p-adic Artin L-function over a CM-field

joint work with Takashi Hara (Tsuda University)

Tadashi Ochiai

Graduate School of Science, Osaka University

August 16, 2022

Table of contents

Main reference

[Gr83] R. Greenberg, On p-adic Artin L-functions, Nagoya Math. J. 89 (1983), pp. 77–87.

[Gr04] R. Greenberg, On p-adic Artin L-functions II, Iwasawa theory 2012, pp. 227–245, Contrib. Math. Comput. Sci., 7, Springer, Heidelberg, 2014.

[HO18] T.Hara and T.Ochiai, The cyclotomic Iwasawa main conjecture for Hilbert cusp forms with complex multiplication, Kyoto J. Math., 58, No. 1 (2018), 1–100.

[HO22] T.Hara and T.Ochiai, p-adic Artin L-Function over CM fields, in preparation.

§1 Artin L-function over $\mathbb{C}$

Hecke L-function

η : algebraic Hecke character $\mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$ of finite order on a number field K

We define the L-function of η to be

$$L(\eta, s) = \prod_{\lambda} \frac{1}{1 - \eta_\lambda(\varpi_\lambda)\mathrm{Norm}(\lambda)^{-s}}$$

where λ runs through the prime ideals of $\mathcal{O}_K$ which are unramified primes of η and ϖ_λ is a uniformizer of the prime λ.

- The Euler product defining $L(\eta, s)$ is absolutely convergent on the region $\mathrm{Re}(s) > 1$.

  (Hence $L(\eta, s)$ is holomorphic and non-zero on the region $\mathrm{Re}(s) > 1$)

- We have the functional equation $\Lambda(\eta, s) = \epsilon(\eta, s)\Lambda(\eta^\vee, 1 - s)$

  where $\Lambda(\eta, s) = L(\eta, s) \times$ (gamma factor) and $\epsilon(\eta, s)$ is the epsilon factor.

- $L(\eta, s)$ is holomorphically continued to the whole $\mathbb{C}$-plane if η is non-trivial.

- $L(\eta, s)$ coincides with the Dedekind zeta function $\zeta_K(s)$ if η is trivial.

Artin L-function

ρ : continuous representation $G_K \longrightarrow GL_n(\overline{\mathbb{Q}}) \cong \mathrm{Aut}_{\overline{\mathbb{Q}}}(V_\rho)$

We define the Artin L-function of ρ to be

$$L(\rho, s) = \prod_{\lambda} \frac{1}{\det(\mathrm{Id} - \mathrm{Frob}_\lambda \mathrm{Norm}(\lambda)^{-s}; V_\rho^{I_\lambda})}$$

where λ runs through the set of prime ideals of $\mathcal{O}_K$.

- The Euler product defining $L(\rho, s)$ is absolutely convergent on the region $\mathrm{Re}(s) > 1$.

  (Hence $L(\rho, s)$ is holomorphic and non-zero on the region $\mathrm{Re}(s) > 1$)

- We have the functional equation $\Lambda(\rho, s) = \epsilon(\rho, s)\Lambda(\rho^\vee, 1 - s)$

  where $\Lambda(\rho, s) = L(\rho, s) \times$ (gamma factor) and $\epsilon(\rho, s)$ is the epsilon factor.

- When ρ is of dimension 1, Artin L-function can be regarded as a Hecke L-function

  by the global class field theory.

- For two Artin representations ρ and ρ', we have $L(\rho \oplus \rho', s) = L(\rho, s)L(\rho', s)$.

By the last remark, any Artin L-function is a product of Artin L-functions associated to irreducible Artin representations. Let us recall the following classical conjecture.

Classical Conjecture

Let ρ be a non-trivial irreducible Artin representation of the absolute Galois group of a finite algebraic number field K. Then $L(\rho, s)$ is holomorphically continued to the whole $\mathbb{C}$-plane.

Recall a classical result as follows.

Theorem 1 (Application of Brauer's induction)

Let ρ be an Artin representation of the absolute Galois group of a finite algebraic number field K. Then $L(\rho, s)$ is meromorphically continued to the whole $\mathbb{C}$-plane.

We will recall briefly why Theorem 1 holds.

Brief sketch of the proof of Theorem 1

Let G be a group, H a finite-index subgroup of G.

Let $\eta : H \longrightarrow GL_n(\mathbb{C})$ be a linear representation

Then the induced representation $\mathrm{Ind}_H^G : G \longrightarrow GL_{n[G:H]}(\mathbb{C})$ is defined to be

the representation such that $V_{\mathrm{Ind}_H^G} \cong V_\eta \otimes_{\mathbb{C}[H]} \mathbb{C}[G]$

Theorem 2 (Brauer's induction theorem)

Let $\rho : G \longrightarrow GL_n(\mathbb{C})$ be a representation of a finite group G. Then there exist

- $H_1, \ldots, H_r$*: subgroups of* G
- $\varphi_i : H_i \longrightarrow \mathbb{C}^\times$
- $a_1, \ldots, a_r \in \mathbb{Z}$

such that $\rho \cong \oplus_{i=1}^r a_i \mathrm{Ind}_{H_i}^G \varphi_i$.

Remark 1

If a_i is positive, $\oplus_{i=1}^r a_i \mathrm{Ind}_{H_i}^G \varphi_i$ means the direct sum $\oplus_{i=1}^r (\mathrm{Ind}_{H_i}^G \varphi_i)^{\oplus a_i}$.
But a_i can be negative here.
So, in Theorem 2, we consider $\oplus_{i=1}^r a_i \mathrm{Ind}_{H_i}^G \varphi_i$ as a virtual sum in the Grothendieck group of representations of G.

Let K_ρ be the finite algebraic number field corresponding to the kernel of ρ.

Then ρ factors through the finite quotient

$$G_K \twoheadrightarrow \mathrm{Gal}(K_\rho/K) \to GL_n(\mathbb{C}).$$

By Brauer's induction theorem, there exist

- $K_1, \ldots, K_r$: intermediate fields of K_ρ/K
- Hecke character of finite order φ_i on K_i $(i = 1, \ldots, r)$ whose associated Galois characters $\varphi_i^{\mathrm{gal}} : G_{K_i} \longrightarrow \mathbb{C}^\times$ factors through $\mathrm{Gal}(K_\rho/K_i)$.
- $a_1, \ldots, a_r \in \mathbb{Z}$

such that $\rho \cong \oplus_{i=1}^r a_i \mathrm{Ind}_{G_{K_i}}^{G_K} \varphi_i^{\mathrm{gal}}$

Since $L(\mathrm{Ind}_{G_{K_i}}^{G_K} \varphi_i^{\mathrm{gal}}, s) = L(\varphi_i^{\mathrm{gal}}, s) = L(\varphi_i, s)$,

we have $L(\rho, s) = \prod_{i=1}^{r} L(\varphi_i, s)^{a_i}$

Since $L(\varphi_i, s)$ is known to be meromorphic for every i, $L(\rho, s)$ is meromorphic.
(Q.E.D)

Remark 2

There are some positive results on Conjecture A as applications of modularity theorems. But we do not discuss this subject here.

§2 Review on the algebraicity of special values

Algebraicity of the special values of Hecke L-function

Let ϕ be a non-trivial algebraic Hecke character of finite order on a number field K.

Suppose that K is a totally real number field.

Let $c \in G_K$ the complex conjugate

Then we have the following result on the algebraicity of special values

- $\dfrac{L(\phi, n)}{\pi^{n[K:\mathbb{Q}]}} \in \overline{\mathbb{Q}}^{\times}$ if $n > 1$ and $(-1)^n \phi^{\mathrm{gal}}(c) = +1$.
- $L(\phi, n) \in \overline{\mathbb{Q}}^{\times}$ if $n < 0$ and $(-1)^n \phi^{\mathrm{gal}}(c) = -1$.

The above integers n are critical points of $L(\phi, s)$.
(the precise definition of being critical is omitted)

When K is not totally real, it is known that $L(\phi, s)$ have no critical points.

For example, we have $L(\phi, n) = 0$ for all negative integers n.

When ψ is an algebraic Hecke character of finite order over a number field K which is not totally real, we can not expect the existence of the (one-variable) cyclotomic p-adic L-function $L_p(\psi)$.

However, we may have another kind of ($d+1$-variable) p-adic L-function when K is a CM-field with $[K:\mathbb{Q}]=2d$.

Let K be a CM field with $[K:\mathbb{Q}] = 2d$.

$I_K = \{\iota : K \hookrightarrow \mathbb{C}\}$: the set of embeddings

$\Sigma_K \subset I_K$ is called a CM type if we have $I_K = \Sigma_K \coprod \overline{\Sigma_K}$

We choose and fix a CM type Σ_K of K.

Let η be an algebraic Hecke character $\mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$
(which is not necessarily of finite order)

Then there exists an element $(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{Z}[\Sigma_K] \times \mathbb{Z}[\overline{\Sigma_K}]$ such that $\eta_\infty : (K \otimes_{\mathbb{Q}} \mathbb{R})^\times \longrightarrow \mathbb{C}^\times$

coincides with the map $(x_\sigma)_{\sigma \in \Sigma_K} \mapsto \prod_{\sigma \in \Sigma_K} x_\sigma^{m_\sigma} \times \prod_{\sigma \in \Sigma_K} \overline{x}_\sigma^{n_\sigma}$.

The element $(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{Z}[\Sigma_K] \times \mathbb{Z}[\overline{\Sigma_K}]$ is called the type of η.

Definition

The algebraic Hecke character η is critical

$\overset{\text{if and only if}}{\Longleftrightarrow}$ *the type $(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{Z}[\Sigma_K] \times \mathbb{Z}[\overline{\Sigma_K}]$ of η satisfies either*

the inequalities " $m_\sigma \leq -1$ and $0 \leq n_\sigma$ for all $\sigma \in \Sigma_K$" (call it the case (A))

or

the inequalities " $n_\sigma \leq -1$ and $0 \leq m_\sigma$ for all $\sigma \in \Sigma_K$ (B)" (call it the case (B)).

Let η be an algebraic Hecke character $\mathbb{A}_K^\times/K^\times \longrightarrow \mathbb{C}^\times$ whose type

$(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{Z}[\Sigma_K] \times \mathbb{Z}[\overline{\Sigma_K}]$ is critical.

Then we have

- $\dfrac{L(\eta, 0)}{\Omega_{\mathrm{CM},\infty}^{\boldsymbol{n}-\boldsymbol{m}}} \in \overline{\mathbb{Q}}^\times$ in the case (A)
- $\dfrac{L(\eta, 0)}{\Omega_{\mathrm{CM},\infty}^{\boldsymbol{m}-\boldsymbol{n}}} \in \overline{\mathbb{Q}}^\times$ in the case (B)

where $\Omega_{\mathrm{CM},\infty} \in (K^+ \otimes_{\mathbb{Q}} \mathbb{C})^\times \cong (\mathbb{C}^\times)^{\Sigma_K}$ is a CM period.

Algebraicity of the special values of Artin L-function

Suppose that K is a totally real number field.

ρ : continuous representation $G_K \longrightarrow GL_n(\overline{\mathbb{Q}}) \cong \mathrm{Aut}_{\overline{\mathbb{Q}}}(V_\rho)$

Suppose that the finite algebraic number field K_ρ corresponding to

the kernel of ρ is totally real.

Then we have the following result on the algebraicity of critical special values

- $\dfrac{L(\rho, n)}{\pi^{n[K:\mathbb{Q}]\dim\rho}} \in \overline{\mathbb{Q}}^\times$ if $n > 1$ and $(-1)^n = +1$.
- $L(\rho, n) \in \overline{\mathbb{Q}}^\times$ if $n < 0$ and $(-1)^n = -1$.

Suppose that K is a CM field.

ρ : continuous representation $G_K \longrightarrow GL_n(\overline{\mathbb{Q}}) \cong \mathrm{Aut}_{\overline{\mathbb{Q}}}(V_\rho)$

Then we have the following result on the algebraicity of special values

for any critical algebraic Hecke character η the type $(\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{Z}[\Sigma_K] \times \mathbb{Z}[\overline{\Sigma_K}]$

- $\dfrac{L(\rho \otimes \eta^{\mathrm{gal}}, 0)}{\Omega_{\mathrm{CM},\infty}^{(\boldsymbol{n}-\boldsymbol{m})\dim\rho}} \in \overline{\mathbb{Q}}^{\times}$ in the case (A).
- $\dfrac{L(\rho \otimes \eta^{\mathrm{gal}}, 0)}{\Omega_{\mathrm{CM},\infty}^{(\boldsymbol{m}-\boldsymbol{n})\dim\rho}} \in \overline{\mathbb{Q}}^{\times}$ in the case (B).

For the later use, we set $|\boldsymbol{m}-\boldsymbol{n}| = \begin{cases} \boldsymbol{m}-\boldsymbol{n} & \text{in the case (B)}, \\ \boldsymbol{n}-\boldsymbol{m} & \text{in the case (A)}. \end{cases}$

§3 Review of p-adic Hecke L-function

Let $\widetilde{K}$ be the composite of all $\mathbb{Z}_p$-extensions of K.

We denote by Γ_K the Galois group $\mathrm{Gal}(\widetilde{K}/K)$.

Leopoldt Conjecture

We have an isomorphism $\Gamma_K \cong (1+p\mathbb{Z}_p)^{r_2(K)+1}$ where $r_2(K)$ is the number of non-real embeddings $K \hookrightarrow \mathbb{C}$ modulo complex conjugate.

Especially, when K is totally real, $\widetilde{K}$ is equal to the cyclotomic $\mathbb{Z}_p$-extension K_{cyc} of K.

(Known to be true if K is a finite abelian extension of $\mathbb{Q}$ or a finite abelian extension of an imaginary quadratic field)

- Let $\chi_{\mathrm{cyc}} : \mathrm{Gal}(K(\mu_{p^\infty})/K) \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character.
- Let $\kappa_{\mathrm{cyc}} = \chi_{\mathrm{cyc}}\omega^{-1} : \mathrm{Gal}(K_{\mathrm{cyc}}/K) \to 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ where $\omega : (\mathbb{Z}/(p))^\times \to \mathbb{Z}_p^\times$ is the Teichmüller character. We have $\mathrm{Gal}(K_{\mathrm{cyc}}/K) \cong \Gamma_K \times (\mathbb{Z}/(p))^\times$ and we can regard κ_{cyc} as a character $\Gamma_K \to 1 + p\mathbb{Z}_p$.
- If $\phi : \Gamma_K \to 1 + p\mathbb{Z}_p$ is a character of finite order, we can regard ϕ as an algebraic Hecke character of finite order on K by the global class field theory.

When K is totally real, we have the following "one-variable" p-adic Hecke L-function.

Theorem 3 (Deligne–Ribet,Cassou-Noguès, Barsky)

Let ψ be an algebraic Hecke character of finite order on a totally real number field K such that K_ψ is totally real (For simplicity, we assume that $\psi \neq \mathbf{1}$). Then we have an element $L_p(\psi) \in \mathbb{Z}_p[\psi][[\Gamma_K]]$ satisfying the interpolation property:

$$\kappa_{\mathrm{cyc}}^j \phi(L_p(\psi)) = \prod_{\mathfrak{p}|(p)} \left(1 - \frac{(\psi\phi^{-1}\omega^{-j})(\mathfrak{p})}{\mathrm{Nr}(\mathfrak{p})^j}\right) \cdot L(\psi\phi^{-1}\omega^{-j}, j)$$

for any integer $j \leq 0$ and any character ϕ of finite order on Γ_K.

When K is a CM number field of degree $2d$, we have the following "$d+1$-variable" p-adic Hecke L-function.

Theorem 4 (Katz, Hida-Tilouine)

Let ψ be an algebraic Hecke character of finite order on a CM number field K of degree $2d$ which is at most tamely ramified at primes over (p). Assume that the CM type Σ_K is p-ordinary (p-ordinary CM type exists only when all primes over (p) of K_+ split at K). Then there exists an element $L_{p,\Sigma_K}(\psi)$ of $\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]$ which is uniquely characterized by the following interpolation property:

$$\frac{\eta^{\mathrm{gal}}(L^{\mathrm{KHT}}_{p,\Sigma_K}(\psi))}{\Omega_{\mathrm{CM},p}^{|\boldsymbol{m}-\boldsymbol{n}|}} = [\mathfrak{r}_K^\times : \mathfrak{r}_{K+}^\times] W_p(\psi^{\mathrm{gal}} \otimes \eta^{\mathrm{gal}}) \frac{(-1)^{w(\eta)d} \prod_{\sigma \in \Sigma_K} (2\pi)^{|r_\sigma(\eta)|} \Gamma_{\Sigma_K}(\boldsymbol{m}, \boldsymbol{n})}{\sqrt{|D_{K+}|} \mathrm{Im}(2\delta)^{\mathsf{r}(\eta)}}$$

$$\cdot \prod_{\mathfrak{P} \in \Sigma_{K,p}} \{(1-\psi^{\mathrm{gal}}\eta^{\mathrm{gal}}(\mathfrak{P}^c))(1-\check{\psi}^{\mathrm{gal}}\check{\eta}^{\mathrm{gal}}(\overline{\mathfrak{P}}))\} \frac{L(\psi\eta, 0)}{\Omega_{\mathrm{CM},\infty}^{|\boldsymbol{m}-\boldsymbol{n}|}}$$

for every critical algebraic Hecke character η of type $(\boldsymbol{m}, \boldsymbol{n})$ such that η^{gal} factors through Γ_K.

Notation of Theorem 4

$\Omega_{\mathrm{CM},p} \in (O_{K^+} \otimes_{\mathbb{Z}} \widehat{\mathcal{O}}^{\mathrm{ur}})^\times$ (resp. $\Omega_{\mathrm{CM},\infty} \in (K^+ \otimes_{\mathbb{Q}} \mathbb{C})^\times$): p-adic (resp. complex) CM period of the CM-field K

$\delta \in K$: polarization divisor associated to a CM abelian variety

$\Gamma_{\Sigma_K}(\boldsymbol{m}, \boldsymbol{n}) = \prod_{\sigma \in \Sigma_K} \Gamma(\max(m_\sigma, n_\sigma)) \in \mathbb{Z}$

$W_p(\eta)$: local root number at p associated to η

$\mathsf{r}(\eta) \in \mathbb{Z}[\Sigma_K]$, $w(\eta) \in \mathbb{Z}$ such that we have

$$\begin{cases} n_{\overline{\sigma}} - w(\eta) = 2r(\eta)_\sigma \text{ and } m_\sigma - w(\eta) = -2r(\eta)_\sigma & \text{(in the case (A))} \\ m_\sigma - w(\eta) = 2r(\eta)_\sigma \text{ and } n_{\overline{\sigma}} - w(\eta) = -2r(\eta)_\sigma & \text{(in the case (B))} \end{cases}$$

for all $\sigma \in \Sigma_K$

§3 Our results on p-adic Artin L-function

Setting

- K: CM number field of degree $2d$ which is absolutely unramified over p.
- ρ: Artin representation of G_K which is unramified at all primes lying over (p) such that K_ρ is also a CM field.

Assumption

- The CM type Σ_K is p-ordinary.
  (Note that p-ordinary CM type exists only when all primes over (p) of K_+ split at K).
- The ($d(K')+1$-variable) Iwasawa Main Conjecture is true for any intermediate field K' of K_ρ/K and for any algebraic Hecke character ψ' of finite order on K' where $d(K') = [K' : \mathbb{Q}]/2$.
  (Note that K' is a CM-field by the assumption that K_ρ is a CM-field.)

In this situation, we have the following "$d+1$-variable" p-adic Artin L-function.

Theorem A (construction of $d+1$-variable p-adic Artin L-function / Hara-O)

There exists an element $L_{p,\Sigma_K}(\rho)$ of $\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]$ uniquely characterized by the following interpolation property:

$$\frac{\eta^{\mathrm{gal}}(L^{\mathrm{KHT}}_{p,\Sigma_K}(\rho))}{\Omega_{\mathrm{CM},p}^{|\boldsymbol{n}-\boldsymbol{m}|\dim\rho}} = W_p(\rho\otimes\eta^{\mathrm{gal}})\left(\frac{(-1)^{w(\eta)d}\prod_{\sigma\in\Sigma_K}(2\pi)^{|r_\sigma(\eta)|}\Gamma_{\Sigma_K}(\boldsymbol{m},\boldsymbol{n})}{\mathrm{Im}(2\delta)^{\mathsf{r}(\eta)}}\right)^{\dim\rho}$$

$$\cdot\prod_{\mathfrak{P}\in\Sigma_{K,p}}\{\det(1-\mathrm{Frob}_{\mathfrak{P}};\rho\otimes\eta^{\mathrm{gal}})\det(1-\mathrm{Frob}_{\overline{\mathfrak{P}}};\check{\rho}\otimes\check{\eta}^{\mathrm{gal}})\}\frac{L(\rho\otimes\eta^{\mathrm{gal}},0)}{\Omega_{\mathrm{CM},\infty}^{|\boldsymbol{n}-\boldsymbol{m}|\dim\rho}}$$

for every critical algebraic Hecke character η of the type $(\boldsymbol{m},\boldsymbol{n})$ such that η^{gal} factors through Γ_K

We define the Selmer group $\mathrm{Sel}_\rho(\widetilde{K})$ to be

$$\mathrm{Sel}_\rho(\widetilde{K}) = \mathrm{Ker}\left[H^1(K_S/\widetilde{K}, A_\rho) \longrightarrow \prod_{\substack{\lambda \in S \\ \lambda \nmid p\infty}} H^1(I_\lambda, A_\rho) \times \prod_{\mathfrak{p} \in \Sigma_{K,p}} H^1(I_{\overline{\mathfrak{p}}}, A_\rho) \right].$$

Here,

- S is a finite set of primes of K which contains the set of primes over (p), infinite primes and ramified primes of ρ.
- The discrete Galois module A_ρ is defined to be $A_\rho = V_\rho / T$ where T is a Galois stable lattice of V_ρ.

We can check that (the isomorphism class of) $\mathrm{Sel}_\rho(\widetilde{K})^\vee$ does not depend on the choice of T under this situation (by using Euler Poincaré characteristic formula).

Below is the $d+1$-variable Iwasawa Main Conjecture for the Artin representation ρ.

Theorem B ($d+1$-variable Iwasawa Main Conjecture / Hara-O)

Under the same setting and the assumptions as Theorem A, we have the following equality between

$$(L_{p,\Sigma_K}^{\mathrm{KHT}}(\rho)) = \mathrm{char}_{\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]} \mathrm{Sel}_\rho(\widetilde{K})^\vee \otimes_{\mathcal{O}[[\Gamma_K]]} \widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]$$

In order prove Theorem A and Theorem B, we will show Proposition A which is a weaker version of Theorem A.

Proposition A

There exists an element $L_{p,\Sigma_K}(\rho) \in \mathrm{Frac}(\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]])$ *which has the following interpolation property:*

$$\frac{\eta^{\mathrm{gal}}(L_{p,\Sigma_K}(\rho))}{\Omega_{\mathrm{CM},p}^{|\boldsymbol{n}-\boldsymbol{m}|\dim\rho}} = W_p(\rho\otimes\eta^{\mathrm{gal}})\left(\frac{(-1)^{w(\eta)d}\prod_{\sigma\in\Sigma_K}(2\pi)^{|r_\sigma(\eta)|}\Gamma_{\Sigma_K}(\boldsymbol{m},\boldsymbol{n})}{\mathrm{Im}(2\delta)^{\mathsf{r}(\eta)}}\right)^{\dim\rho}$$

$$\cdot\prod_{\mathfrak{P}\in\Sigma_{K,p}}\{\det(1-\mathrm{Frob}_{\mathfrak{P}};\rho\otimes\eta^{\mathrm{gal}})\det(1-\mathrm{Frob}_{\overline{\mathfrak{P}}};\check{\rho}\otimes\check{\eta}^{\mathrm{gal}})\}\frac{L(\rho\otimes\eta^{\mathrm{gal}},0)}{\Omega_{\mathrm{CM},\infty}^{|\boldsymbol{n}-\boldsymbol{m}|\dim\rho}}$$

for every critical algebraic Hecke character η *of the type* $(\boldsymbol{m},\boldsymbol{n})$ *such that* η^{gal} *factors through* Γ_K

Sketch of the proof of Proposition A

By Brauer's induction theorem, there exist

- $K_1, \ldots, K_r$: intermediate fields of K_ρ/K
- Hecke character of finite order φ_i on K_i $(i = 1, \ldots, r)$ whose associated Galois characters $\varphi_i^{\mathrm{gal}} : G_{K_i} \longrightarrow \mathbb{C}^\times$ factors through $\mathrm{Gal}(K_\rho/K_i)$.
- $a_1, \ldots, a_r \in \mathbb{Z}$

such that $\rho \cong \oplus_{i=1}^r a_i \mathrm{Ind}_{G_{K_i}}^{G_K} \varphi_i^{\mathrm{gal}}$.

By applying Theorem 4 to $\varphi_1, \ldots \varphi_r$, we have $L_{p,\Sigma_{K_i}}^{\mathrm{KHT}}(\varphi_i) \in \widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_{K_i}]]$. We define $L'_{p,\Sigma_K}(\rho) \in \mathrm{Frac}(\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]])$ to be

$$L'_{p,\Sigma_K}(\rho) = \prod_{i=1}^r p_i(L_{p,\Sigma_{K_i}}^{\mathrm{KHT}}(\varphi_i))^{a_i}$$

where p_i is the natural surjection $\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_{K_i}]] \twoheadrightarrow \widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]$ $(i = 1, \ldots, r)$.

By definition, we have

$$\begin{aligned}
\eta^{\mathrm{gal}}(L'_{p,\Sigma_K}(\rho)) &= \prod_{i=1}^{r} \eta^{\mathrm{gal}}(L^{\mathrm{KHT}}_{p,\Sigma_{K_i}}(\varphi_i))^{a_i} \\
&= \prod_{i=1}^{r} \Bigg(\Omega^{|\boldsymbol{m}-\boldsymbol{n}|}_{\mathrm{CM},K_i,p} [\mathfrak{r}^{\times}_{K_i} : \mathfrak{r}^{\times}_{K_i^+}] W_p(\eta^{\mathrm{gal}}|_{K_i}) \\
&\quad \cdot \frac{(-1)^{w(\eta)d_i} \prod_{\sigma \in \Sigma_{K_i}} (2\pi)^{|r_\sigma(\eta)|} \Gamma_{\Sigma_{K_i}}(\boldsymbol{m}_i, \boldsymbol{n}_i)}{\sqrt{|D_{K_i^+}|} \mathrm{Im}(2\delta)^{\mathsf{r}_i(\eta)}} \\
&\quad \cdot \prod_{\mathfrak{P} \in \Sigma_{K_i,p}} \{\det(1 - \mathrm{Frob}_{\mathfrak{P}}; \varphi_i \otimes \eta^{\mathrm{gal}}) \det(1 - \mathrm{Frob}_{\overline{\mathfrak{P}}}; \check{\varphi}_i \otimes \check{\eta}^{\mathrm{gal}})\} \\
&\quad \cdot \frac{L(\varphi_i^{\mathrm{gal}} \otimes \eta^{\mathrm{gal}}, 0)}{\Omega^{|\boldsymbol{m}-\boldsymbol{n}|}_{\mathrm{CM},K_i,\infty}} \Bigg)^{a_i}
\end{aligned}$$

We show that the interpolation property of $L'_{p,\Sigma_K}(\rho)$ "almost" fits into

the expected interpolation property of $L_{p,\Sigma_K}(\rho)$ (call this "matching").

For example, verifying the "matching" of the special valiues $L(\varphi_i \otimes \eta^{\mathrm{gal}}, 0)^{a_i}$ is rather straight-forward as follows:

$$\begin{aligned}\prod_{i=1}^{r} L(\varphi_i^{\mathrm{gal}} \otimes \eta^{\mathrm{gal}}, 0)^{a_i} &= L(\oplus_{i=1}^{r} \varphi_i^{\mathrm{gal}})^{\oplus a_i} \otimes \eta^{\mathrm{gal}}, 0) \\ &= L(\oplus_{i=1}^{r} a_i (\mathrm{Ind}_{G_K}^{G_{K_i}} \varphi_i^{\mathrm{gal}}) \otimes \eta^{\mathrm{gal}}, 0) \\ &= L(\rho \otimes \eta^{\mathrm{gal}}, 0).\end{aligned}$$

On the other hand, verifying the "matching" of the epsilon factors is not so straight-forward.

We need to verify:

$$\prod_{i=1}^{r} W_p(\varphi_i \otimes \eta^{\mathrm{gal}}, 0)^{a_i} = W_p(\rho \otimes \eta^{\mathrm{gal}}, 0).$$

This is essentially reduced to a certain relation between generalized Gauss sums and we will

prove a certain kind of generalization of Davenport-Hasse relation of Gauss sums.

Similarly we will verify the "matching" of Gamma factors, complex periods, p-adic peiods, polarization divisors, etc.

Then, multiplying a certain constant to $L'_{p,\Sigma_K}(\rho)$, we obtain the p-adic L-function $L_{p,\Sigma_K}(\rho)$.

(Q.E.D)

The key proposition on the algebraic side is as follows.

Proposition B

We have

$$\mathrm{char}_{\mathcal{O}[[\Gamma_K]]}\mathrm{Sel}_\rho(\widetilde{K}K_\rho)^\vee = \prod_{i=1}^{r} p_i(\mathrm{char}_{\mathcal{O}[[\Gamma_{K_i}]]}\mathrm{Sel}_{\varphi_i}(\widetilde{K}_iK_\rho)^\vee)^{a_i}$$

where p_i *is the natural surjection* $\widehat{\mathcal{O}}[[\Gamma_{K_i}]] \twoheadrightarrow \widehat{\mathcal{O}}[[\Gamma_K]]$ $(i = 1, \ldots, r)$.

We call the above proposition the "descent argument".

Remark 3

The descent argument for the p-ordinary CM-type algebraic Iwasawa module was studied in [HO18]. In the proof of Proposition B, we apply techniques developed in [HO18] and their further refinement.

Sketch of the proof of Proposition B

By the Brauer induction isomorphism $\rho \cong \oplus_{i=1}^{r} a_i \mathrm{Ind}_{G_{K_i}}^{G_K} \varphi_i^{\mathrm{gal}}$ given in the proof of Proposition A, we have:

$$\mathrm{Sel}_{\rho}(\widetilde{K}K_{\rho}) = \oplus_{i=1}^{r} \mathrm{Sel}_{\varphi_i}(\widetilde{K}K_{\rho})^{\oplus a_i}.$$

Note that $\mathrm{Sel}_{\varphi_i}(\widetilde{K}K_{\rho})^{\vee}$ is a finitely generated torsion $\mathcal{O}[[\Gamma_K]]$-module.

To prove the desired descent, we need to show the following descent:

(DES) $$\mathrm{char}_{\mathcal{O}[[\Gamma_K]]}\mathrm{Sel}_{\varphi_i}(\widetilde{K}K_{\rho})^{\vee} = p_i(\mathrm{char}_{\mathcal{O}[[\Gamma_{K_i}]]}\mathrm{Sel}_{\varphi_i}(\widetilde{K}_i K_{\rho})^{\vee}).$$

The most important key to prove (DES) is the use of the strict Selmer group which is defined as follows:
We define the Selmer group $\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}K_\rho)$ to be

$$\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}K_\rho) = \mathrm{Ker}\left[H^1(K_S/\widetilde{K}K_\rho, A_{\varphi_i}) \longrightarrow \prod_{\substack{\lambda \in S_{K_\rho} \\ \lambda \nmid p\infty}} H^1(I_\lambda, A_{\varphi_i}) \times \prod_{\mathfrak{p} \in \Sigma_{K_\rho, p}} H^1(D_{\overline{\mathfrak{p}}}, A_{\varphi_i}) \right].$$

where $D_{\overline{\mathfrak{p}}}$ are decomposition groups.

Note that $\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}K_\rho)$ is a subgroup of $\mathrm{Sel}_{\varphi_i}(\widetilde{K}K_\rho)$.

The strict Selmer groups $\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}'K_\rho)$over other intermediate fields K' are defined in the same way (replacing $\widetilde{K}$ by $\widetilde{K}'$ in the right-hand side of the above definition).

Under our setting, we can show that

$$\mathrm{Sel}_{\varphi_i}(\widetilde{K}_i K_\rho)^\vee \twoheadrightarrow \mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}_i K_\rho)^\vee$$

$$\mathrm{Sel}_{\varphi_i}(\widetilde{K} K_\rho)^\vee \twoheadrightarrow \mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K} K_\rho)^\vee$$

is a pseudo-isomorphism of $\mathcal{O}[[\Gamma_{K_i}]]$-modules (resp. $\mathcal{O}[[\Gamma_K]]$-modules)

Hence the descent (DES) is reduced to the following descent:

(DES)′ $$\mathrm{char}_{\mathcal{O}[[\Gamma_K]]}\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K} K_\rho)^\vee = p_i(\mathrm{char}_{\mathcal{O}[[\Gamma_{K_i}]]}\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}_i K_\rho)^\vee).$$

The advantage of working with the strict Selmer group (compared to the usual Selmer group) is as follows:

- Easier to show the control theorem for the restriction map

$$\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}K_\rho) \longrightarrow \mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}_iK_\rho)[\mathrm{Ker}(p_i)].$$

- Easier to prove the non-existence of the pseudo-null $\mathcal{O}[[\Gamma_{K'}]]$-submodule of

  $\mathrm{Sel}^{\mathrm{str}}_{\varphi_i}(\widetilde{K}'K_\rho)^\vee$ for any intermediate fields K' of K_ρ/K.

Working on this, we prove (DES)$'$ and we complete the proof of Proposition B.

(Q.E.D)

Proof of Theorem A and Theorem B

Proposition A + Proposition B $\Rightarrow$ Theorem A + Theorem B

More precisely, by proposition A and Proposition B, we prove the equality of the fractional ideal of $\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]$ as follows:

$$(L^{\mathrm{KHT}}_{p,\Sigma_K}(\rho)) = \mathrm{char}_{\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]}\mathrm{Sel}_\rho(\widetilde{K})^\vee \otimes_{\mathcal{O}[[\Gamma_K]]} \widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]].$$

Since the right-hand side of this equality is an integral ideal of $\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]$, the left-hand side of the above equality is also an an integral ideal of $\widehat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma_K]]$. Thus we prove Theorem A and Theorem B simultaneously.

(Q.E.D)

Greenberg [Gr83], [Gr04] proved the prototype of this work over a totally real field.

Finally, we explain some technical difficulties compared to the work of Greenberg

- In [Gr83] and [Gr04], we did not need the "matching of the interpolation property".

  The interpolation property of the p-adic L-function is much simpler.
  (no complex periods, no p-adic periods, no Gamma-facors, no epsilon-factors, etc. in [Gr83] and [Gr04])

- In [Gr83] and [Gr04], we did not need the "descent argument".

  The Selmer groups and the p-adic L-functions of number of the intermediate fields of $\mathrm{Ker}\rho/K$ is always one-variable when K is totally real. Hence, in [Gr83], [Gr04], we did not need the descent argument.