Local-Global Principles for Reductive Groups over Finitely Generated Fields

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Classical examples: Hasse Norm Theorem

We consider:

- K = a number field
- V = set of all places of K
- K_v = completion of K at $v \in V$.

Hasse Norm Theorem

Let L/K be a cyclic extension. An element $x \in K^{\times}$ is norm from L (i.e. $x \in N_{L/K}(L^{\times})$) if and only if x as an element of $K_v^{\times}$ is a norm from $L_{\bar{v}}$ $(\bar{v}|v)$ for all $v \in V$.

Equivalently: For a cyclic extension L/K, the natural map

$$K^{\times}/N_{L/K}(L^{\times}) \to \prod_{v \in V} K_v^{\times}/N_{L_{\bar{v}}/K_v}(L_{\bar{v}}^{\times})$$

is injective.

Classical examples: Albert-Brauer-Hasse-Noether theorem

Recall: For any field F and any cyclic extension T/F,

$$F^\times / N_{T/F}(T^\times) \simeq \mathrm{Br}(T/F) := \ker\left(\mathrm{Br}(F) \to \mathrm{Br}(T)\right).$$

Thus, Hasse Norm Theorem implies that the natural map

$$\mathrm{Br}(L/K) \to \prod_{v \in V} \mathrm{Br}(L_{\bar{v}}/K_v), \qquad [A] \mapsto [A \otimes_K K_v]$$

is injective for any cyclic extension L/K.

More generally, Albert-Brauer-Hasse-Noether Theorem (ABHN) yields injectivity of the natural map

$$\mathrm{Br}(K) \to \prod_{v \in V} \mathrm{Br}(K_v).$$

Cohomological perspective

Recall: For any cyclic extension P/F, one has

$$F^\times/N_{P/F}(P^\times) = \widehat{H}^0(P/F, P^\times) \quad \text{(Tate cohomology)}$$

and $\widehat{H}^0(P/F, P^\times) \simeq \widehat{H}^2(P/F, P^\times)$ by periodicity of cohomology.

Thus, Hasse Norm Theorem implies global-to-local maps

$$\widehat{H}^0(L/K, L^\times) \to \prod_{v \in V} \widehat{H}^0(L_{\bar{v}}/K_v, L_{\bar{v}}^\times) \quad \text{and} \quad \widehat{H}^2(L/K, L^\times) \to \prod_{v \in V} \widehat{H}^2(L_{\bar{v}}/K_v, L_{\bar{v}}^\times)$$

are injective.

Furthermore, since $\mathrm{Br}(F) \simeq H^2(F, (F^{\mathrm{sep}})^\times)$ for any field F, (ABHN) yields injectivity of global-to-local map

$$H^2(K, (K^{\mathrm{sep}})^\times) \to \prod_{v \in V} H^2(K_v, (K_v^{\mathrm{sep}})^\times).$$

Thus, Hasse Norm Theorem and (ABHN) can be interpreted as cohomological local-global principles.

Cohomological perspective (cont.)

In fact, both results can be viewed in the general context of the Galois cohomology of algebraic groups.

For any finite sep. extension P/F, there exists an algebraic group $T = R^{(1)}_{P/F}(\mathbb{G}_m)$ defined over F with the property that

$$T(F) = \{s \in P^{\times} \mid N_{P/F}(s) = 1\} \quad \text{(norm torus)}.$$

Setting $H^1(F, T) = H^1(\mathrm{Gal}(F^{\mathrm{sep}}/F), T(F^{\mathrm{sep}}))$, one shows that

$$H^1(F, T) \simeq F^{\times}/N_{P/F}(P^{\times}).$$

Thus, Hasse Norm Theorem means that global-to-local map

$$H^1(K, T) \to \prod_{v \in V} H^1(K_v, T)$$

is injective for any cyclic extension of number fields.

To go further, one needs to consider cohomology of noncommutative algebraic groups.

Cohomological perspective (cont.)

Recall: For any field F and alg. F-gp G, one sets

$$H^1(F,G) = \{\text{cont. 1-cocycles } f\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to G(F^{\mathrm{sep}})\}/\sim$$

Note: $H^1(F,G)$ generally not a group, only a pointed set.

$H^1(F,G)$ can often be interpreted in terms of twisted forms.

For any field F, we have

$$\mathrm{Aut}_{F\text{-alg.}}(M_n(F)) = \mathrm{PGL}_n(F) \qquad \text{(Skolem-Noether Theorem).}$$

Also, for any central simple F-algebra A with $\deg_F A = n$, one has

$$A \otimes_F F^{\mathrm{sep}} \simeq M_n(F^{\mathrm{sep}}).$$

Using this, one shows

$$H^1(F, \mathrm{PGL}_n) \leftrightarrow \{F\text{-isom. classes of CSA } A/F \mid \deg_F A = n\}.$$

Cohomological perspective (cont.)

Thus, (ABHN) implies that for any number field K, global-to-local map

$$H^1(K, \mathrm{PGL}_n) \to \prod_{v \in V} H^1(K_v, \mathrm{PGL}_n)$$

is injective for all $n \geqslant 2$.

Another example: Let f be n-dim non-deg. quadratic form over F. Then

$$H^1(F, \mathrm{O}_n(f)) \leftrightarrow \{F\text{-equiv. classes of non-deg. } f' \mid \dim_F f' = n\}$$

Thus, Hasse-Minkowski Theorem implies that for any number field K, global-to-local map

$$H^1(K, \mathrm{O}_n(f)) \to \prod_{v \in V} H^1(K_v, \mathrm{O}_n(f))$$

is injective for any non-deg. quadratic form f over K.

General set-up and terminology

Let

- K be a field
- V a set of rank 1 valuations of K
- K_v = completion of K at v
- G an algebraic group over K.

One says that *the Hasse principle holds* if global-to-local map

$$\theta_{G,V}\colon H^1(K,G) \to \prod_{v \in V} H^1(K_v,G)$$

is *injective*.

Kernel of $\theta_{G,V}$ is called *Tate-Shafarevich set*

$$Ш(G,V) := \ker \theta_{G,V}.$$

Some known cases of Hasse principle

K = number field, V = set of all places of K.

- We have seen Hasse principle holds for $G = R^{(1)}_{L/K}(\mathbb{G}_m)$ (L/K cyclic), PGL_n, $\mathrm{O}_n(f)$.
- In general, it is known that if G is *simply-connected* or *adjoint* alg. K-group, then
$$\theta_{G,V} \colon H^1(k, G) \to \prod_{v \in V} H^1(k_v, G)$$
is injective (i.e. Hasse principle holds).

K = func. field of p-adic curve, V = set of all discr. vals of K.

- Hasse principle known to hold for simply-connected K-groups of classical types (by work of Parimala, Suresh, Preeti, Hu).

Properness of $\theta_{G,V}$

- Hasse principle may fail for arbitrary alg. K-gps, even over number fields.

- However, over number fields, $\theta_{G,V}$ is always *proper* (i.e. pre-image of finite set is finite); in particular, $Ш(G,V)$ is finite.

- Our recent results suggest that properness should hold for reductive groups over arbitrary *finitely generated fields* with respect to *divisorial* sets of places.

Divisorial valuations

We consider the following situation.

- Let K be a finitely generated field.
- Pick a normal integral affine model $\mathfrak{X}$ for K.
- Let V = set of discrete valuations of K associated with prime divisors on $\mathfrak{X}$ (*divisorial* set).

Algebraically: We find $R \subset K$ such that $K = \mathrm{Frac}(R)$ and

- R is a finitely generated $\mathbb{Z}$-algebra (or $\mathbb{F}_p$-algebra);
- R is integrally closed in K.

Then: V corresponds to height one prime ideals of R.

Divisorial valuations: Example

Take $K = \mathbb{Q}(x)$ and $R = \mathbb{Z}[x]$.

Height one primes in R are principal and are of two types:

- $\mathfrak{p} = (p(x))$, with $p(x) \in \mathbb{Z}[x]$ irreducible of content 1;
- $\mathfrak{p} = (p)$, $p \in \mathbb{Z}$ a prime.

Two corresponding types of discrete valuations:

- “geometric places” V_0;
- “arithmetic places” V_1.

Then $V = V_0 \cup V_1$ is divisorial set of discrete valuations associated with the model $\mathfrak{X} = \mathrm{Spec}(R)$ of K.

Properness conjecture

Suppose

- K a finitely generated field;
- V a divisorial set of places of K.

Conjecture.

If G is a (connected) reductive algebraic K-group, then $\theta_{G,V}$ is proper. In particular, the Tate-Shafarevich set $\text{Ш}(G, V)$ is finite.

We have resolved conjecture for:

- all algebraic tori over arbitrary finitely generated fields;
- certain semisimple groups over 2-dim. global fields (i.e. $K = k(C)$ for number field k), and some other cases.

For semisimple adjoint groups, conjecture is closely related to study of groups with good reduction.

Theorem 1.

Suppose K is a finitely generated field and V is a divisorial set of places. **Then** *for any K-torus T, the global-to-local map*

$$\theta_{T,V} \colon H^1(K,T) \to \prod_{v \in V} H^1(K_v,T)$$

is proper. Equivalently, the Tate-Shafarevich group $Ш(T,V) = \ker \theta_{T,V}$ is finite.

- Same result holds for any alg. K-group whose connected component is a torus.
- Classical proof for tori over number fields relies on Tate-Nakayama duality, which is not available in general.
- Our proof uses adelic methods. In particular, it shows that finiteness of $Ш(T,V)$ for a torus T over a number field follows from finiteness of class number and finite generation of group of S-units.

Adelic set-up

Let

- K be a field
- V a set of discrete valuations of K such that
  (*) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite.

For such V, we define *Picard group* $\mathrm{Pic}(V) = \mathrm{Div}(V)/\mathrm{P}(V)$,
$\mathrm{Div}(V)$ = free abelian group on $v \in V$,
$\mathrm{P}(V)$ = subgp of "principal divisors" $\sum_{v \in V} v(a)v$, $a \in K^{\times}$.

Example: If K is a number field and V is set of all finite places of K, then $\mathrm{Pic}(V) = \mathrm{Cl}(\mathcal{O}_K)$.

Adelic set-up (cont.)

The ring of adeles of K with respect to V is

$$\mathbb{A}_K(V) = \prod_{v \in V}{}' K_v = \left\{ (x_v) \in \prod_{v \in V} K_v \mid x_v \in \mathcal{O}_v \text{ for almost all } v \in V \right\}.$$

Group $\mathbb{I}_K(V)$ of invertible elements is idele group.

Let

- $\mathbb{A}_K^\infty(V) = \prod_{v \in V} \mathcal{O}_v$ (subring of integral adeles)
- $\mathbb{I}_K^\infty(V) = \prod_{v \in V} \mathcal{O}_v^\times$ (subgroup of integral ideles)

Observe that condition (*) yields diagonal embeddings

$$K \hookrightarrow \mathbb{A}_K(V) \quad \text{and} \quad K^\times \hookrightarrow \mathbb{I}_K(V).$$

Note: The map $\mathbb{I}_K(V) \to \mathrm{Div}(V)$, $(x_v) \mapsto \sum_{v \in V} v(x_v) \cdot v$ induces isom. $\mathbb{I}_K(V)/(\mathbb{I}_K^\infty(V) \cdot K^\times) \xrightarrow{\sim} \mathrm{Pic}(V)$.

Adelic set-up (cont.)

Let

- L/K finite field extension
- $\bar{V}$ = set of all extensions of valuations in V to L

Then $\mathbb{A}_K(V) \otimes_K L \simeq \mathbb{A}_L(\bar{V})$.

$\Rightarrow$ for Galois L/K, $\mathbb{A}_L(\bar{V})$ has natural $\mathrm{Gal}(L/K)$-action.

Next:

- T = a K-torus (thus, $T \otimes_K K^{\mathrm{sep}} \simeq (\mathbb{G}_{m,K^{\mathrm{sep}}})^d$)
- for each $v \in V$, let $T(\mathcal{O}_v) \subset T(K_v)$ be the unique max. bounded subgp.

Then

$$T(\mathbb{A}_K(V)) = \left\{ (x_v) \in \prod_{v \in V} T(K_v) \mid x_v \in T(\mathcal{O}_v) \text{ for almost all } v \right\}.$$

Adelic set-up (cont.)

As before, we have

- $T(\mathbb{A}_K^\infty(V)) = \prod_{v \in V} T(\mathcal{O}_v)$ (subgroup of integral adeles)
- diagonal embedding $T(K) \hookrightarrow T(\mathbb{A}_K(V))$.

Furthermore,

- $T(\mathbb{A}_L(\bar{V}))$ and $T(\mathbb{A}_L^\infty(\bar{V}))$ defined analogously for any finite extension L/K;
- for L/K Galois, $\mathrm{Gal}(L/K)$-action on $\mathbb{A}_L(\bar{V})$ induces $\mathrm{Gal}(L/K)$-action on $T(\mathbb{A}_L(\bar{V}))$;
- in particular, diagonal embedding $T(L) \hookrightarrow T(\mathbb{A}_L(\bar{V}))$ induces hom.

$$\lambda_{L/K} \colon H^1(L/K, T) \to H^1(L/K, T(\mathbb{A}_L(\bar{V}))).$$

Sketch of argument over number fields

Let

- K a number field
- V = set of all finite places of K
- T a K-torus
- L/K finite Galois splitting field of T (i.e. $T \otimes_K L \simeq (\mathbb{G}_{m,L})^d$)

Our goal is to show that

$$Ш(T, V) := \ker \left(H^1(K, T) \to \prod_{v \in V} H^1(K_v, T) \right)$$

is finite.

Sketch of argument over number fields (cont.)

Step 1: Show that

$$Ш(T, V) = Ш(L/K, T, V) := \ker\left(H^1(L/K, T) \to \prod_{v \in V} H^1(L_{\bar{v}}/K_v, T)\right).$$

This follows from Hilbert's Theorem 90.

Step 2: Show that

$$Ш(L/K, T, V') = \ker\left(H^1(L/K, T) \xrightarrow{\lambda_{L/K,V'}} H^1(L/K, T(\mathbb{A}_L(\bar{V}')))\right)$$

for any set $V' \subset V$.

Note: Both steps work for any field K.

Sketch of argument over number fields (cont.)

Step 3: Show that there exists cofinite $V' \subset V$ such that

$$T(\mathbb{A}_L(\bar{V}')) = T(\mathbb{A}_L^\infty(\bar{V}')) \cdot T(L).$$

Since $\mathrm{Cl}(\mathcal{O}_L)$ is finite, we can find finite $S \subset V$ such that for $\bar{S} = \{\text{all extensions of } v \in S \text{ to } L\}$, we have $\mathrm{Cl}(\mathcal{O}_{L,\bar{S}}) = \{e\}$.

Setting $V' = V \setminus S$, we obtain

$$\mathbb{I}_L(\bar{V}')/(\mathbb{I}_L^\infty(\bar{V}') \cdot L^\times) \simeq \mathrm{Pic}(L, \bar{V}') = \mathrm{Cl}(\mathcal{O}_{L,\bar{S}}) = \{e\}.$$

$$\Rightarrow \mathbb{I}_L(\bar{V}') = \mathbb{I}_L^\infty(\bar{V}') \cdot L^\times$$

Since T is split over L, we have

$$T(\mathbb{A}_L(\bar{V}')) = T(\mathbb{A}_L^\infty(\bar{V}')) \cdot T(L).$$

Sketch of argument over number fields (cont.)

Step 4: Let $E(T, V', L) = T(L) \cap T(\mathbb{A}_L^{\infty}(\bar{V}'))$ for V' as in Step 3. Show that $Ш(T, V')$ is contained in the image of the map

$$\nu \colon H^1(L/K, E(T, V', L)) \to H^1(L/K, T).$$

This follows from a direct computation.

Step 5: Observe that $E(T, V', L) = T(L) \cap T(\mathbb{A}_L^{\infty}(\bar{V}'))$ is finitely generated, hence $H^1(L/K, E(T, V', L))$ is finite.

We have $E(T, V', L) \simeq U^d$ $(d = \dim T)$, where

$$U = \{x \in L^{\times} \mid v(x) = 0 \text{ for all } v \in \bar{V}'\},$$

which is finitely generated by Dirichlet's S-unit theorem.

Thus, $H^1(L/K, E(T, V', L))$ is a finitely generated torsion abelian group, hence finite.

Passing to arbitrary finitely generated fields

Suppose K is an arbitrary finitely generated field and V is a divisorial set of places associated to a normal affine model $\mathfrak{X} = \operatorname{Spec} A$ over $\mathbb{Z}$.

We have two important facts:

- The group of units $A^\times$ is finitely generated (P. Samuel)
- The Picard group $\operatorname{Pic}(V) \simeq \operatorname{Pic}(\mathfrak{X})$ is finitely generated (B. Kahn)

Theorem 1.

Suppose K is a finitely generated field and V is a divisorial set of places. **Then** *for any K-torus T, the global-to-local map*

$$\theta_{T,V} \colon H^1(K,T) \to \prod_{v \in V} H^1(K_v, T)$$

is proper (equivalently, $Ш(T,V)$ is finite.)

Some modifications of the argument using Bertini-type theorems yield:

Theorem 2.

Let X be a smooth geometrically integral variety over a number field k and let V be the set of discrete valuations of function field $K = k(X)$ associated with prime divisors of X. Then $\text{Ш}(T, V)$ is finite for any k-defined torus T.

Remarks:

- Case where X is a curve first studied by Harari and Szamuely.
- If T is not assumed to be k-defined, then finiteness of $\text{Ш}(T, V)$ is an open probem even when X is a curve.

Serre introduced condition (F) for *perfect* fields to study finiteness properties of Galois cohomology.

Recall: A perfect field k is of type (F) if

(F) For every $m \geqslant 1$, $\mathrm{Gal}(\bar{k}/k)$ has finitely many open subgroups of index m.

Examples: finite fields, p-adic fields, $\mathbb{C}((t))$, etc.

Theorem 3.

Suppose k is a field of char. 0 that is of type (F). Let X be a smooth geometrically integral variety over k and V be the set of discrete valuations of function field $K = k(X)$ associated with prime divisors of X. Then $\text{Ш}(T, V)$ is finite for any K-defined torus T.

Proof depends on purity results and finiteness statements for étale cohomology.

Since the Properness Conjecture is known for tori, a natural problem is the following:

Problem.
For finitely generated fields of char. 0, reduce Properness Conjecture from arbitrary reductive groups to (adjoint) semisimple groups.

Conceptually, such a reduction is desirable due to the fact that Properness Conjecture for adjoint semisimple groups is closely related to analysis of groups with good reduction.

Let K be a field equipped with discrete valuation v.

Definition

A reductive K-group G has *good reduction* at v if there exists a reductive group scheme $\mathcal{G}$ (which is unique) over valuation ring $\mathcal{O}_v \subset K_v$ such that

$$\textit{generic fiber } \mathcal{G} \otimes_{\mathcal{O}_v} K_v \text{ is isomorphic to } G \otimes_K K_v.$$

Then *special fiber* (reduction)

$$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$$

is *a connected reductive* group over residue field $K^{(v)}$.

Examples.

0. If G is K-split then G has a good reduction at *any* v (follows from Chevalley's construction).

1. For a central simple K-algebra A, group $G = \mathrm{SL}_{1,A}$ has good reduction at v if there exists an Azumaya algebra $\mathcal{A}$ over $\mathcal{O}_v$ such that

$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$

(in other words, A is *unramified* at v).

2. $G = \mathrm{Spin}_n(q)$ has good reduction at v if (over K_v)

$$q \sim \lambda(a_1 x_1^2 + \cdots + a_n x_n^2) \quad \text{with} \quad \lambda \in K_v^\times, \ a_i \in \mathcal{O}_v^\times$$

(assuming that $\mathrm{char}\, K^{(v)} \neq 2$).

A K-group G' is a *K-form* (or $\overline{K}/K$-form) of G if

$$G' \otimes_K \overline{K} \simeq G \otimes_K \overline{K} \quad \text{(where } \overline{K} \text{ is a sep. closure of } K\text{).}$$

Examples.

1. If A is a central simple algebra of degree n over K, **then**

$$A \otimes_K \overline{K} \simeq M_n(\overline{K})$$

and $G' = \mathrm{SL}_{1,A}$ is a K-form of $G = \mathrm{SL}_n$.

Examples (cont.).

2. If q is a nondegenerate quadratic form in n variables over K $(\mathrm{char}\, K \neq 2)$ and

$$G = \mathrm{Spin}_n(q),$$

then for any other nondegenerate quadratic form q' in n variables,

$$G' = \mathrm{Spin}_n(q')$$

is a K-form of G.

If n is *odd*, then these are **all** K-forms.

Otherwise, there may be K-forms coming from hermitian forms over noncommutative division algebras.

Main Finiteness Conjecture

Let:

- K a finitely generated field;
- V a divisorial set of places of K;
- G a (connected) reductive K-group.

Main Conjecture for Groups with Good Reduction

If $\mathrm{char}\, K$ *is "good," then the set of K-isomorphism classes of (inner) K-forms G' of G having good reduction at all $v \in V$ is finite.*

(If G is absolutely almost simple, $\mathrm{char}\, K = p$ is "good" for G if $p = 0$ or p does not divide order of Weyl group of G. For non-semisimple reductive groups only char. 0 is "good.")

Connection to Properness Conjecture

Suppose

- K a finitely generated field;
- V a divisorial set of places of K.

Recall that we have the global-to-local map

$$\theta_{G,V}\colon H^1(K,G) \to \prod_{v \in V} H^1(K_v,G)$$

for any algebraic K-group G.

Proposition 4.

Assume Main Conjecture holds for an absolutely almost simple simply connected K-group G and all divisorial sets of places of K. **Then** *$\theta_{\overline{G},V}$ is proper for corresponding adjoint group $\overline{G}$ and any divisorial set V.*

Some results on the Main Conjecture

We have resolved the Main Conjecture in several cases.

For K an arbitrary finitely generated field:

- Algebraic tori
- Inner forms of type A_n (i.e. $G = \mathrm{SL}_{1,A}$) — uses finiteness results for unramified Brauer group.

For $K = k(C)$ a 2-dimensional global field:

- $G = \mathrm{Spin}_n(q)$ $(n \geqslant 5)$, G of type G_2;
- $G = \mathrm{SU}_n(L/K, h)$, L/K quadratic extension, h nondegenerate hermitian form of $\dim \geqslant 2$;
- $G =$ universal cover of $\mathrm{SU}_n(D, h)$, D quaternion algebra over K, h nondeg. skew-hermitian form over D (I.R.).

These results depend on finiteness of certain unramified cohomology groups.

Some results on the Properness Conjecture

Consider the global-to-local map

$$\theta_{G,V} \colon H^1(K,G) \to \prod_{v \in V} H^1(K_v,G).$$

Several cases where we have established properness of $\theta_{G,V}$:

- $\mathrm{PSL}_{1,A}$ over arbitrary finitely generated fields.

- K a 2-dimensional global field and
  - $G = \mathrm{SO}_n(q)$ $(n \geqslant 5)$;
  - G of type G_2;
  - $G = \mathrm{SU}_n(L/K,h)$, L/K quadratic extension, h nondegenerate hermitian form of $\dim \geqslant 2$;
  - $G = \mathrm{SL}_{1,A}$, A a c.s.a$/K$ of square-free degree.

- K a purely transcendental extension or function field of Severi-Brauer variety over number field and G of type G_2.

The genus problem was inspired by Amitsur's Theorem and initially dealt with division algebras having the same maximal subfields.

Definition.

Let D be a finite-dimensional central division algebra over K.

The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ division algebra with same maximal subfields as } D\}.$$

Key result: The genus $\mathbf{gen}(D)$ is finite for any central division K-algebra D over a finitely generated field K.

Genus of an algebraic group

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semisimple groups over a field K.

We say: G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

- Let G be an absolutely almost simple K-group.

$\mathbf{gen}_K(G) =$ set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori as G.

Question 1. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 5. (G. Prasad-A. Rapinchuk)

Let G *be an absolutely almost simple simply connected algebraic group over a number field* K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If* G *is not of type* A_n, D_{2n+1}, *or* E_6, *then* $|\mathbf{gen}_K(G)| = 1$.

Conjecture.

(1) *For* $K = k(x)$, k *a number field, and* G *an absolutely almost simple simply connected* K*-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;

(2) *If* G *is an absolutely almost simple group over a finitely generated field* K *of "good" characteristic, then* $\mathbf{gen}_K(G)$ *is finite.*

Connections to groups with good reduction

Theorem 6.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that residue field $K^{(v)}$ is finitely generated, and G has good reduction at v.

Then *every $G' \in \mathbf{gen}_K(G)$ has good reduction at v, and reduction $\underline{G}'^{(v)} \in \mathbf{gen}_{K^{(v)}}(\underline{G}^{(v)})$.*

In particular, the Main Conjecture yields finiteness results for the genus.

A sampling of results

Theorem 7.

(1) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where k is a number field or a finite field of characteristic $\neq 2$. Then for $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$, we have $|\mathbf{gen}_K(G)| = 1$.*

(2) *Let $G = \mathrm{SL}_{m,D}$, where D is a central division algebra over a finitely generated field K. Then $\mathbf{gen}_K(G)$ is finite.*

Theorem 8.

Let $K = k(C)$ be a 2-dimensional global field for a number field k, and set $G = \mathrm{Spin}_n(q)$. Then for any $n \geqslant 10$, the genus $\mathbf{gen}_K(G)$ is finite.

Remark: We previously showed this for odd n. My current work establishes finiteness of genus for all n.

Applications of the Main Finiteness Conjecture

The conjecture has close connections to:

- Local-global principles for algebraic groups.
- The genus problem for simple algebraic groups.
- Finiteness properties of unramified cohomology.
- Analysis of weakly commensurable Zariski-dense subgps and applications to classical problems on locally symmetric spaces.

Thus, study of groups with good reduction occupies a central place in the emerging arithmetic theory of algebraic groups over higher-dimensional fields.

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