".
- Try to divide into two groups as above.

(Draw your model below)

Questions:

- What method helps in successful division?
- What (if anything) remains invariant as you swap members between groups?

Activity 5: Iterated Differences – Shrinking Squares

Background:

Given four positive integers (a, b, c, d), repeatedly replace with $(|a - b|, |b - c|, |c - d|, |d - a|)$.

Instructions:

- Start with an example like $(0, 3, 10, 13)$ and record the process.
- Try another quadruple; record.

Step	Tuple
0	
1	
2	
3	
...	

Questions:

- What do you observe about the process?
- Is there any invariant or does some quantity always decrease?

3. Reflection & Further Questions

- What patterns or invariants did you discover in each activity?
- Did you notice anything surprising?
- Can you create your own mathematical experiment or puzzle that uses invariance?
- With which activity did you have the most difficulty? Why?

4. For the Curious

- Try inventing a new "invariance game" inspired by these activities.
- Share your findings, puzzling cases, or conjectures with someone else.

5. Teacher/Facilitator Notes

- Allow learners to experiment on paper or physically (e.g. counters, cards).
- Encourage precise recording and comparison.
- Prompt students to verbalize or write their conjectures about invariants.
- Guide exploration but allow time for frustration, experimentation, and "aha" moments.