IFS and Fractals

Definition (**Metric**): A metric d on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y \in X$:

(1) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ (symmetry);

(3) $d(x,y) \leq d(x,z) + d(z,x)$ (triangle inequality)

A **metric space** (X,d) is a set X with a metric d defined on X.

Question 1 *Let* $M = \mathbb{R}^2$, *define* $d : M \times M \to \mathbb{R}$ *by*

$$d((x_1,y_1),(x_2,y_2)) := \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

Show that this defines a metric on $\mathbb{R}^2$. *Can we find some other examples of metrics on* $\mathbb{R}^2$*?*

Definition 2 *(**Contraction**):*
Let (X,d) *be a metric space. A mapping* $T : X \to X$ *is a contraction mapping or contraction, if there exists a constant c (contractivity factor), with* $0 \leq c < 1$ *such that*

$$d(T(x),T(y)) \leq cd(x,y)$$

for all $x, y \in X$.

Remark 3 *If* $T : X \to X$, *then a point* $x \in X$ *such that* $T(x) = x$, *is called a fixed point of* T. *One can show that if* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *is a contraction, then there is a unique fixed point of* T.

Question 4 *Can you see some examples?*

Definition 5 *(**IFS**) An Iterated function system (IFS) on* $\mathbb{R}^2$ *consists of a finite set of contraction mappings* $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ *with respective contractivity factors* s_n, *for* $n = 1, 2, ...N$. *The notation for IFS is* $\{f_n, n = 1, 2, ...N\}$ *and its contractivity factor is* $s = max\{s_n : n = 1, 2, ...N\}$.

Remark 6 *The Collage theorem says that there is a unique non-empty bounded subset* $A \subset \mathbb{R}^2$ *(called, **attractor**), such that*

$$A = f_1(A) \cup f_2(A) \cup .... \cup f_n(A)$$

One definition of a fractal is:

Definition 7 *A geometric figure or natural object is said to be **fractal** if it combines the following characteristics: (a) its parts have the same form or structure as the whole, except that they are at different scales and may be slightly deformed; (b) its form is extremely irregular, or extremely interrupted or fragmented, and remains so, whatever the scale of examination; (c) it contains 'distinct elements' whose scales are very varied and cover a large range.*

Remark 8 *One of the most common ways of generating fractals is as the fixed attractor set of an iterated function system.*

Sierpinski Gasket

The construction goes as follows: Start with an equilateral triangle S_0 (Figure 1.1). Divide this into four smaller equilateral triangles using the midpoints of the three sides of the original triangle as the new vertices. Remove the interior of the middle triangle (that is, do not remove the boundary) to get S_1 (Figure 1.2). Now repeat this procedure on each of the three remaining solid equilateral triangles to obtain S_2.

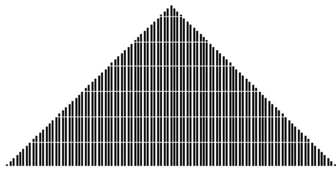


Figure 1.1: S_0 Original Triangle

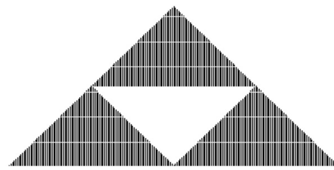


Figure 1.2: First Iteration

Figure 1.3: Second Iteration

Figure 1.4: Third Iteration

Figure 1.5: Fourth Iteration

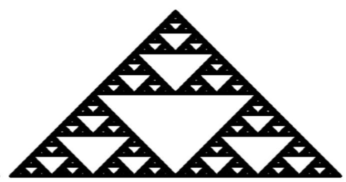


Figure 1.6: Fifth Iteration

Question 9 *Check the previous construction with the following IFS:*
Let S_0 denote the original isosceles triangle with vertices at $(0,0), (1,0), (\frac{1}{2}, 1)$. Define T_1, T_2, T_3 as follows:

$$T_1(S) := \frac{1}{2}S \quad or \quad T_1(x,y) = (\frac{x}{2}, \frac{y}{2})$$

$$T_2(S) := \frac{1}{2}S + (\frac{1}{2}, 0) \quad or \quad T_2(x,y) = (\frac{x}{2} + \frac{1}{2}, \frac{y}{2})$$

$$T_3(S) := \frac{1}{2}S + (\frac{1}{4}, \frac{1}{2}) \quad or \quad T_2(x,y) = (\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{1}{2})$$

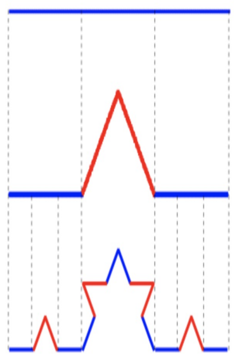
Then, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ on bounded subsets by

$$T(S) := T_1(S) \cup T_2(S) \cup T_3(S)$$

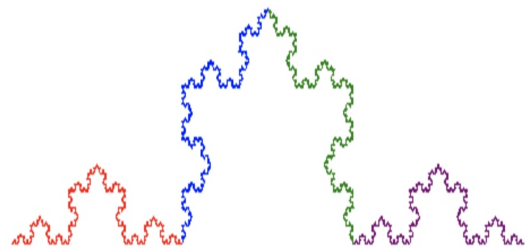
and define $S_n := T(S_{n-1})$.

Koch Curve

Begin with a straight line (the blue segment in the top figure). Divide it into three equal segments and replace the middle segment by the two sides of an equilateral triangle of the same length as the



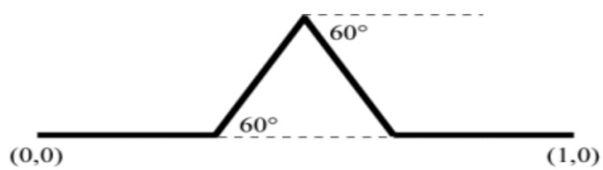
(a) literations



(b) The fixed attractor of this IFS is the Koch curve.

segment being removed (the two red segments in the middle figure). Now repeat, taking each of the four resulting segments, dividing them into three equal parts and replacing each of the middle segments by two sides of an equilateral triangle (the red segments in the bottom figure). Continue this construction.

Question 10 *The first iteration for the Koch curve consists of taking four copies of the unit horizontal line segment, each scaled by $r = 1/3$. Two segments must be rotated by 60°, one counter-clockwise and one clockwise.*



Check that one gets the following IFS:

$$f_1(\mathbf{x}) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{x} \qquad \text{scale by } \mathbf{r}$$

$$f_2(\mathbf{x}) = \begin{bmatrix} 1/6 & -\sqrt{3}/6 \\ \sqrt{3}/6 & 1/6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \qquad \text{scale by } \mathbf{r}\text{, rotate by } 60^\circ$$

$$f_3(\mathbf{x}) = \begin{bmatrix} 1/6 & \sqrt{3}/6 \\ -\sqrt{3}/6 & 1/6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix} \qquad \text{scale by } \mathbf{r}\text{, rotate by } -60^\circ$$

$$f_4(\mathbf{x}) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \qquad \text{scale by } \mathbf{r}$$

Remark 11 *For other interesting fractals like Mandelbrot sets, one can check the following* ***link****.*