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Scaling of Extreme Events in Self-Organized Critical Systems

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Outline

1. Introduction
2. The Stochastic Manna model and the BTW model
3. Generalised extreme value distributions
4. Results & Conclusion

Introduction

- A complex system refers to a system composed of numerous interconnected components or elements. The interactions among these components give rise to emergent behavior that is often difficult to predict from the properties of the individual parts.

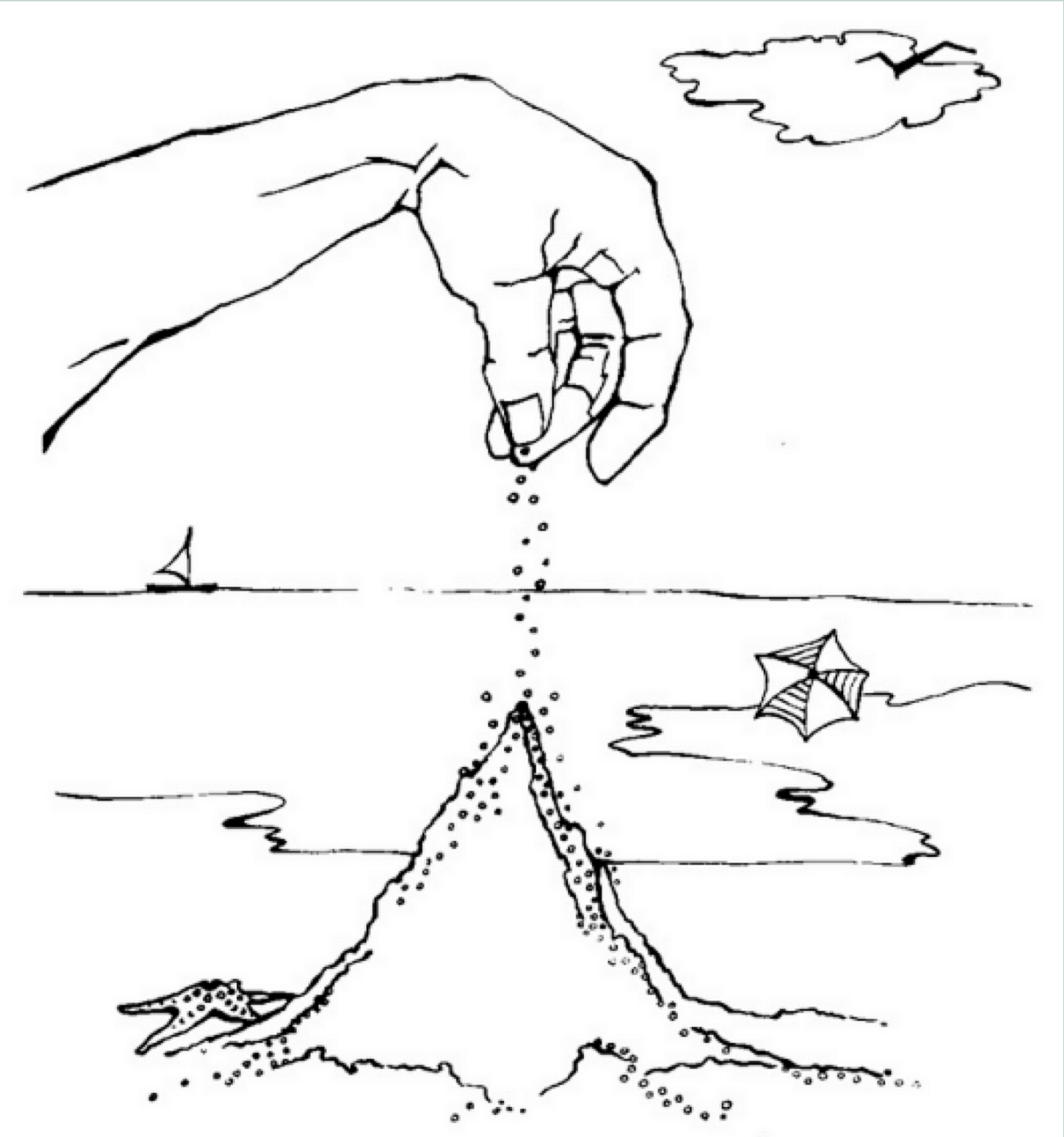
- The dynamics of complex systems may give rise to large fluctuations in a relevant variable, resulting in extreme events that may be defined as events exceeding a predefined large threshold.

- These events may have cascading effects throughout the system and may deviate significantly from a system's average (or usual) patterns. They are often rare but can have profound and disproportionate impacts namely, the breakdown of a mechanical structure, an earthquake, flooding or crashes in financial markets.

- Other instances where extreme events have been reported are the self organized critical systems.

✓ **<u>Bak-Tang-Weisenfeld's Insight</u> (1987,1988)**: *What if systems naturally evolve into a critical state without tuning?*

- Introduced the concept of Self-Organized Criticality (SOC):
  - A mechanism by which dynamical systems self-tune to a critical point.
    - ✓ The Sandpile Model: A minimal lattice model where:
      - Grains are added slowly.
      - Sites topple deterministically when a threshold is reached.
      - Avalanches emerge with no characteristic size.



System evolves into a scale-invariant state—criticality emerges naturally.

P. Bak, C. Tang, and K. Wiesenfeld, **Phys. Rev. Lett. 59, 381 (1987); Phys. Rev. A 38, 364 (1988).**

C. Tang and P. Bak, **Phys. Rev. Lett. 60, 2347 (1988); J. Stat. Phys. 51, 797 (1988).**

D. Dhar and R. Ramaswamy, **Phys. Rev. Lett. 63, 1659 (1989)**.

S. S. Manna, **J. Phys. A: Math. Gen. 24 L363 (1991)**.

Why Study Extreme Events in Self-Organized Critical Models?

- ✓ **Intrinsic Criticality:** Self-organized models naturally evolve to critical state that are useful to study extremal dynamics.
- ✓ **Understanding Rare but Impactful Events:** Extreme avalanches, though rare, can dominate system behavior - crucial for risk assessment in real-world analogues (e.g., earthquakes, blackouts, financial crashes). Providing windows into the resilience and fragility of complex systems.
- ✓ **Universal Behavior:** Studying extreme events reveal universal scaling laws beyond average behaviour - shedding light on deep statistical structure.
- ✓ **Bridging different fields:** bridges statistical mechanics and extreme value theory for understanding and predicting real-world phenomena.

Generalised Extreme Value Distribution

- The cumulative distribution function (CDF) for the maxima x results in the GEV distribution

$$\mathscr{F}(x, \mu, \beta, \xi) = \exp\left\{ -\left[1 + \xi\left(\frac{x-\mu}{\beta}\right)\right]^{-1/\xi} \right\}$$

where μ, β and ξ are location (or mode), scale, and shape parameters, respectively.

- The corresponding probability distribution function (PDF) is given by

$$f(x, \mu, \beta, \xi) = \frac{1}{\beta}\left(1 + \xi\frac{x-\mu}{\beta}\right)^{-(\xi+1)/\xi} \exp\left\{ -\left[1 + \xi\left(\frac{x-\mu}{\beta}\right)\right]^{-1/\xi} \right\}$$

➡ CDF: Gumbel distribution $\xi = 0$

$$\mathscr{F}(x, \mu, \beta) = \exp\left\{ -\exp\left(\frac{x-\mu}{\beta}\right) \right\}$$

➡ PDF: Gumbel distribution $\xi = 0$

$$f(x, \mu, \beta) = \frac{1}{\beta}\exp\left\{ -\exp\left(-\frac{x-\mu}{\beta}\right) - \left(\frac{x-\mu}{\beta}\right) \right\}$$

The Stochastic Manna Model & the BTW Model

1. Consider 2D square lattice of size $N = L^2$, where L is the linear extent.
2. Choose a random site (i, j) and add a grain: $h(i, j) \to h(i, j) + 1$
3. If any site having $h(i, j) \geq h_c$, it topples

$$h_i \to h_i - h_c,$$
$$h_j \to h_j + 1.$$

where $j's$ are the nearest neighbours for

a) The **stochastic Manna Model**: two randomly chosen sites from left, right, top and bottom.
b) The **BTW Model**: all four nearest sites, *i.e.*, left, right, top and bottom.

In both the models, the observables of interest are the avalanche size (s) and area (a), namely $x \in \{s, a\}$.

The maxima are obtained by dividing the dataset into intervals of fixed length. Then the maximum value from each block is considered.

Results

- Mean

$$\mu = \frac{1}{N}\sum x_i \sim N^{\alpha_1}$$

- Variance

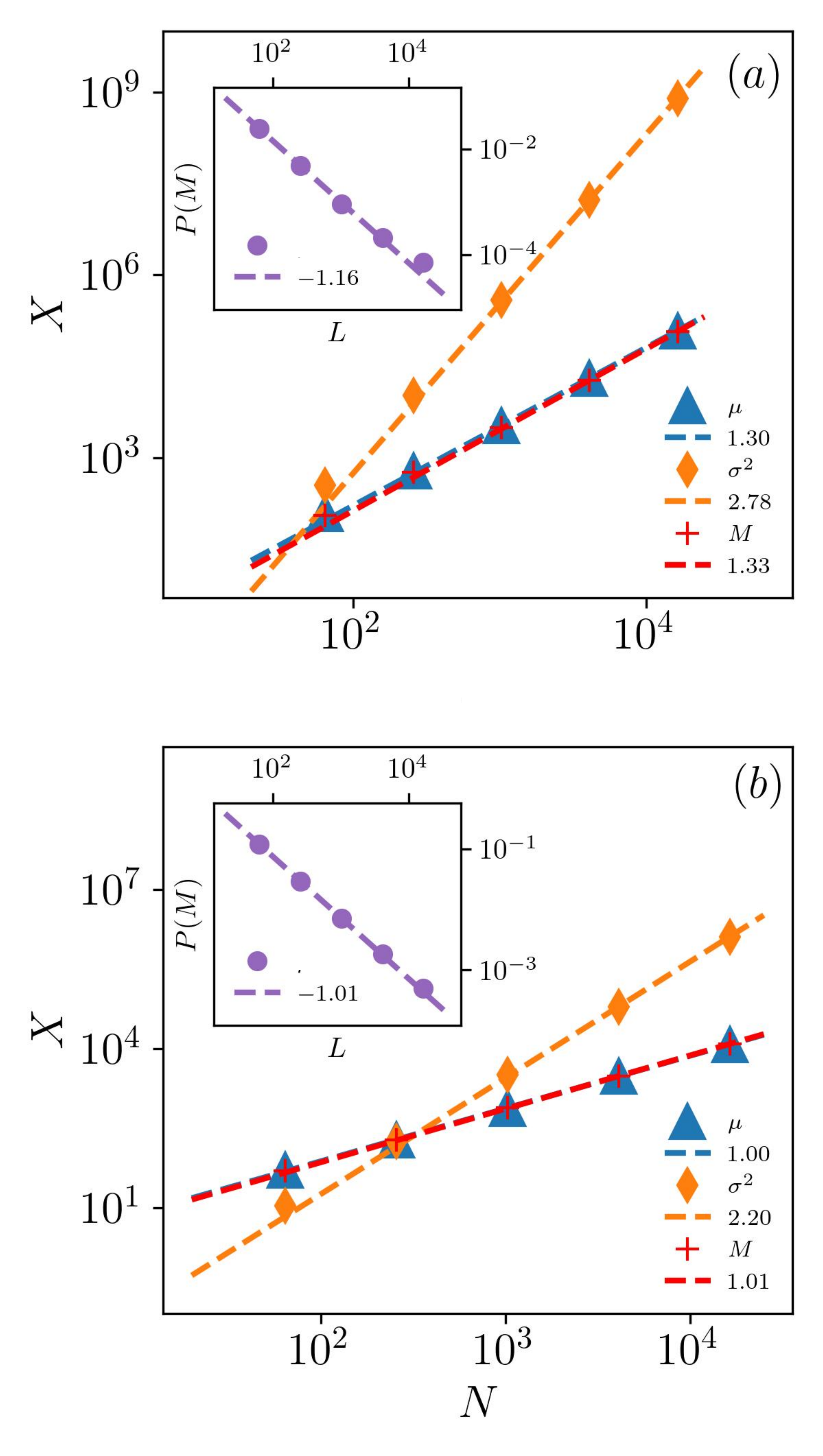
$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \sim N^{2\alpha_2}$$

- Mode of the extreme activities shows a unimodal behaviour

$$M \sim N^{\alpha_3}$$

- Probability distribution of extreme activities

$$P(M) \sim N^{-\alpha_4}$$



Introducing a scaling variable

$$u = \frac{x - M}{\sigma} = \frac{\Delta x}{\sigma}$$

Proposing a scaling function as

$$F(u) = c\frac{P(x)}{P(M)}$$

Normalised probability distribution implies

$$\int P(x)dx = 1$$

Scaling functions

$$P(x) \sim \frac{1}{N^\alpha}\mathscr{F}\left(\frac{x - M}{N^\alpha}\right) = \frac{1}{N^\alpha}\mathscr{F}(u)$$

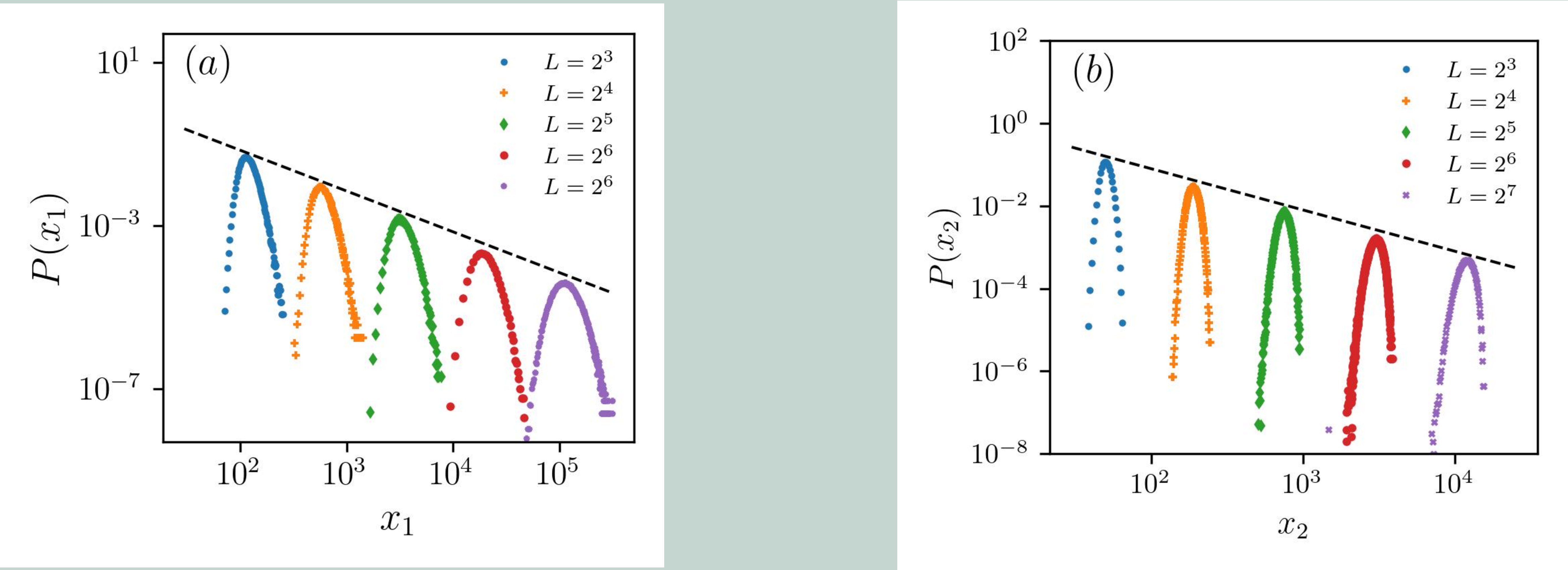
$$\int P(x)dx \sim N^{\alpha_2 - \alpha_4} = 1 \qquad \alpha_2 = \alpha_4$$

$$\langle \Delta x \rangle = N^{2\alpha_2 - \alpha_4} \sim N^{\alpha_1} \qquad 2\alpha_2 - \alpha_4 = \alpha_1 \implies \alpha_1 = \alpha_2$$

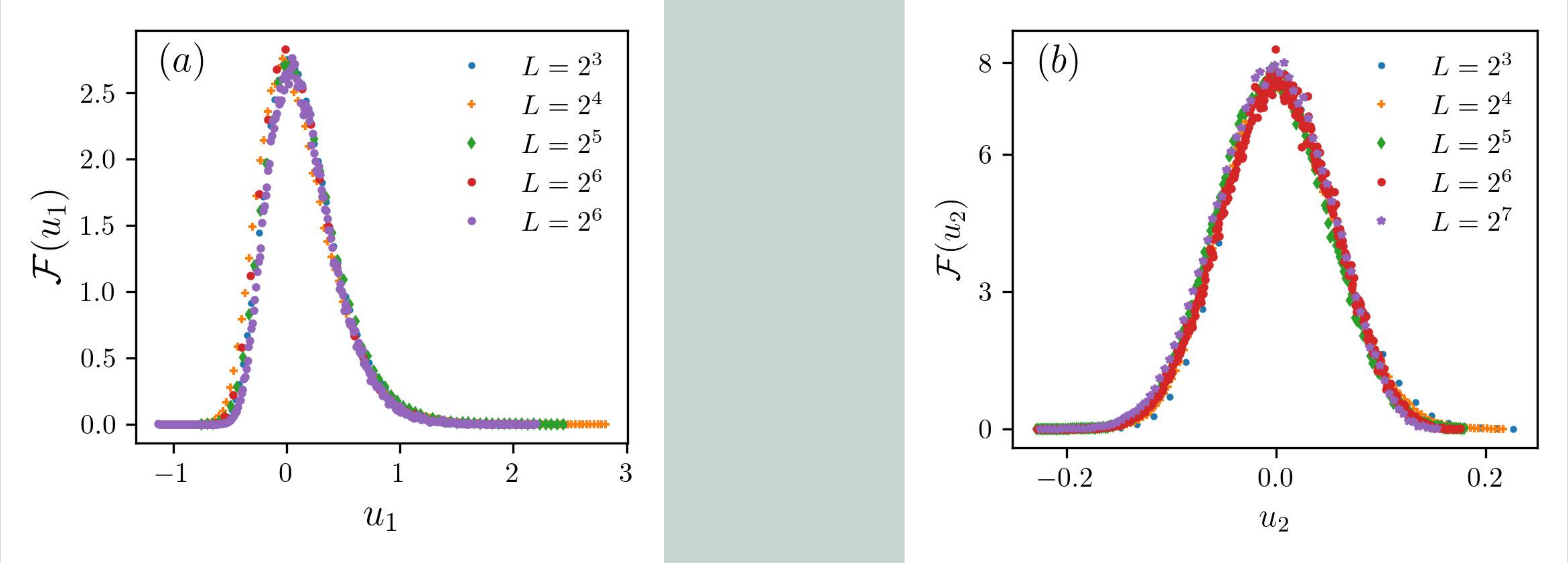
$$M \sim N^{\alpha_3 = \alpha_4} \qquad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$$

Model	**Extreme activity**	α_1	α_2	α_3	α_4
SMM	Size	1.30	1.34	1.33	1.16
	Area	1.00	1.10	1.01	1.01
BTW	Size	1.19	1.36	1.13	1.25
	Area	1.00	1.10	0.99	1.10

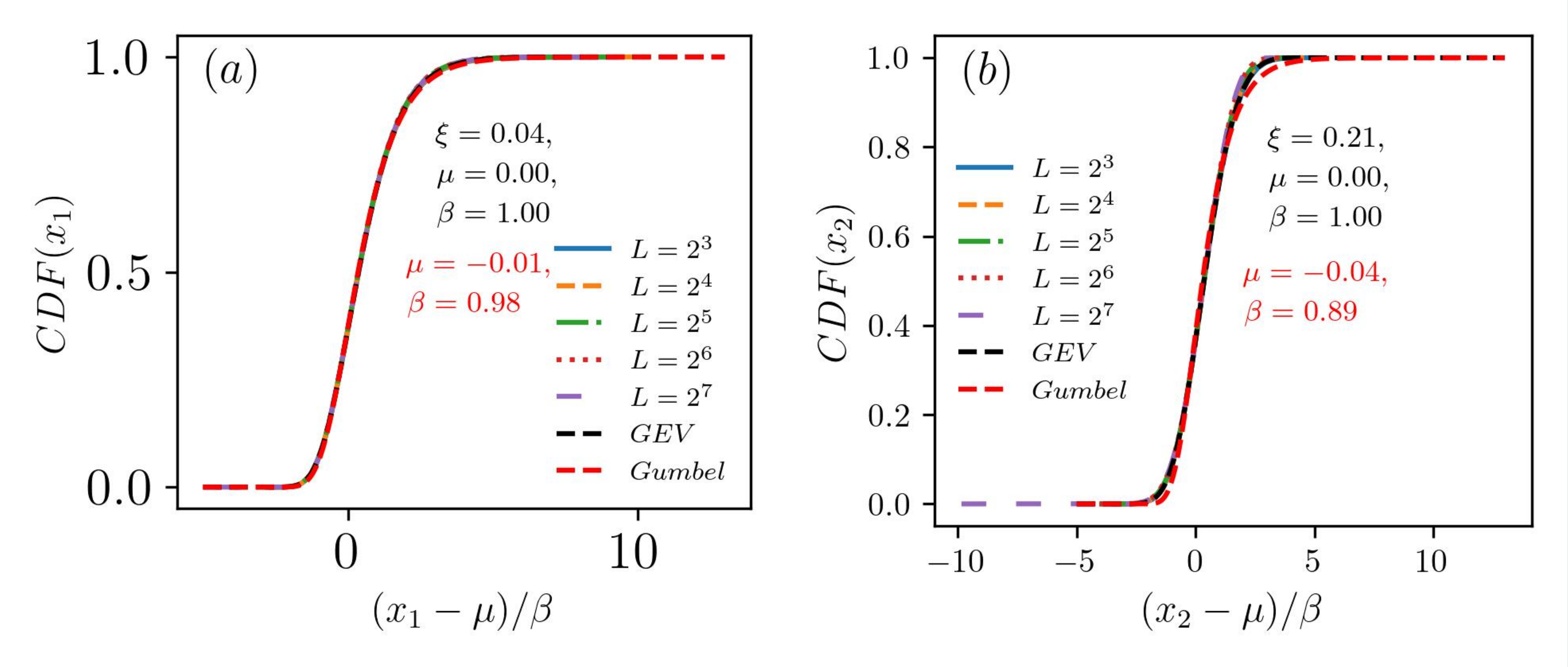
Critical exponents for extreme avalanche activities in case of the SMM and the BTW model.



The probability distribution $P(x)$ for (a) extreme avalanche size x_1 and (b) extreme avalanche area x_2 in case of the SMM for different system sizes $N = L^2$. The black dashed line has slope -1 which represents $P(M) \sim x^{-1}$.



The data collapse corresponding to the probability distribution



For the SMM, the data collapse of the CDF is plotted for the extreme avalanche size and area. The red dashed line represents the fitted data with Gumbel family ($\xi = 0.0$) while the black dashed line represents the fitted GEV family with parameters (μ, σ, ξ). In both cases the goodness of fit is $R^2 > 0.99$

Model	Extreme activity	GEV parameters		
		ξ	μ	β
SMM	Size	0.04	0.00	1.00
	Area	0.21	0.00	1.00
BTW	Size	0.003	-0.09	1.00
	Area	0.153	-0.11	0.99

The fitted parameters, describing the scaling function for the probability distribution of extreme avalanche activities for different SOC models. In all the cases, the goodness of fit is $R^2 > 0.99$.

Conclusion

With finite system size, the avalanche activities in the SMM and BTW models follow a power law with upper cut-off.

The statistical characteristics of extreme events such as mean, variance, and the mode follow the system size scaling characterized by the same critical exponent.

Employing the FS scaling with GEV theory, we obtain the scaling function (or data collapse) for the PDF of extreme events.

For extreme avalanche size, the data collapse fits with the Gumbel distribution whereas it is Frèchet for extreme avalanche area.

Thank you.