Lie brackets and functional analysis for control

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Let $f_0, f_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$. We consider

$$\dot{x}(t) = f_0(x(t)) + u(t) f_1(x(t)). \qquad (\star)$$

Proposition

- *Let $T > 0$, $x_0 \in \mathbb{R}^n$ and $u \in L^1((0,T),\mathbb{R})$. There exists a unique maximal solution $x \in \mathcal{C}^0([0,T'],\mathbb{R}^n)$ such that $x(0) = x_0$, i.e.*

$$\forall t \in [0,T'], \quad x(t) = x_0 + \int_0^t \Big(f_0(x(s)) + u(s) f_1(x(s)) \Big) \mathrm{d}s.$$

Moreover, if $x_0 = 0$ and $\|u\|_{L^1}$ is small enough, then $T' = T$.
- *The end-point map $u \in L^1(0,T) \mapsto x(T; u, 0)$ is C^1.*

Proof : Fixed point argument. Implicit function theorem.

Small time local controllability : $W^{m,\infty}$-STLC

$$\dot{x} = f_0(x) + u(t)f_1(x) \qquad f_0(0) = 0$$

Definition ($W^{m,\infty}$-STLC)

Let $m \in \mathbb{N}$. The system is $W^{m,\infty}$-*Small-Time Locally Controllable* if for every $T, \eta > 0$, there exists $\delta > 0$ such that, for every $x_f \in B_{\mathbb{R}^n}(0, \delta)$, there exists $u : [0, T] \to \mathbb{R}$ such that $\|u\|_{W^{m,\infty}} \leqslant \eta$ and $x(T; u, 0) = x_f$.

- Nonlinear open mapping + continuity of $x_f \mapsto u$ at 0.
- Starting from $x(0) = 0$ is not restrictive (under LARC).
- The historical definition of STLC corresponds to $m = 0$.
- For nonlinear systems, the choice of norm influences the answer.

$\forall m \in \mathbb{N}^*$, ($W^{m,\infty}$-STLC) $\Rightarrow$ (L^∞-STLC) $\Rightarrow$ small-state-STLC.
Any reciprocal implication is false.

- Specifying the norm prepares the transfer to PDEs.

Definition

smooth-STLC = $W^{m,\infty}$-STLC for any $m \in \mathbb{N}$

- We are also interested in Hölder cost estimates $\|u\| \leq C|x_f|^\alpha$.

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^2 - x_2^2 \end{cases}$$

Poincaré-Wirtinger : $\int_0^T \phi^2 \leq \left(\frac{T}{\pi}\right)^2 \int_0^T (\phi')^2$ is $=$ for $\phi(t) = \sin(\frac{\pi t}{T})$.

• **The system is not controllable in time** $T \leq \pi$ because $(\phi \leftarrow x_2)$

$$x_3(T) \geq \left(1 - \left(\frac{T}{\pi}\right)^2\right) \int_0^T x_1^2 \geq 0.$$

• **The system is controllable in any time** $T > \pi$: if $u(t) := \dot{x}_1(t)$ where $x_1(t) = \rho^\epsilon(t) \cos(\frac{\pi t}{T})$ and ρ^ϵ is a cut-off, then

$$x_3(T) = \int_0^T \left(x_1(t)^2 - \left(\int_0^t x_1 \right)^2 \right) dt \underset{\epsilon \to 0}{\longrightarrow} \left(1 - \left(\frac{T}{\pi}\right)^2\right) \frac{T}{2\pi} < 0$$

Example : influence of the norm on the local controllability

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2^2 + x_1^3 \end{cases}$$

• **The system is not $W^{1,\infty}$-STLC :** if $x_2(T) = 0$ then $\int_0^T x_1^3 = -\int_0^T 2x_2 x_1 u = +\int_0^T \dot{u} x_2^2$ thus, if $\|\dot{u}\|_{L^\infty} \leq 1$ then

$$x_3(T) \geq (1 - \|\dot{u}\|_{L^\infty}) \int_0^T x_2^2 \geq 0$$

• **The system is L^∞-STLC** because : if $u(t) = \epsilon \phi'' \left(\frac{t}{\lambda}\right)$ where $\phi \in C_c^2((0,1), \mathbb{R})$ and $T = \frac{1}{\lambda}$ then

$$x_3(T, u) = \lambda^5 \epsilon^2 \int_0^1 \phi^2 + \lambda^4 \epsilon^3 \int_0^1 (\phi')^3$$

With $\lambda << \epsilon$ then $x_3(T, u) < 0$.

Example : influence of the norm on the Hölder exponent

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2^2 + x_1^3 \end{cases}$$

Hölder exponent : for L^∞-STLC $= \frac{1}{6}$, for $W^{-1,\infty}$-STLC $= \frac{1}{3}$

• With $x_2(t) = \epsilon^3 \chi(t) \theta\left(\frac{t}{\epsilon}\right)$ where $\chi \in C_c^\infty(0,T)$ and θ is 1-periodic, we obtain a control $u \approx \epsilon \chi(t) \theta''\left(\frac{t}{\epsilon}\right)$ such that

$$x(T,u) \approx \epsilon^6 \left(\int_0^T \chi^2 \int_0^1 \theta + \int_0^T \chi^3 \int_0^1 (\theta')^2 \right) e_3 .$$

By choosing χ such that $\int \chi^3 = 1$ and $\theta(s) = \overline{\theta}(ns)$ we get $x_3(T) < 0$.
Moreover $\quad \|u\|_{L^\infty} \approx \epsilon \leq |x_f|^{\frac{1}{6}} \quad$ and $\quad \|u_1\|_{L^\infty} \approx \epsilon^2 \leq |x_f|^{\frac{1}{3}}$.

• Let $\sigma > \frac{1}{6}$. Assume that, for $\lambda > 0$ small, $\exists u^\lambda \in L^\infty(0,T)$ such that $x(T,u^\lambda) = -\lambda e_3$ and $\|u^\lambda\|_{L^\infty} \leq C\lambda^\sigma$. By Gagliardo Nirenberg inequality

$$\|x_1^\lambda\|_{L^3}^3 \lesssim \|u^\lambda\|_{L^6}^{3/2} \|x_2^\lambda\|_{L^2}^{3/2} \lesssim \lambda^{\frac{3\sigma}{2}} \|x_2^\lambda\|_{L^2}^{3/2} \leq C' \lambda^{6\sigma} + \frac{1}{2} \|x_2^\lambda\|_{L^2}^2$$

$$\lambda + \|x_2^\lambda\|_{L^2}^2 = -\int_0^T (x_1^\lambda)^3 \leq C' \lambda^{6\sigma} + \frac{1}{2} \|x_2^\lambda\|_{L^2}^2 \text{ : contradiction.}$$

Structure of this course

1. Linear theory and Kalman rank condition
2. Linear test and linear cost-estimate
3. Power series expansion and Hölder-cost estimate
4. Lie brackets
5. A new representation formula of the state
6. Proof of necessary conditions to STLC
7. Extension to a PDE : the bilinear Schrödinger equation

Linear theory and Kalman rank condition

$$\dot{y} = Ay + u(t)b$$

where $y(t) \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$, $u(t) \in \mathbb{R}$ and $b \in \mathbb{R}^n$.

Theorem

Smooth-STLC : $\forall T > 0, y_f \in \mathbb{R}^n, \exists u \in C^\infty((0, T), \mathbb{R})$ *such that* $y(T; u, 0) = y_f$ *and* $\|u\|_{W^{m,\infty}} \leq C(m, T)|y_f|$.
$\Leftrightarrow$ ***Kalman condition*** : $\mathrm{rank}\left\{b, Ab, \ldots A^{n-1}b\right\} = n$

Proof of $\Leftarrow$: For every $T > 0$, the Grammian matrix is invertible.

$$G := \int_0^T e^{A(T-\tau)}\, b\, b^*\, e^{A^*(T-\tau)}\, d\tau$$

For $y_f \in \mathbb{R}^n$, the explicit control $u : t \in (0, T) \mapsto b^* e^{A^*(T-t)} G^{-1} y_f$ belongs to $C^\infty(0, T)$ and gives the conclusion.

Linear test and linear cost-estimate

Linear test and linear cost-estimate

Let $f_0, f_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $f_0(0) = 0$, $A := \partial_x f_0(0)$ and $b := f_1(0)$

$$\dot{x} = f_0(x) + u(t) f_1(x) \qquad (\star)$$

Theorem

1. *If the linearized system $\dot{y} = Ay + u(t)b$ is controllable then the nonlinear system $(\star)$ is **smooth**-STLC.*
2. *Moreover, $\forall m \in \mathbb{N}$, $T > 0$, $\exists C, \delta > 0$, such that $\forall x_f \in B_{\mathbb{R}^n}(0, \delta)$, there exists $u : [0, T] \to \mathbb{R}$ with $\|u\|_{W^{m,\infty}} \leq C|x_f|$ such that $x(T; u, 0) = x_f$.*
3. *Kalman is necessary for STLC ****with**** linear cost estimate.*

Proof : Apply the inverse mapping theorem to the C^1-end-point map $\Theta : u \in W^{m,\infty}(0, T) \mapsto x(T; u, 0)$. The right inverse Θ^{-1} is locally lipschitz : $\|\Theta^{-1}(x_f)\|_{W^{m,\infty}} = \|\Theta^{-1}(x_f) - \Theta^{-1}(0)\|_{W^{m,\infty}} \leq C|x_f|$.

Kalman is not necessary for STLC : $\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^3 \end{cases}$ $\quad \begin{cases} \dot{y}_1 = u \\ \dot{y}_2 = 0 \end{cases}$

Local controllability **with** linear cost estimate $\Rightarrow$ Kalman

$$\begin{cases} \dot{x} = f_0(x) + u(t) f_1(x), \\ x(0) = 0, \end{cases} \qquad \begin{cases} \dot{y} = Ay + u(t) b, \\ y(0) = 0 \end{cases}$$

Let $T > 0$. We assume $\exists C, \delta > 0, \forall x_f \in B_{\mathbb{R}^n}(0,\delta), \exists u \in L^\infty(0,T)$ such that $x(T;u) = x_f$ and $\|u\|_{L^\infty} \leq C|x_f|$.

Goal : $\forall y_f \in \mathbb{R}^n, \exists u \in L^\infty(0,T)$ such that $y(T;u) = y_f$.

Let $y_f \in \mathbb{R}^n$. For $\epsilon > 0$ small enough, there exists $u^\epsilon \in L^\infty(0,T)$ such that $x(T;u^\epsilon) = \epsilon y_f$ and $\|u^\epsilon\|_{L^\infty} \leq C\epsilon |y_f|$. There exists $u \in L^\infty(0,T)$ such that, up to an extraction, $\frac{u^\epsilon}{\epsilon} \overset{*}{\rightharpoonup} u$ in $\sigma(L^\infty, L^1)$. Then

$$\begin{aligned} |y_f - y(T;u)| &= \left| \frac{x(T;u^\epsilon) - y(T;u^\epsilon)}{\epsilon} + y(T; \tfrac{u^\epsilon}{\epsilon}) - y(T;u) \right| \\ &\leq \frac{1}{\epsilon} \underset{\epsilon \to 0}{o}(\|u^\epsilon\|) + \left| \int_0^T e^{A(T-t)} b \left(\frac{u^\epsilon(t)}{\epsilon} - u(t) \right) dt \right| \\ &= o(1) \end{aligned}$$

Power series expansion and Hölder-cost estimate

When the linear test fails : power series expansion

$$\dot{x} = f_0(x) + uf_1(x)\,, \qquad f_0(0) = 0\,, A = \partial_x f_0(0)\,, b = f_1(0)$$

We assume that the linearized system misses one direction e_1

$$\mathbb{R}^n = \mathbb{R}e_1 \oplus S \qquad \text{where} \qquad S := \text{Vect}\{A^k b; 0 \leqslant k \leqslant n-1\}$$

that we try to recover at the quadratic order. We make a formal power series expansion of (x, u)

$$u = 0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots \qquad\qquad x = 0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots$$

$$\dot{y}_1 = \partial_x f_0(0) y_1 + u_1 f_1(0) = Ay_1 + u_1 b$$

$$\begin{aligned}\dot{y}_2 &= \partial_x f_0(0) y_2 + u_2 f_1(0) + \frac{1}{2}\partial_x^2 f_0(0).(y_1, y_1) + u_1 \partial_x f_1(0).y_1 \\ &= Ay_2 + u_2 b + \frac{1}{2}\partial_x^2 f_0(0).(y_1, y_1) + u_1 \partial_x f_1(0).y_1\end{aligned}$$

Assume there exists $u_1^{\pm}, u_2^{\pm} \in L^\infty(0, T)$ such that the associated solutions with $y_1^{\pm}(0) = y_2^{\pm}(0) = 0$ satisfy $y_1^{\pm}(T) = 0$ and $y_2^{\pm}(T) = \pm e_1$. Then, the nonlinear system is locally controllable in time T.

Theorem

Let $T > 0$.

1. *If there exists $u_1^{\pm}, u_2^{\pm} \in L^{\infty}(0,T)$ that steer the linearized system from 0 to 0 and the quadratic system from 0 to $\pm e$, then the nonlinear system is STLC.*
2. *Moreover, $\exists C_T, \delta_T > 0$, such that $\forall x_f \in B_{\mathbb{R}^n}(0, \delta_T)$, there exists $u : [0,T] \to \mathbb{R}$ such that $x(T) = x_f$ and*
$$\|u\|_{\infty} < C_T |x_f|^{1/2}.$$
3. *Controlling at the quadratic order the component along e_1 with controls **leaving the linear order invariant** is necessary for the $\frac{1}{2}$-Holder cost estimate.*

Proof : SC for STLC with $\frac{1}{2}$-Hölder cost-estimate

We consider $\dot{x} = f_0(x) + u(t) f_1(x)$ where $\mathbb{R}^n = \mathbb{R}e_1 \oplus S_1$
$S = \text{Span}\{e_2, \dots, e_n\}$ and $\exists u_1^{\pm}, u_2^{\pm}, v_2, \dots, v_n$ such that

$$\begin{aligned} &\dot{y}_1 = Ay + u_1(t) b, && \dot{y}_2 = Ay_2 + u_2(t) b + Q(y_1) + u_1 L(y_1), \\ &y_1(T, u_1^{\pm}) = 0, && y_2(T, u_1^{\pm}, u_2^{\pm}) = \pm e_1, \\ &y_1(T, v_j) = e_j. && \end{aligned}$$

Goal : For x_f small enough, find u st $x(T; u) = x_f$ and $\|u\| \leq C|x_f|^{1/2}$.

For $b = \sum_{j=1}^n b_j e_j \in \mathbb{R}^n$, the control

$$u_b(t) := \sqrt{|b_1|} u_1^{\text{sg}(b_1)}(t) + |b_1| u_2^{\text{sg}(b_2)}(t) + \sum_{j=2}^n b_j v_j(t)$$

satisfies $x(T; u_b) = b + o(b)$. By applying the Brouwer fixed point theorem to the map $\mathcal{F} : b \mapsto b - x(T; u_b) + x_f$ we obtain b^* such that $x_f = x(T; u_{b^*})$. Then $b^* = \mathcal{F}(b^*) = o(b^*) - x_f$ proves $|b^*| \leq C|x_f|$ thus $\|u_{b^*}\| \leq C|b^*|^{1/2} \leq C|x_f|^{1/2}$.

NC for $\frac{1}{2}$-Hölder cost-estimate

non linear	linear	quadratic
$\dot{x} = f_0(x) + uf_1(x)$	$\dot{y}_1 = Ay + u_1 b$	$\dot{y}_2 = Ay_2 + Q(y_1) + u_1 L(y_1)$

Let $T > 0$. We assume $\exists C, \delta > 0, \forall x_f \in B_{\mathbb{R}^n}(0,\delta), \exists u \in L^\infty(0,T)$ such that $x(T;u) = x_f$ and $\|u\|_{L^\infty} \leq C|x_f|^{1/2}$.

Goal : $\exists u^\pm \in L^\infty$ that leaves the linear order invariant : $y_1(T, u^\pm) = 0$, and moves the second order along $\pm e_1$: $\mathbb{P}_{e_1} y_2(T,u) = \pm 1$

$\exists u^\epsilon \in L^\infty(0,T)$ such that $x(T;u^\epsilon) = \pm\epsilon e_1$ and $\|u^\epsilon\|_{L^\infty} \leq C\sqrt{\epsilon}$.
$\exists u \in L^\infty(0,T)$ such that $\frac{u^\epsilon}{\sqrt{\epsilon}} \overset{*}{\rightharpoonup} u$ in $\sigma(L^\infty, L^1)$.

$$y_1^\epsilon(t) := y_1(t, \tfrac{u^\epsilon}{\sqrt{\epsilon}}) = \int_0^t e^{A(t-s)} b \frac{u^\epsilon(s)}{\sqrt{\epsilon}} ds \overset{\text{pointwise } \& L^2}{\longrightarrow} y_1(t,u),$$

$$y_1^\epsilon(T) = \frac{1}{\sqrt{\epsilon}} \mathbb{P}_{S_1}(y_1 - x)(T, u^\epsilon) = \frac{1}{\sqrt{\epsilon}} O(\|u^\epsilon\|^2) = O(\sqrt{\epsilon}),$$

$$y_2(T, \tfrac{u^\epsilon}{\sqrt{\epsilon}}) = \int_0^T e^{A(T-s)} \left(Q(y_1^\epsilon(s)) + \frac{u^\epsilon(s)}{\sqrt{\epsilon}} L y_1^\epsilon(s) \right) ds \longrightarrow y_2(T,u),$$

$$\mathbb{P}_e y_2(T, \tfrac{u^\epsilon}{\sqrt{\epsilon}}) - \pm 1 = \frac{1}{\epsilon} \mathbb{P}_e (y_1 + y_2 - x)(T, u^\epsilon) = \frac{1}{\epsilon} o(\|u^\epsilon\|^2) = o(1).$$

Examples

- Local controllability in time $T > \pi$ with $\frac{1}{2}$-Hölder cost estimate

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^2 - x_2^2 \end{cases}$$

- STLC with $\frac{1}{3}$-Hölder cost-estimate :

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^3 \end{cases}$$

- $\frac{1}{3}$-Hölder cost estimate does not hold for

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2^2 + x_1^3 \end{cases}$$

Here, the optimal exponent for L^∞-STLC is $\frac{1}{6}$.

When 2 nonlinear terms are in competition, determining the optimal Holder exponent can be complicated.

Goal and structure of the 2nd course

$$\dot{x} = f_0(x) + u(t) f_1(x)$$

- Prove necessary conditions of STLC formulated in terms of Lie brackets of f_0 and f_1 evaluated at 0
- With a new strategy :
  - to go further on ODEs
  - to prepare the transfer to PDEs

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Lie brackets

An important tool : iterated Lie brackets

Definition

Let f and g be smooth vector fields on $\mathbb{R}^n$. The Lie bracket $[f, g]$ of f and g is the smooth vector field defined by :

$$[f, g](x) := f'(x)g(x) - g'(x)f(x)\,.$$

We define by induction on $k \in \mathbb{N}$:

$$\mathrm{Ad}_f^0(g) = g\,, \qquad \mathrm{Ad}_f^{k+1}(g) = \left[f, \mathrm{Ad}_f^k(Y)\right].$$

When $f(x) = Ax$ and $g(x) = Bx$ with $A, B \in \mathcal{M}_n(\mathbb{R})$, then

$$[f, g](x) = (BA - AB)x.$$

Lie brackets measure the lack of commutativity between motions.

Jacobi : $\mathrm{Ad}_f([g, h]) = [\mathrm{Ad}_f(g), h] + [g, \mathrm{Ad}_f(h)]$

Convenient notations for Lie brackets

- Let $X := \{X_0, X_1\}$ be non-commutative **indeterminates**
- Let $\mathcal{A}(X)$ be the **free algebra** over X
  i.e. the vector space of non-commutative polynomials
  ex : $7X_0^2 + 3X_1X_0 + 2X_0X_1 \in \mathcal{A}(X)$
- Let $\mathcal{L}(X)$ the **free Lie algebra** over X,
  i.e. the smallest vector subspace of $\mathcal{A}(X)$ containing X_0, X_1, and stable by the Lie bracket (commutator) operation $[a,b] := ab - ba$
  ex : $X_0 + 2[X_0, X_1] + 8[X_1, [X_1, X_0]] \in \mathcal{L}(X)$
- $n_j(b)$ is the nb of occurrences of X_j in b, for a Lie bracket $b \in \mathcal{L}(X)$
  ex : for $b = [X_1, [X_1, X_0]]$ then $n_0(b) = 1$ and $n_1(b) = 2$.
- One can "**evaluate**" (although not injective)

$$b \in \mathcal{L}(X) \hookrightarrow f_b \in C^\omega(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow f_b(0) \in \mathbb{R}^n$$

$$[X_1, X_0] = X_1X_0 - X_0X_1 \to [f_1, f_0] = (Df_0)f_1 - (Df_1)f_0 \to [f_1, f_0](0)$$

For analytic vector fields f_0, f_1 on a neighborhood of 0, the LARC is

$$\mathrm{Lie}(f_0, f_1)(0) := \mathrm{span}\left\{ f_b(0); \;\; b \in \mathcal{L}(X) \right\} = \mathbb{R}^n. \qquad (1)$$

• For driftless syst $\dot{x} = u_0(t) f_0(x) + u_1(t) f_1(x)$: LARC $\Leftrightarrow$ smooth-STLC.
($\Rightarrow$ uses piecewise cst controls with max $2n$ switches, smoothing OK)
The solutions live in a submanifold M such that $T_x M = Lie(f_0, f_1)(x)$.

• **For systems $\dot{x} = f_0(x) + u(t) f_1(x)$, small-state-STLC $\Rightarrow$ LARC.**
[Hermann 1963, Nagano 1966].
The analyticity of f_0, f_1 is necessary : $\dot{x} = u e^{-1/u^2}$.

• **But for non-zero drift $f_0 \neq 0$, LARC is not sufficient.**

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2, \end{cases} \qquad \begin{array}{l} f_{X_1}(0) = f_1(0) = e_1 \\ f_{W_1}(0) = [f_1, [f_1, f_0]](0) = 2e_2 \end{array}$$

The quadratic Lie bracket $W_1 := [X_1, [X_1, X_0]]$ looks like a 'bad' :
associated with a signed motion in an oriented direction.

The goal is to determine good/bad brackets.

Why Lie brackets ? (#2)

Consider analytic systems $\dot{x} = f_0(x) + u(t) f_1(x)$ with $f_0(0) = 0$
$\dot{y} = g_0(y) + u(t) g_1(y)$ with $g_0(0) = 0$

Theorem (Nagano 1968, Krener 1973, Sussmann 1974, 1985)

The systems are loc. diffeomorphic : $\exists \Phi, \forall u, y(t,u) = \Phi(x(t,u))$
$\Leftrightarrow$ *their Lie brackets at* 0 *have the same vectorial structure :*

$$\{b \in \mathcal{L}(X); f_b(0) = 0\} = \{b \in \mathcal{L}(X); g_b(0) = 0\}.$$

Proof of $\Leftarrow$: Let $b_1, \dots, b_n \in \mathcal{L}(X)$ such that $\mathbb{R}^n = \text{Span}\{f_{b_j}(0)\}$.
Define loc. coordinates $(\alpha_1, \dots, \alpha_n)$ of $x \in \mathbb{R}^n$ by $x = e^{\alpha_1 f_{b_1}} \dots e^{\alpha_n f_{b_n}}(0)$.
Then $\Phi(x) = e^{\alpha_1 g_{b_1}} \dots e^{\alpha_n g_{b_n}}(0)$ gives the conclusion.
If $\Psi(t,p) = e^{tf} p$ then $(\partial_p \Psi(t,p))^{-1} g(\Psi(t,p)) = \sum_{k=0}^{+\infty} \frac{t^k}{k!} \mathrm{Ad}_f^k(g)(p)$.

Hence, **the vectors** $f_b(0)$ **contain all the information for STLC**.

A new representation formula of the state

Computing the state using Lie brackets

$$\dot{x} = f_0(x) + u(t) f_1(x) \qquad\qquad x(0) = 0$$

Theorem (Beauchard, Le Borgne, Marbach 2020)

$$x(t;u) = \sum_b \eta_b(t,u) f_b(0) + O(\text{“}remainders\text{”}) + o(x(t;u)).$$

The sum

- ranges over elements b of a basis of $\mathcal{L}(X)$
- involves system-dependent vectors $f_b(0) \in \mathbb{R}^n$
- universal functionals $\eta_b(t,u)$

Caution : The full sum does not converge, even with analyticity. One has to consider (possibly infinite) truncations (wrt t, or u, or a parameter). And well chosen bases of $\mathcal{L}(X)$. **This is not a Taylor expansion, but a csq of a Magnus-type formula.**

Proof of necessary conditions for STLC

A naive strategy to prove obstructions

$$x(T;u) = \sum_{b\in\mathcal{B}_{[\mathbf{1},M]}} \eta_b(T,u) f_b(0) + O\left(\|u\|_{W^{-\mathbf{1},M+\mathbf{1}}}^{M+1} + |x(T;u)|^{1+\frac{\mathbf{1}}{M}} \right)$$

where $\mathcal{B}_{[1,M]} = \mathcal{B} \cap \{n_1 \leq M\}$, $\|u\|_{W^{-\mathbf{1},p}} = \|u_1\|_{L^p}$ and $u_1(t) = \int_0^t u$.

- Choose $B \in \mathcal{B}$ st the functionnal $\eta_B(T,.)$ is signed for T small.
- Find $M \in \mathbb{N}$ st $\|u\|_{W^{-\mathbf{1},M+\mathbf{1}}}^{M+1} = o(\eta_B(T,u))$ when $(T, \|u\|_{W^{m,\infty}}) \to 0$.

Then a necessary condition for STLC is

$$f_B(0) \in \text{Span}\left\{ f_b(0); b \in \mathcal{B}_{[1,M]} \setminus \{B\} \right\}$$

Indeed otherwise, $x(T;u)$ drifts along $f_B(0)$:

$$\mathbb{P}x(T,u) = \eta_B(T,u) + o(|\eta_B(T,u)| + |x(T,u)|).$$

Motions of the form $x(T,u) = -\epsilon f_B(0)$ are impossible.

Drawback : The coordinates η_B are not signed in general
• a principal part ξ_B ("coordinates of the second kind") :
easily computable by recursion, nice for $\mathcal{B}^\star$ i.e. obvious signs
• cross terms of other $\xi_{b'}$
ex : $\eta_{W_\mathbf{1}}(t,u) = \int_0^t u_1^2 - u_1(t)u_2(t)$

Our unified approach for obstructions to STLC

$$x(T;u) = \sum_{b \in \mathcal{B}^\star_{[\mathbf{1},M]}} \xi_b(T,u) f_b(0) + \text{cross terms} + O\left(\|u\|^{M+1}_{W^{-\mathbf{1},M+\mathbf{1}}} + |x(T;u)|^{1+\frac{\mathbf{1}}{M}} \right)$$

- Choose $B \in \mathcal{B}^\star$ such that the functional $\xi_B(T,.)$ is signed.
- Find $M \in \mathbb{N}$ st $\|u\|^{M+1}_{W^{-\mathbf{1},M+\mathbf{1}}} = o(\xi_B(T,u))$ when $(T, \|u\|_{W^{m,\infty}}) \to 0$ using interpolation inequality.
- Prove cross terms $= o(|\xi_B(T,u)| + |x(T;u)|)$ when

$$f_B(0) \notin \text{Span}\left\{ f_b(0); b \in \mathcal{B}^\star_{[1,M]} \setminus \{B\} \right\} \qquad (\star)$$

using closed loop estimates + interpolation

If $(\star)$ and T, $\|u\|_{W^{m,\infty}}$ are small enough then $x(T;u)$ drifts along $f_B(0)$.

Thus a NC for $W^{m,\infty}$-STLC is $\quad f_B(0) \in \text{Span}\left\{ f_b(0); b \in \mathcal{B}^\star_{[1,M]} \setminus \{B\} \right\}$.

$\mathcal{B}_1^\star$: $M_\nu := X_1 0^\nu$ $\quad u_{\nu+1}(t) = \int_0^t \frac{(t-\tau)^\nu}{\nu!} u(\tau) d\tau$

$\mathcal{B}_2^\star$: $W_{j,\nu} := (M_{j-1}, M_j) 0^\nu$ $\quad \int_0^t \frac{(t-\tau)^\nu}{\nu!} u_j(\tau)^2 d\tau$

where $b0^\nu = [\dots[b, X_0], \dots, X_0]$ and X_0 appears ν times.

Let us prove the following results.

Theorem

L^∞-$STLC \Rightarrow f_{W_1}(0) \in Span\{f_b(0); b \in \mathcal{B}^\star_{[1,2]} \setminus \{W_1\}\}$

$f_{W_2}(0) \in Span\{f_b(0); b \in \mathcal{B}^\star_{[1,3]} \setminus \{W_2\}\}$

ex : The following systems are not L^∞-STLC

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1^2 + x_1^3 \end{cases} \qquad \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_2 \\ \dot{x}_3 = x_2^2 - x_1^4 \end{cases}$$

Proof of the necessary condition on $W_1 = [X_1, [X_1, X_0]]$

We assume

$$f_{W_1}(0) \notin F := \text{Span}\{f_b(0); b \in \mathcal{B}^\star_{[1,2]} \setminus \{W_1\}\} \qquad (\star)$$

Let $\mathbb{P} : \mathbb{R}^n \to \mathbb{R}$ be a projection on $f_{W_1}(0)$ parallel to F.
We apply $\mathbb{P}$ to our representation formula

$$x(T; u) = \sum_{b \in \mathcal{B}^\star_{[1,2]}} \eta_b(T, u) f_b(0) + O\left(\|u_1\|^3_{L^3} + |x(T; u)|^{\frac{3}{2}} \right)$$

$$\begin{aligned} \mathbb{P}x(T; u) &= \eta_{W_1}(T, u) + O\left(\|u_1\|^3_{L^3} + |x(T; u)|^{\frac{3}{2}} \right) \\ &= \int_0^T \frac{u_1^2}{2} + \frac{1}{2} u_1(T) u_2(T) + O\left(\int_0^T |u_1|^3 + |x(T; u)|^{\frac{3}{2}} \right) \\ &= \int_0^T \frac{u_1^2}{2} + O\left(|u_1(T)|^2 + (T + \|u_1\|_{L^\infty}) \int_0^T u_1^2 + |x(T; u)|^{\frac{3}{2}} \right) \end{aligned}$$

because $|u_2(T)|^2 = \left| \int_0^T u_1 \right| \leq T \int_0^T u_1^2$ by Cauchy-Schwarz.

Proof of a closed loop estimate on $|u_1(T)|$ by higher order terms :

The assumption $(\star)$ implies that $f_{M_0}(0) \notin F' = \text{Span}\{f_{M_j}(0); j \geq 1\}$.
Let $\mathbb{P}'$ be a projection on $f_{M_0}(0)$ parallel to F'.
We apply $\mathbb{P}'$ to our representation formula

$$x(T,u) = \sum_{j=1}^{\infty} u_j(T) f_{M_{j-1}}(0) + O\left(\|u_1\|_{L^2}^2 + |x(T,u)|^{\frac{3}{2}}\right)$$

$$\mathbb{P}' x(T,u) = u_1(T) + O\left(\|u_1\|_{L^2}^2 + |x(T,u)|^2\right)$$

thus

$$u_1(T) = O\left(\|u_1\|_{L^2}^2 + |x(T,u)|\right).$$

Proof of the necessary condition on W_1 (3)

We have proved

$$\mathbb{P}x(T;u) = \int_0^T \frac{u_1^2}{2} + O\left(|u_1(T)|^2 + (T + \|u_1\|_{L^\infty}) \int_0^T u_1^2 + |x(T;u)|^{\frac{3}{2}}\right),$$

$$u_1(T) = O(\|u_1\|_{L^2}^2 + |x(T,u)|).$$

Thus

$$\mathbb{P}x(T;u) = \int_0^T \frac{u_1^2}{2} + O\left((T + \|u_1\|_{L^\infty}) \int_0^T u_1^2 + |x(T;u)|^{\frac{3}{2}}\right)$$

This estimate prevents motions of the form $x(T;u) = -\epsilon f_{W_1}(0)$. because they would imply $-\epsilon \geq -C\epsilon^{\frac{3}{2}}$.

Sharp necessary condition on $\mathrm{Ad}^{2\ell}_{X_1}(X_0)$

By refining/extending the previous proof, we obtain

1. $W^{-1,\infty}$-STLC $\Rightarrow$ $f_{W_1}(0) \in \mathrm{Span}\{f_b(0); b \in \mathcal{B}_1^\star\}$
   [Sussmann 1983]
2. $W^{-1,\infty}$-STLC $\Rightarrow$ $\mathrm{Ad}^{2\ell}_{f_1}(f_0)(0) \in \mathrm{Span}\{f_b(0); b \in \mathcal{B}^\star_{[1,2\ell-1]}\}$
   for all $k \in \mathbb{N}^*$ [Stefani 1986]

$$x(t;u,0) \approx \sum_{n_1(b)<2\ell} \eta_b(t,u) f_b(0) + \frac{1}{(2\ell)!} \overbrace{\left(\int_0^t u_1^{2\ell}\right)}^{\text{coercive}} f_{\mathrm{Ad}^{2\ell}_{X_1}(X_0)}(0) + \mathcal{O}\left(t\int_0^t u_1^{2\ell} + \int_0^t |u_1|^{2\ell+1}\right).$$

Proof of the necessary condition on $W_2 = [X_1 0, X_1 0^2]$

We assume

$$f_{W_2}(0) \notin F := \text{Span}\{f_b(0); b \in \mathcal{B}^\star_{[1,3]} \setminus \{W_2\}\} \qquad (\star)$$

Let $\mathbb{P} : \mathbb{R}^n \to \mathbb{R}$ be a projection on $f_{W_2}(0)$ parallel to F.
We apply $\mathbb{P}$ to our representation formula

$$x(T;u) = \sum_{b \in \mathcal{B}^\star_{[1,3]}} \eta_b(T,u) f_b(0) + O\left(\|u_1\|^4_{L^4} + |x(T;u)|^{\frac{4}{3}} \right)$$

$$\begin{aligned} \mathbb{P}x(T;u) &= \eta_{W_2}(T,u) + O\left(\|u_1\|^4_{L^4} + |x(T;u)|^{\frac{4}{3}} \right) \\ &= \int_0^T \frac{u_2^2}{2} + \frac{1}{2}(u_2 u_3 - u_1 u_4)(T) + O\left(\|u_1\|^4_{L^4} + |x(T;u)|^{\frac{4}{3}} \right) \\ &= \int_0^T \frac{u_2^2}{2} + O\left(|(u_1, u_2)(T)|^2 + T\|u_2\|^2_{L^2} + \|u_1\|^4_{L^4} + |x(T;u)|^{\frac{4}{3}} \right) \end{aligned}$$

Proof of the necessary condition on W_2 (2)

Closed loop estimate on $|(u_1, u_2)(T)|$ by higher order terms :

The assumption $(\star)$ implies that $f_{M_0}(0), f_{M_1}(0)$ are linearly independent and $\text{Span}\{f_{M_0}(0), f_{M_1}(0)\} \cap F' = \{0\}$ where $F' = \text{Span}\{f_{M_j}(0); j \geq 2\}$.
Let $\mathbb{P}'$ be a projection on $\text{Span}\{f_{M_0}(0), f_{M_1}(0)\}$ parallel to F'.
We apply $\mathbb{P}'$ to our representation formula

$$x(T,u) = \sum_{j=1}^{\infty} u_j(T) f_{M_{j-1}}(0) + O\left(\|u_1\|_{L^2}^2 + |x(T,u)|^2\right)$$

$$\mathbb{P}' x(T,u) = u_1(t) f_{M_0}(0) + u_2(t) f_{M_1}(0) + O\left(\|u_1\|_{L^2}^2 + |x(T,u)|^2\right)$$

thus

$$|(u_1, u_2)(t)| = O\left(|x(T,u)| + \|u_1\|_{L^2}^2\right)$$

We have proved

$$\mathbb{P}x(T;u) = \int_0^T \frac{u_2^2}{2} + O\left(|(u_1,u_2)(T)|^2 + T\|u_2\|_{L^2}^2 + \|u_1\|_{L^4}^4 + |x(T;u)|^{\frac{4}{3}} \right)$$

$$|(u_1,u_2)(t)| = O\left(|x(T,u)| + \|u_1\|_{L^2}^2 \right)$$

thus

$$\mathbb{P}x(T;u) = \int_0^T \frac{u_2^2}{2} + O\left(T\|u_2\|_{L^2}^2 + \|u_1\|_{L^4}^4 + |x(T;u)|^{\frac{4}{3}} \right)$$

Gagliardo-Nirenberg inequality : $\|u_1\|_{L^4}^4 \lesssim \|u\|_{L^\infty}^2 \|u_2\|_{L^2}^2$ implies

$$\mathbb{P}x(T;u) = \int_0^T \frac{u_2^2}{2} + O\left((T + \|u\|_{L^\infty}^2)\|u_2\|_{L^2}^2 + |x(T;u)|^{\frac{4}{3}} \right)$$

which prevents motions of the form $x(T;u) = -\epsilon f_{W_2}(0)$.

Sharp necessary condition on W_2 and extensions

By refining/extending the previous proof, we obtain

- L^∞-STLC $\quad\Rightarrow\quad f_{W_2}(0) \in \text{Span}\{f_b(0); b \in \mathcal{B}_1^\star \cup \{\mathrm{Ad}^3_{X_1}(X_0)0^\nu\}\}$ [Kawski 1987]
- L^∞-STLC $\quad\Rightarrow\quad f_{W_3}(0) \in \text{Span}\{f_b(0); b \in$ sharp list of $\mathcal{B}^\star_{[1,5]}\}$
- L^∞-STLC $\quad\Rightarrow\quad f_{W_k}(0) \in \text{Span}\{f_b(0); b \in \mathcal{B}^\star_{[1,2k-1]\setminus\{2\}}\}$ [Kawski's conjecture 1986]
- $W^{m,\infty}$-STLC $\quad\Rightarrow\quad f_{W_k}(0) \in \text{Span}\{f_b(0); b \in \mathcal{B}^\star_{[1,\pi(k,m)]\setminus\{2\}}\}$ where $\pi(k,m) = 1 + \lceil \frac{2k-2}{m+1} \rceil$ is optimal.
- necessary condition on quartic/sextic brackets for $W^{m,\infty}$-STLC

[KB, Marbach]

$$\dot{x} = f_0(x) + u(t) f_1(x)$$

We have proposed methodology ingredients to prove NC for STLC :

- approximate formula for the state from the $f_b(0)$,
- a new Hall basis $\mathcal{B}^\star$ of $\mathcal{L}(X)$, designed for this purpose,
- interpolation inequalities to absorb the remainder by the coercive signed drift and the smallness of the control

Perspectives :

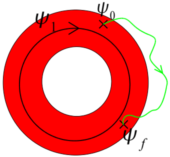
• "splitting" between good/bad brackets $\mathcal{B}^\star = \mathcal{B}^\star_{good} \cup \mathcal{B}^\star_{bad}$
$\longrightarrow$ OK at the level of $\{n_1 \leq 4\}$ [KB-Marbach]
• multi-input systems [Gherdaoui]

Transfer to PDEs : the bilinear Schrödinger equation

Example of transfer to Schrödinger PDE

$$i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi \qquad \psi(t,0) = \psi(t,1) = 0$$

Ground state :
$\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}$



Depending on the assumption on μ :

- linear test + smoothing effect [KB-Laurent 2010]
- 1 direction lost on the linearized syst and [Bournissou 2022]
  - quadratic obstruction in some regimes
  - STLC in complementary regimes : $A_3 \int_0^T u_3^2 dt + C \int_0^T u_1^2 u_2$
    This is the first positive STLC result for a PDE with a nonlinear competition.

Perspectives : Does it work for other equations ? KdV ?
How behave the high order terms for multi-input syst ? [Gherdaoui]