Open-system analysis of thermalization in isolated quantum systems

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ICTS program “Stability of Quantum Matter in and out of Equilibrium at Various Scales”

outline

- **Introduction**
  thermalization in isolated quantum systems
  physics of open quantum many-body systems

- **Main result on Floquet systems**
  Liouvillian gap and thermalization (heating to infinite temperature)

- **Main result on static systems**
  Projected Liouvillian gap and thermalization

arXiv: 2311.10304

timescale of thermalization

theoretically, eigenstate thermalization hypothesis (ETH) explains thermalization

J. von Neumann, Z. Phys. (1929); M. Srednicki, Phys. Rev. E (1994); J. M. Deutsch, Phys. Rev. A (1991)

local operator has random matrix structure

$$A_{nm} := \langle n | \hat{A} | m \rangle \quad \hat{H} | n \rangle = E_n | n \rangle$$

$$\boxed{A_{nm} = A_{\text{eq}}(\varepsilon_{nm})\delta_{nm} + e^{-S(\varepsilon_{nm})/2} f(\omega_{nm}, \varepsilon_{nm}) r_{nm}}$$

$$\varepsilon_{nm} = \frac{E_n + E_m}{2N} \quad \omega_{nm} = E_n - E_m$$

r_{nm}: random variables

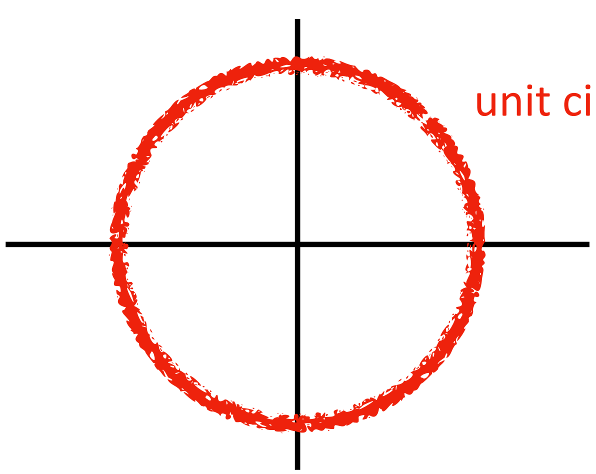
long-time average = ensemble average

long-time fluctuations are negligible

finite-time nonequilibrium dynamics (e.g. relaxation time)

ETH does not tell us much about timescale of the onset of thermalization

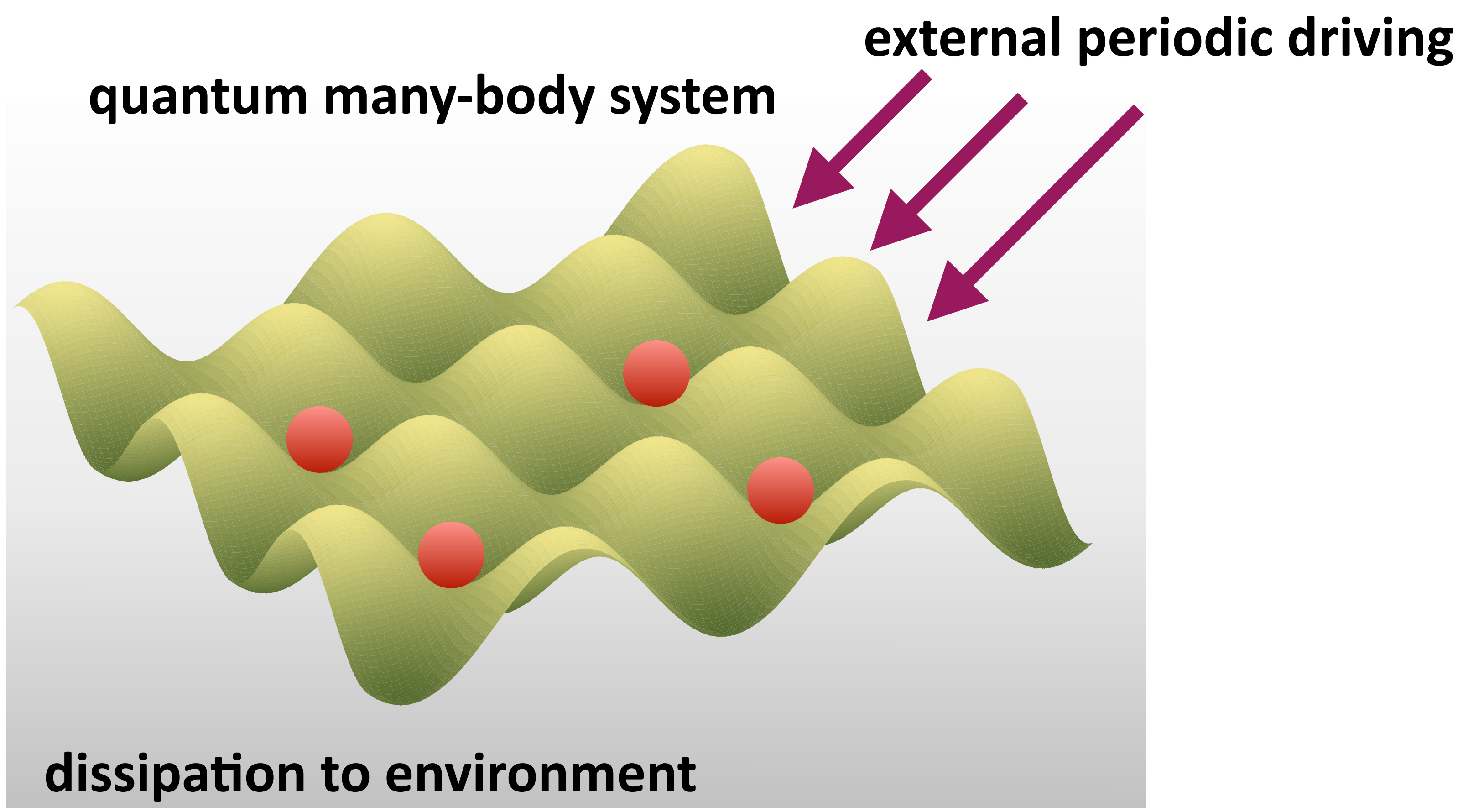
eigenvalues of the time evolution operator



thermalization proceeds via complicated dephasing process among energy eigenstates

important problem: evaluating timescale of thermalization

open quantum many-body systems



in Markovian regime: Lindblad equation

$$\frac{d\rho}{dt} = -i[\hat{H}(t), \rho] + \gamma \sum_{i=1}^{N} \left(\hat{L}_i \rho \hat{L}_i^\dagger - \frac{1}{2} \left\{ \hat{L}_i^\dagger \hat{L}_i, \rho \right\} \right)$$

cold-atom experiment: well-controlled dissipation in quantum many-body systems

dissipation engineering: create novel phases of matter by utilizing dissipation

novel physics usually emerges in the *strong* dissipation regime

motivation of this work

generic properties in the *weak* dissipation regime are not paid much attention

strong dissipation can be used to create a novel quantum material

↕

weak dissipation can be used as a probe of intrinsic properties of a quantum system

better understanding of thermalization dynamics of an *isolated* quantum system?

theoretical setup: model

Lindblad equation describing a quantum many-body system under bulk dissipation

$$\frac{d\rho}{dt} = -i[\hat{H}(t), \rho] + \gamma \sum_{i=1}^{L} \left(\hat{L}_i \rho \hat{L}_i^\dagger - \frac{1}{2} \left\{ \hat{L}_i^\dagger \hat{L}_i, \rho \right\} \right) =: \mathscr{L}(t)\rho$$

$\mathscr{L}(t) = \mathscr{L}(t+\tau)$: Liouvillian or Lindbladian

specific model: kicked Ising chain under bulk dephasing

$$\hat{H}(t) = -\sum_{i=1}^{L} \left(J\sigma_i^z \sigma_{i+1}^z + h_z \sigma_i^z \right) + \sum_{k=-\infty}^{\infty} \delta(t - k\tau)\hat{V} \qquad \hat{V} = -h_x \sum_{i=1}^{L} \sigma_i^x$$

bulk dephasing $L_i = \sigma_i^z$

$$\mathscr{L}(t)\rho = -i[\hat{H}(t), \rho] + \gamma \sum_{i=1}^{L} \left(\hat{\sigma}_i^z \rho \hat{\sigma}_i^z - \rho \right)$$

$J = 1, h_z = 0.8090, h_x = 0.9045$ (quantum chaotic regime)

theoretical setup: quantity of our interest

Liouvillian gap g

time evolution operator over one cycle $\mathcal{U}_F = \mathcal{T} e^{\int_0^\tau \mathcal{L}(t)dt}$

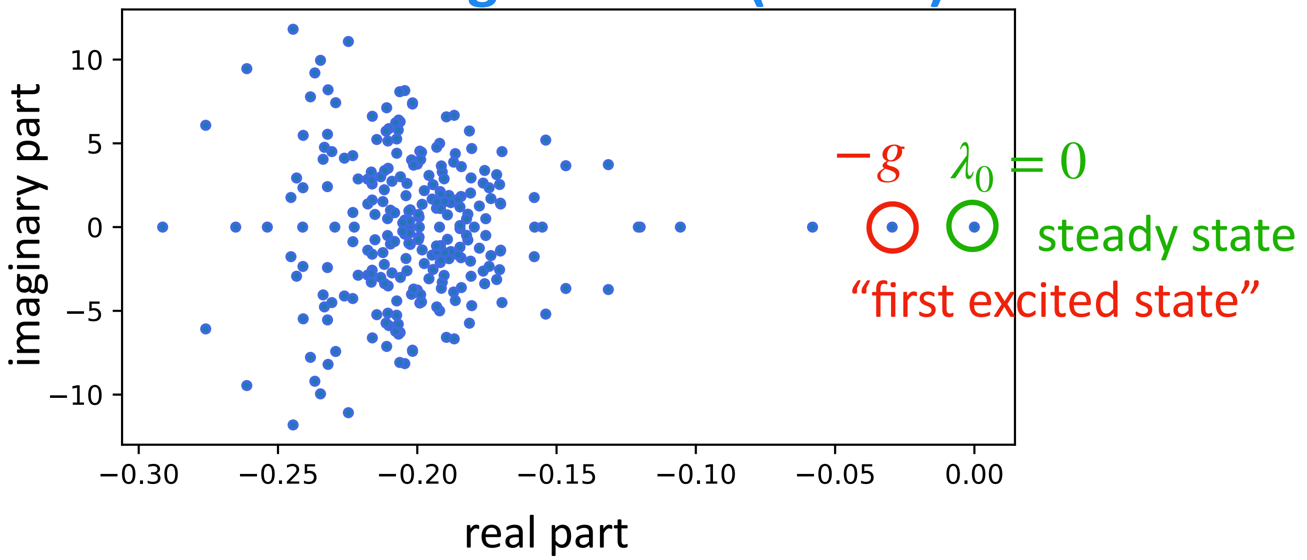
$$\mathcal{U}_F \rho_\alpha = e^{\lambda_\alpha \tau} \rho_\alpha$$

$\{\lambda_\alpha\}$: "Liouvillian eigenvalues"

$$0 = \lambda_0 > \mathrm{Re}\,\lambda_1 \geq \mathrm{Re}\,\lambda_2 \geq \ldots$$

$$g = -\mathrm{Re}\,\lambda_1$$

Liouvillian eigenvalues ($L = 4$)



Liouvillian gap has several important properties

- Liouvillian gap gives the asymptotic decay rate

E. M. Kessler et al., Phys. Rev. A 86, 012116 (2012)

$$t = n\tau$$

$$\rho(t) - \rho_0 = \sum_{\alpha>0} C_\alpha e^{\lambda_\alpha t} \rho_\alpha \overset{t\to\infty}{\sim} e^{-gt}$$

- A finite Liouvillian gap implies exponential decay of spatial correlations in the steady state

M. J. Kastoryano and J. Eisert, J. Math. Phys. 54, 102201 (2013)

- Dissipative phase transition is associated with the closing of the Liouvillian gap

F. Minganti et al., Phys. Rev. A 98, 042118 (2018)

main result: singularity of the Liouvillian gap

general behavior of the Liouvillian gap for small γ (weak dissipation)

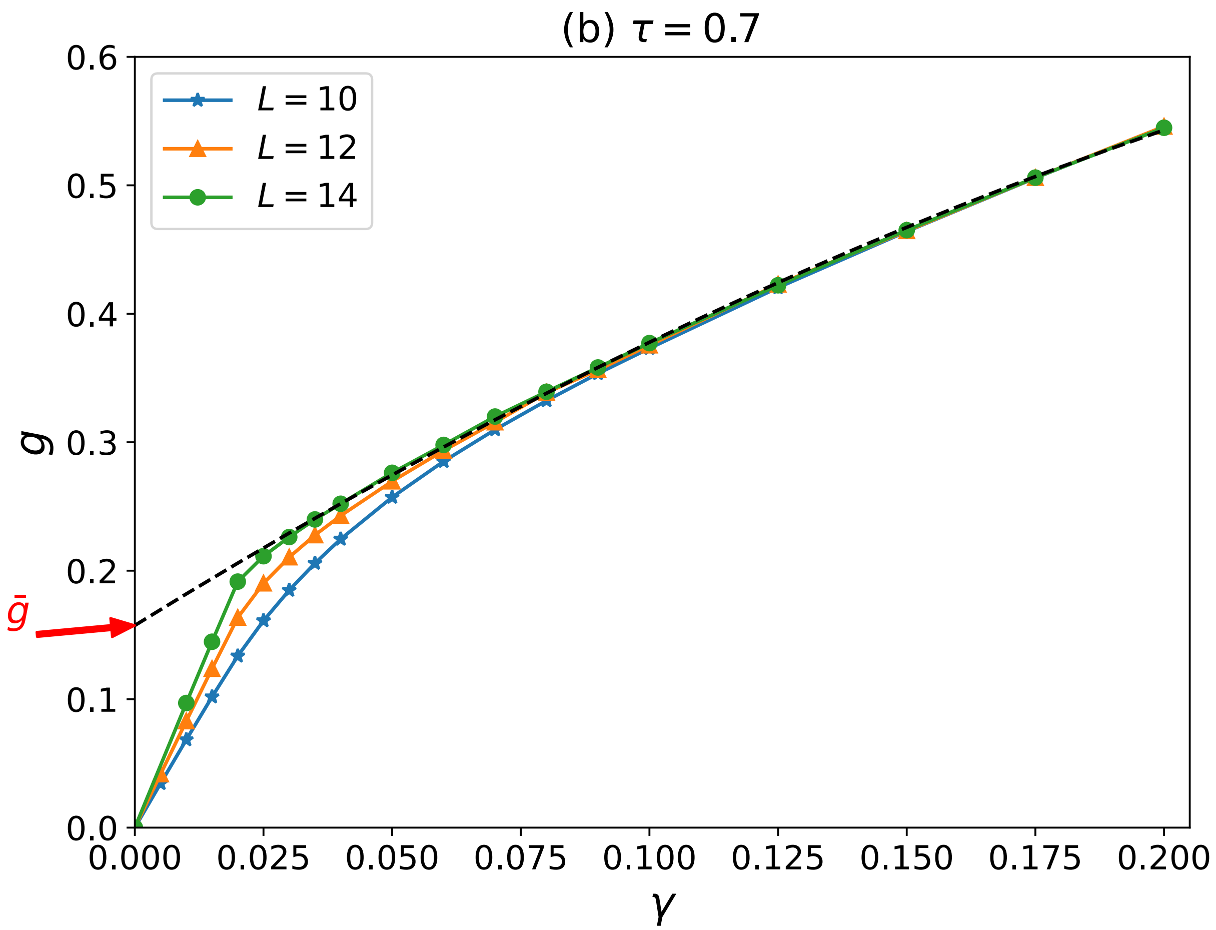
$$g \sim \begin{cases} \gamma L & \text{for } \gamma \lesssim v/L \\ \bar{g} + O(\gamma) & \text{for } v/L \lesssim \gamma \ll v \end{cases}$$

v: Lieb-Robinson velocity
(local energy scale of the Hamiltonian)

The thermodynamic limit of the Liouvillian gap remains finite in the limit of $\gamma \to +0$

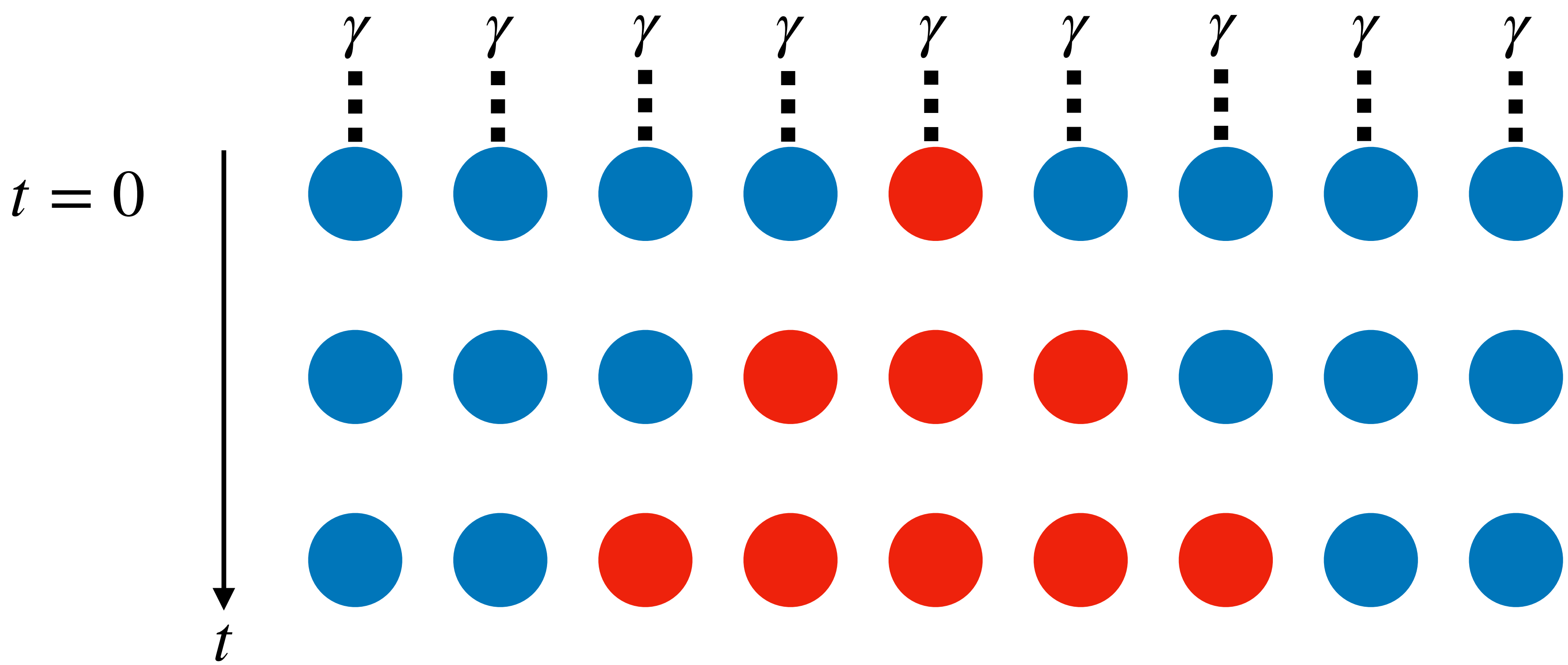
$$\lim_{\gamma\to+0}\lim_{L\to\infty} g =: \bar{g} > 0 \qquad \lim_{L\to\infty}\lim_{\gamma\to+0} g = 0$$

numerical evidence



operator spreading and Liouvillian gap

time evolution of an operator in the Heisenberg picture $\hat{O}(t) = e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t}$



decay

$\hat{O}_i \to e^{-\gamma\Delta t}$

$\hat{O}_{i-1}\hat{O}_i\hat{O}_{i+1} \to e^{-3\gamma\Delta t}$

$\hat{O}_{i-2}\hat{O}_{i-1}\hat{O}_i\hat{O}_{i+1}\hat{O}_{i+2} \to e^{-5\gamma\Delta t}$

instantaneous decay rate is proportional to $\gamma \times$ (operator size)

TM and T. Shirai, arXiv:2309.03485

In a many-body system, weak dissipation is amplified by the operator spreading

operator spreading and Liouvillian gap

For a small $\gamma(\ll v)$, the effect of dissipation can be ignored up to time $t_\gamma \sim \gamma^{-1}$

In the absence of dissipation, the operator size increases as vt

The operator size is saturated after t_γ

$$\text{(operator size in the long-time limit)} \sim \min\{vt_\gamma, L\} \sim \begin{cases} L & \text{for } \gamma \lesssim v/L \\ v\gamma^{-1} & \text{for } \gamma \gtrsim v/L \end{cases}$$

Liouvillian gap = asymptotic decay rate

$$\gamma \times \text{(operator size in the long-time lmit)} \sim \begin{cases} \gamma L & \text{for } \gamma \lesssim v/L \\ \bar{g} + O(\gamma) & \text{for } v/L \lesssim \gamma \ll v \end{cases}$$

$$\bar{g} \sim v$$

$$\lim_{\gamma\to+0} \lim_{L\to\infty} g =: \bar{g} > 0$$

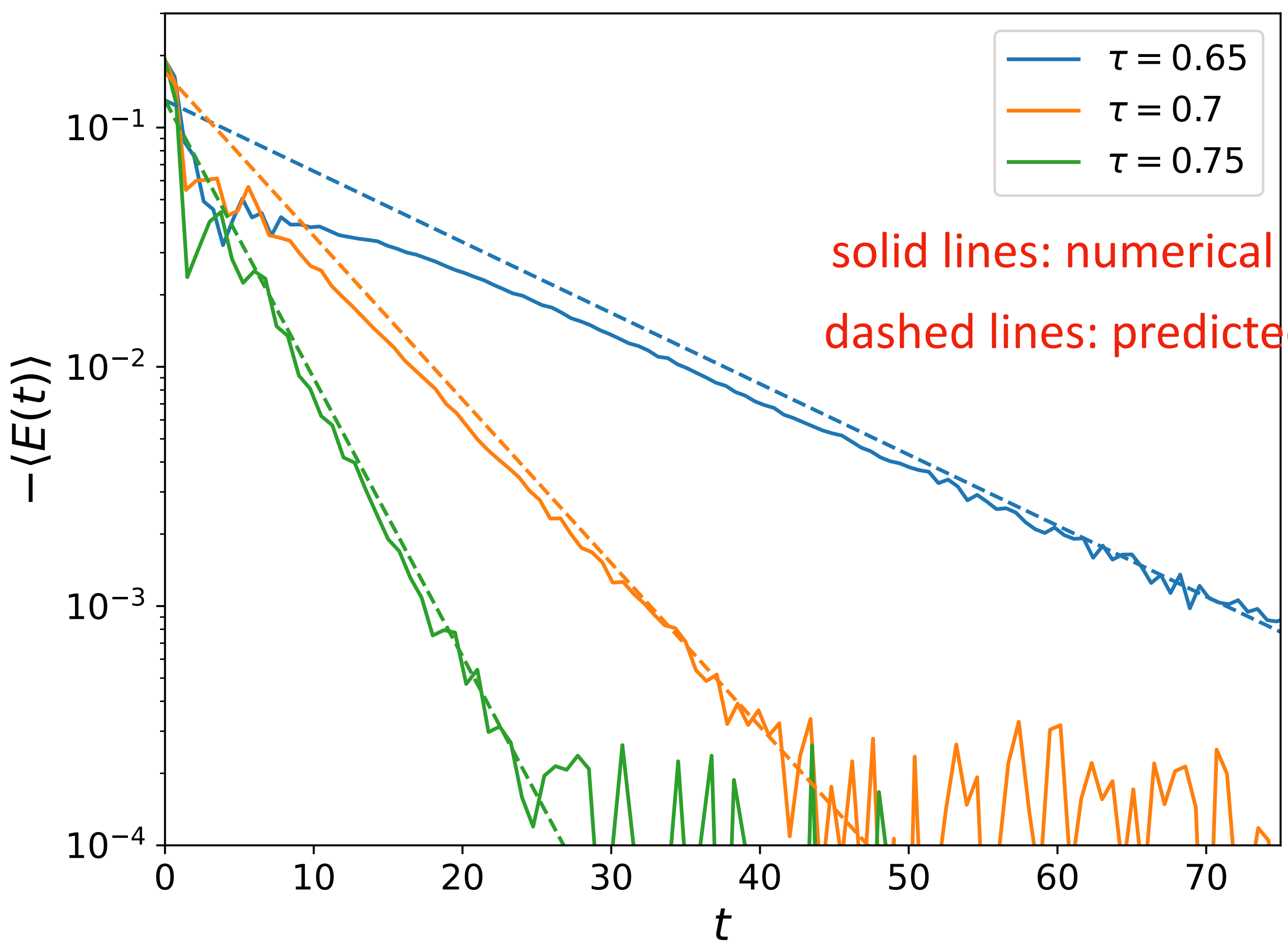
physical meaning of $\bar{g}$: Ruelle-Pollicott resonance

conjecture: $\bar{g}$ corresponds to a quantum analog of *the leading Ruelle-Pollicott resonance* in classical chaos theory

exponential decay rate hidden in the unitary time evolution

timescale of thermalization

numerical result for the kicked Ising chain under bulk dephasing



solid lines: numerical solution of the Schrödinger eq.

dashed lines: predicted exponential decays $\exp(-\bar{g}t)$

Ruelle-Pollicott resonance in classical chaos

trajectory in the phase space $\Gamma_t = (q_1(t), q_2(t), \ldots, q_N(t), p_1(t), p_2(t), \ldots, p_N(t))$

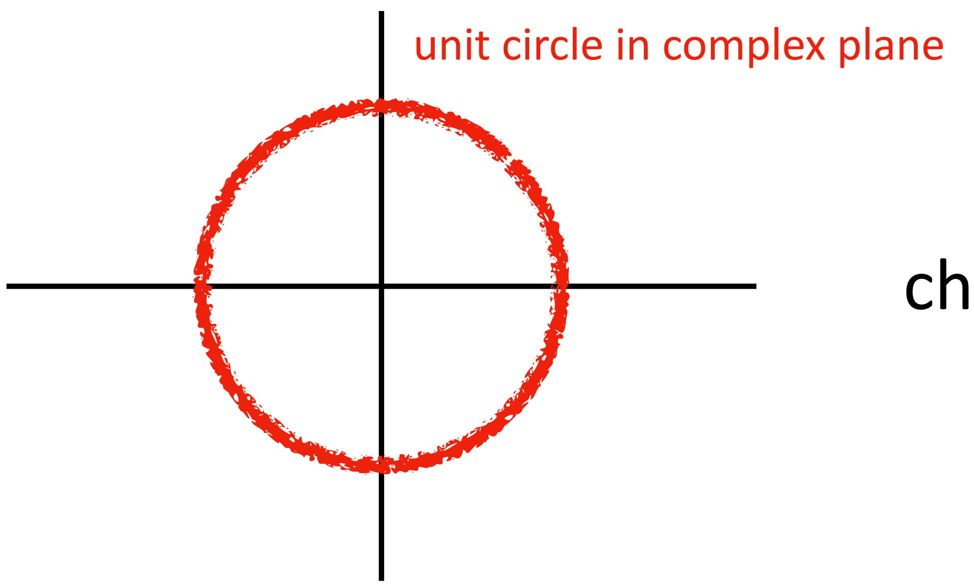
probability distribution $P_t(\Gamma) = P(\Gamma_{-t})$ $\dot{q}_i(t) = \dfrac{\partial H}{\partial p_i}, \quad \dot{p}_i(t) = -\dfrac{\partial H}{\partial q_i}$

time evolution operator (Frobenius-Perron operator) $\mathscr{U}_t$ $P_t(\Gamma) = \mathscr{U}_t P_0(\Gamma)$ unitary

spectrum of the Frobenius-Perron operator is defined
by singularities of the resolvent

$$R(z) = \frac{1}{z - \mathscr{U}_t}$$

e.g. eigenvalues = poles of $R(z)$
continuous spectrum = brunch cut of $R(z)$



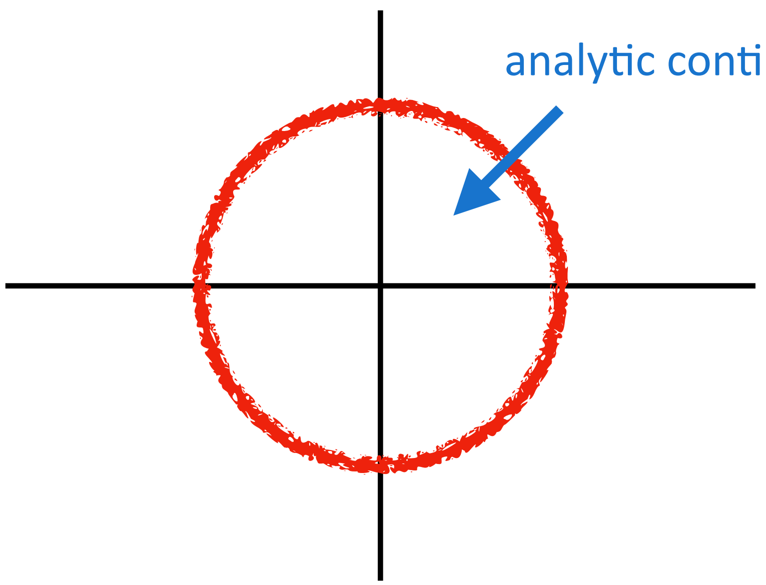
chaotic dynamics: continuous spectrum

Ruelle-Pollicott resonance in classical chaos

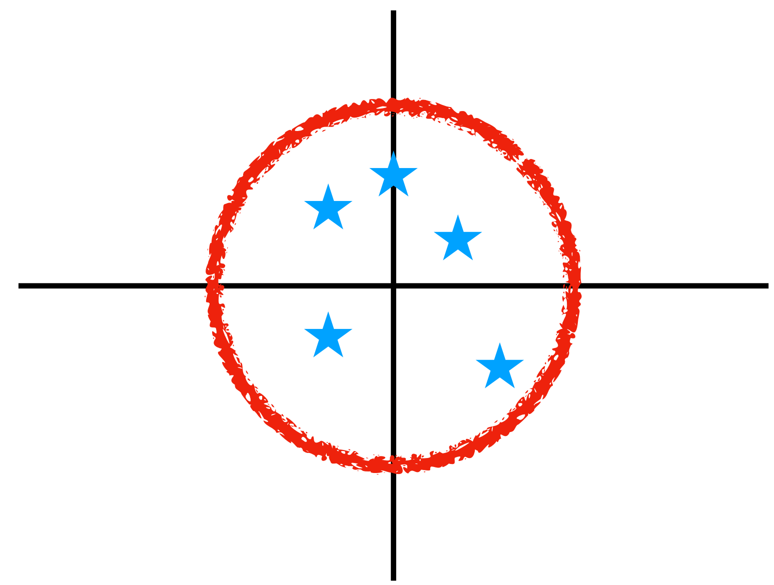
H. H. Hasegawa and W. C. Saphir, Phys. Rev. A (1992)

consider a chaotic system: continuous spectrum corresponds to a brunch cut of $R(z)$

analytic continuation of $R(z)$



2nd Riemann sheet



★ poles of $R(z)$ in the second Riemann sheet

$$e^{\nu_i t} \quad \mathrm{Re}\,\nu_i < 0$$

$\{\nu_i\}$ "Ruelle-Pollicott resonances"

D. Ruelle, Phys. Rev. Lett. (1986)
M. Pollicott, Invent. Math. (1985)

leading Ruelle-Pollicott resonance: ν_i with the maximum value of $\mathrm{Re}\,\nu_i$

(closest to the unit circle)

Ruelle-Pollicott resonance in classical chaos

express the time evolution as a sum of exponential decays

$$\langle A \rangle_t := \int A(\Gamma) P_t(\Gamma) d\Gamma \qquad \langle A \rangle_t \approx \sum_i C_i e^{\nu_i t} \sim e^{\nu_* t} \quad (t \to \infty)$$

ν_*: the leading RP resonance

$\{\nu_i\}$ are solely determined by the Frobenius-Perron operator
(independent of physical quantities)

any physical quantity exhibits an exponential decay $e^{\nu_* t}$
governed by the leading RP resonance

RP resonances can describe exponential decays hidden in the Hamiltonian dynamics

We can estimate the timescale of thermalization

weak noise limit and Ruelle-Pollicott resonance

Ruelle-Pollicott resonances can be obtained by introducing weak stochastic noise, instead of directly performing the analytic continuation of the resolvent

P. Gaspard, G. Nicolis, A. Provata, and S. Tasaki, Phys. Rev. E (1995)

J. Kurchan, arXiv:0901.1271

Hamilton equations of motion $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$

add random Gaussian noise (Langevin eq) $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \xi_i(t)$

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = \epsilon \delta_{ij} \delta(t - t')$$

time evolution of the probability distribution: Kramers equation

$$\frac{\partial P}{\partial t} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial P}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial P}{\partial q_i} + \frac{\epsilon}{2} \frac{\partial^2 P}{\partial p_i^2} \right) =: \mathscr{L}_K P$$

some eigenvalues of the time evolution operator $e^{\mathscr{L}_K t}$ are inside the unit circle even in the weak-noise limit $\epsilon \to +0$; they are nothing but RP resonances!

analogy

classical chaotic systems

P. Gaspard, G. Nicolis, A. Provata, and S. Tasaki, Phys. Rev. E (1995)

J. Kurchan, arXiv:0901.1271

$$\frac{\partial P}{\partial t} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial P}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial P}{\partial q_i} + \frac{\epsilon}{2}\frac{\partial^2 P}{\partial p_i^2} \right) =: \mathscr{L}_K P$$

RP resonances appear as eigenvalues of $\mathscr{L}_K$ in the weak noise limit $\epsilon \to +0$

quantum chaotic systems

$$\frac{d\rho}{dt} = -i[\hat{H}, \rho] + \gamma \sum_{i=1}^{L} \left(\hat{L}_i \rho \hat{L}_i^\dagger - \frac{1}{2}\left\{ \hat{L}_i^\dagger \hat{L}_i, \rho \right\} \right) =: \mathscr{L}\rho$$

(our conjecture) RP resonances appear as eigenvalues of $\mathscr{L}$ in the weak dissipation limit $\gamma \to +0$

bulk dissipation + thermodynamic limit (before $\gamma \to +0$) is crucial in quantum case

continuous spectrum is important for Ruelle-Pollicott resonances

quantum spin-1/2 system has continuous spectrum *only in the thermodynamic limit*

summary up to here

Thermodynamic limit of the Liouvillian gap is discontinuous at $\gamma = 0$

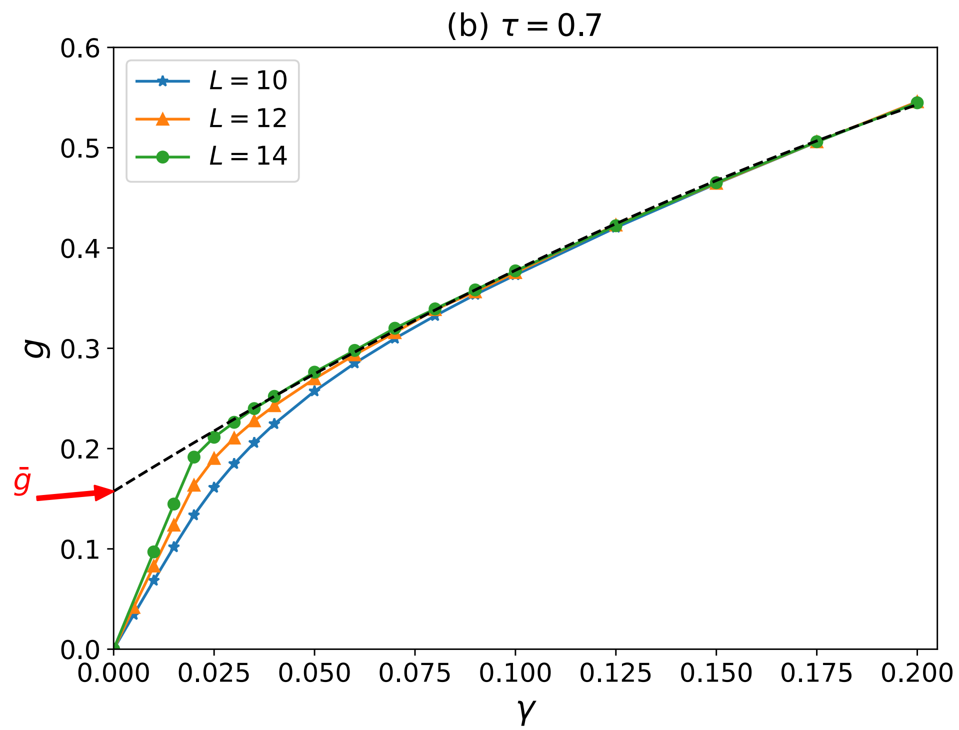
$$\bar{g} := \lim_{\gamma \to +0} \lim_{L \to \infty} g > 0$$

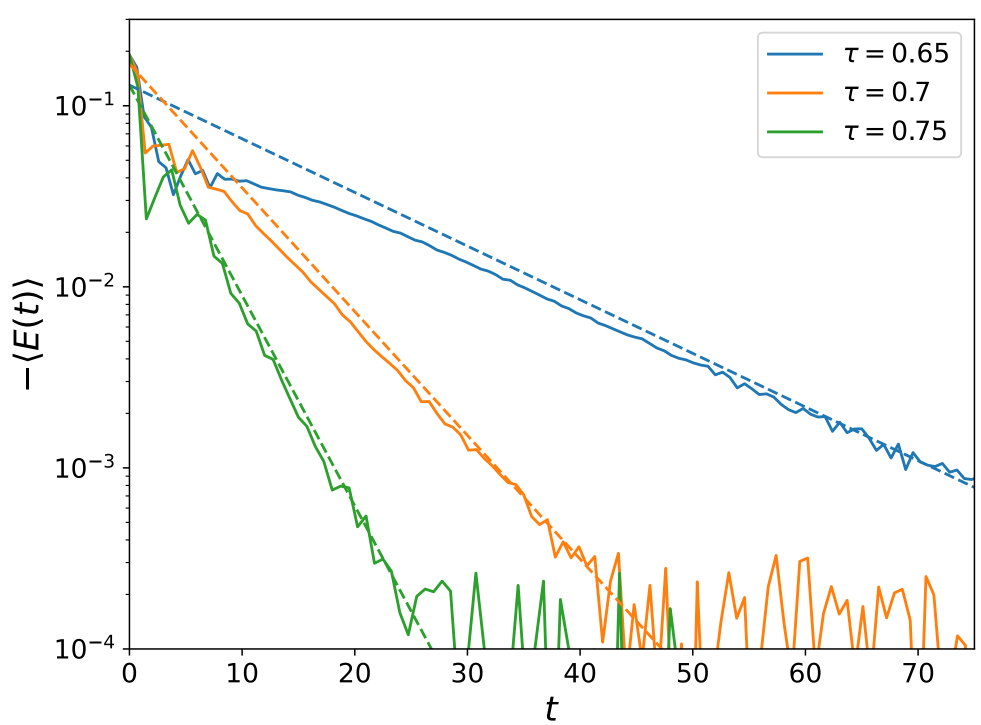
This result is explained by the operator spreading

(decay rate under bulk dissipation) $\propto \gamma \times$ (operator size)

$\bar{g}$ corresponds to the leading quantum Ruelle-Pollicott resonance: it determines the exponential decay rate of the *isolated* system

These results are numerically verified in the kicked-Ising chain under bulk dephasing





static systems

In static systems, the Liouvillian gap simply vanishes in the weak dissipation limit

$$\lim_{\gamma\to+0}\lim_{L\to\infty} g = 0$$

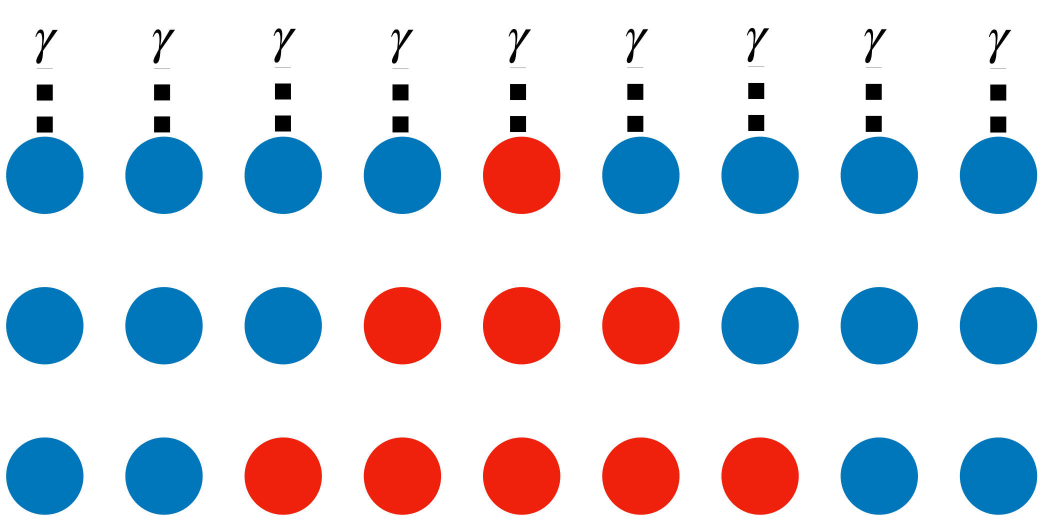
N. Shibata and H. Katsura, Phys. Rev. B 99, 174303 (2019)

1D quantum compass model + bulk dephasing $\lim_{L\to\infty} g = 2\gamma$ for $\gamma < \gamma_c$

N. Shibata and H. Katsura, Phys. Rev. B 99, 224432 (2019)

quantum Ising chain + bulk dissipation $\lim_{L\to\infty} g = 4\gamma$ for $\gamma < \gamma_c'$

large decay rates $\propto L\gamma$ emerge as a consequence of the operator spreading



Hamiltonian is conserved under unitary evolution:
no operator spreading

Liouvillian gap describes the relaxation of the energy due to dissipation
(not related with the intrinsic relaxation)

idea of extracting RP resonances in static systems

In static systems, the Liouvillian gap does not correspond to an intrinsic decay rate of the system because of the local conservation law in the isolated system

Conservation laws in the isolated system are fully encoded in the diagonal matrix elements in the energy basis

We might be able to extract intrinsic decay rates (i.e. RP resonances) by discarding all the diagonal elements

idea: project out the diagonal elements

$$\mathscr{P}\rho = \rho - \sum_n \rho_{nn} |n\rangle\langle n| \qquad \hat{H}|n\rangle = E_n|n\rangle$$

projected Liouvillian $\mathscr{L}_P := \mathscr{P}\mathscr{L}\mathscr{P}$

projected Liouvillian gap $g_P = -\min_n \{\mathrm{Re}\,\lambda_n : \exists \rho_n \text{ s.t. } \mathscr{L}_P\rho_n = \lambda_n\rho_n, \mathscr{P}\rho_n = \rho_n\}$

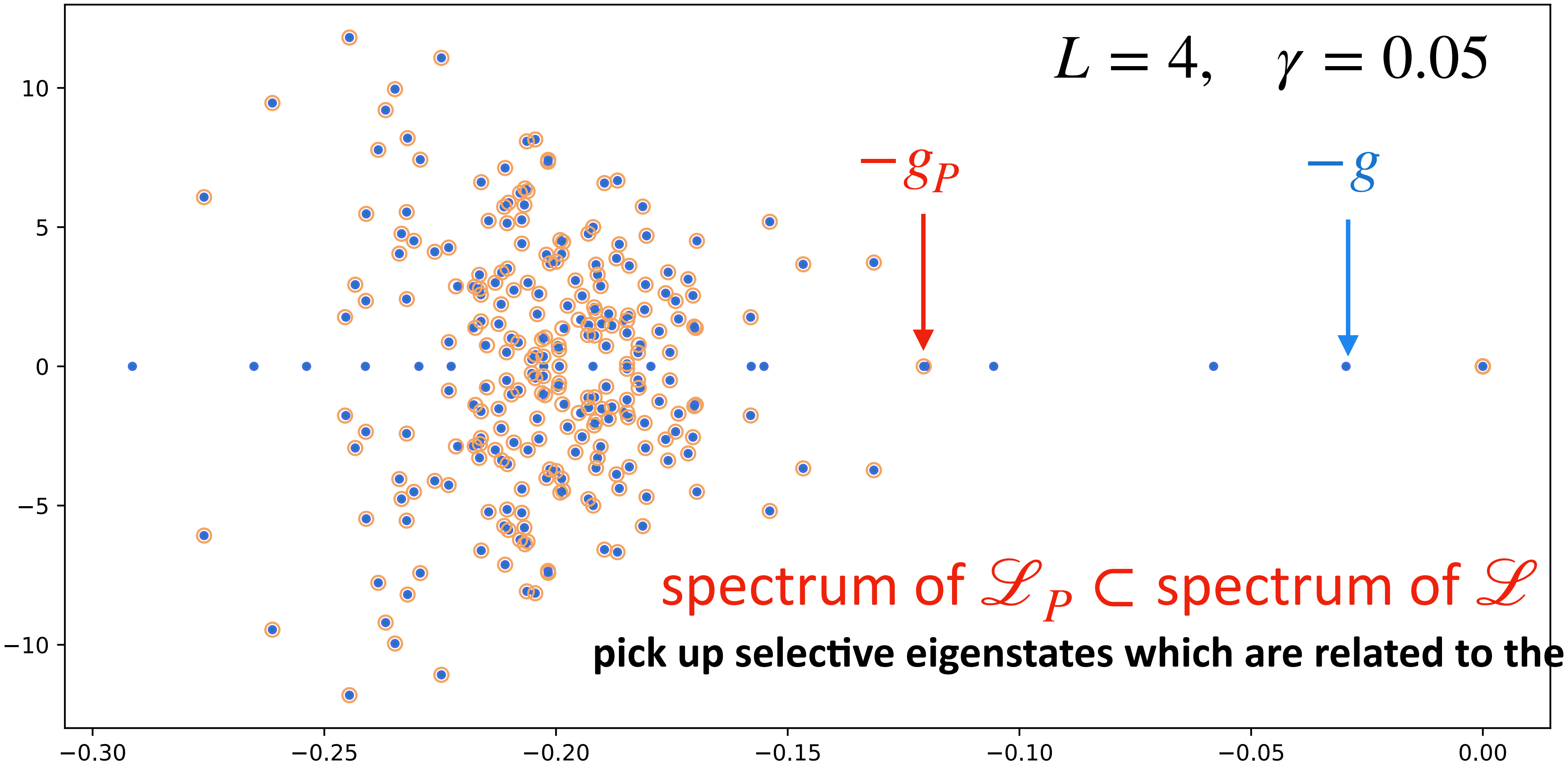
comparison between eigenvalues of $\mathscr{L}$ and $\mathscr{L}_P$

model: quantum Ising chain under bulk dephasing

$$\hat{H} = -J\sum_{i=1}^{L-1}\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z - h_z\sum_{i=1}^{L}\hat{\sigma}_i^z - h_x\sum_{i=1}^{L}\hat{\sigma}_i^x \qquad \hat{L}_i = \hat{\sigma}_i^z$$

$$J = 1, h_z = 0.8090, h_x = 0.9045$$

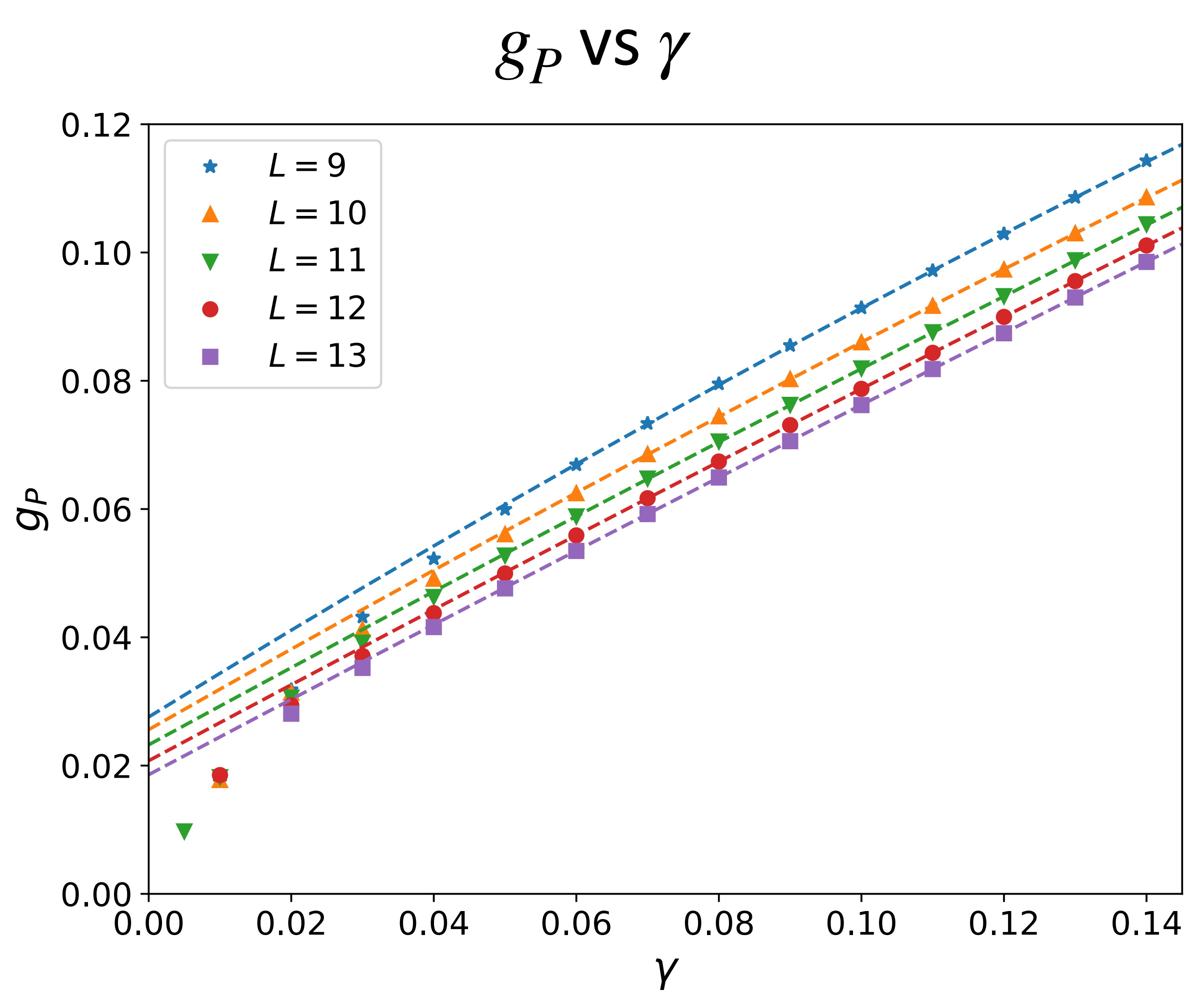
eigenvalues of $\mathscr{L}$ eigenvalues of $\mathscr{L}_P$

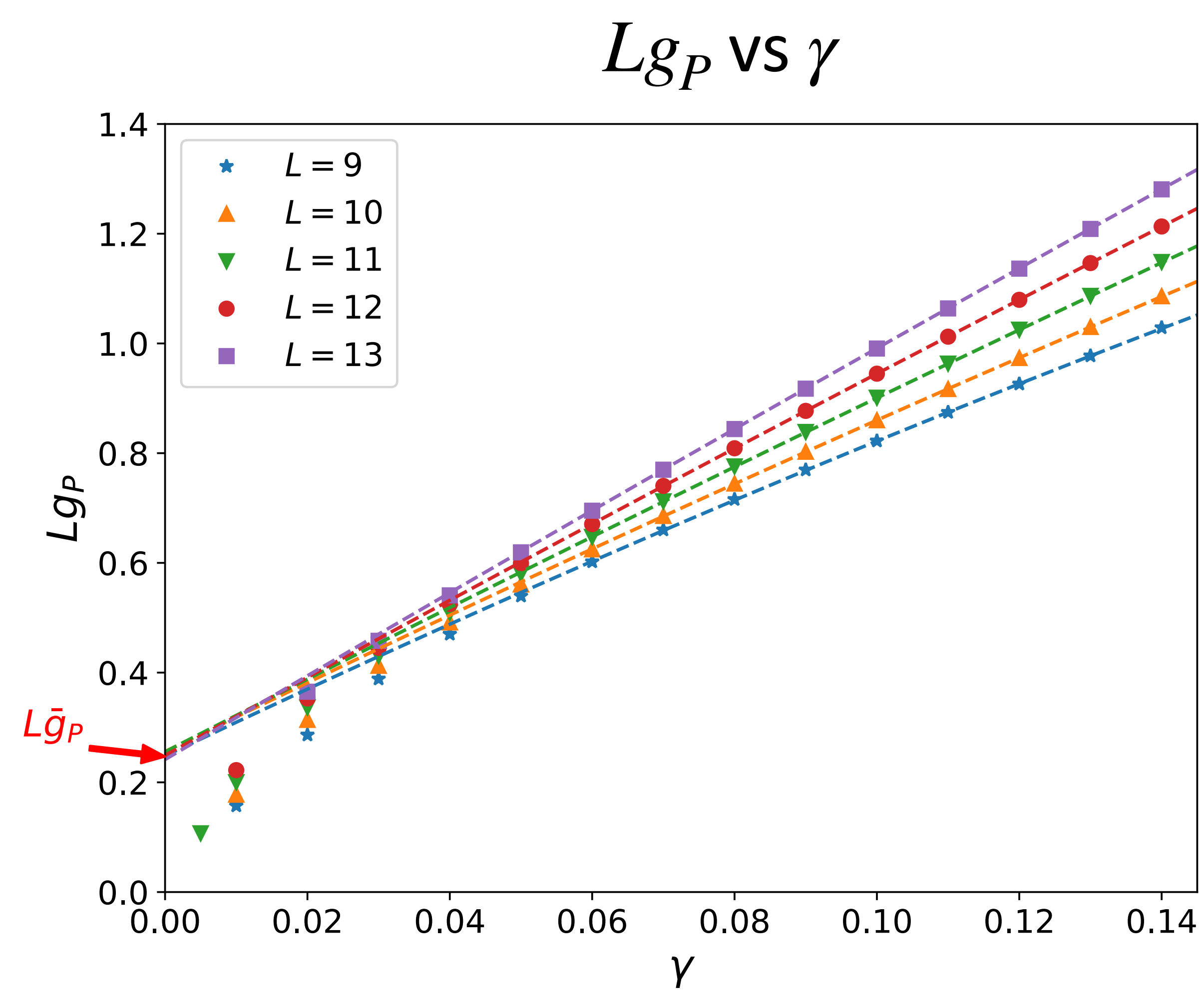


spectrum of $\mathscr{L}_P \subset$ spectrum of $\mathscr{L}$

pick up selective eigenstates which are related to the intrinsic relaxation

numerical results: size-dependent RP resonances





size-dependent value $\bar{g}_P$ that is obtained by extrapolating the data to $\gamma = 0$

(fitting a quadratic function to data of $\gamma \geq 0.05$)

numerical data suggest $\bar{g}_P \propto 1/L$ (size-dependent Ruelle-Pollicott resonance)

numerical estimate: $\bar{g}_P = 0.247/L$

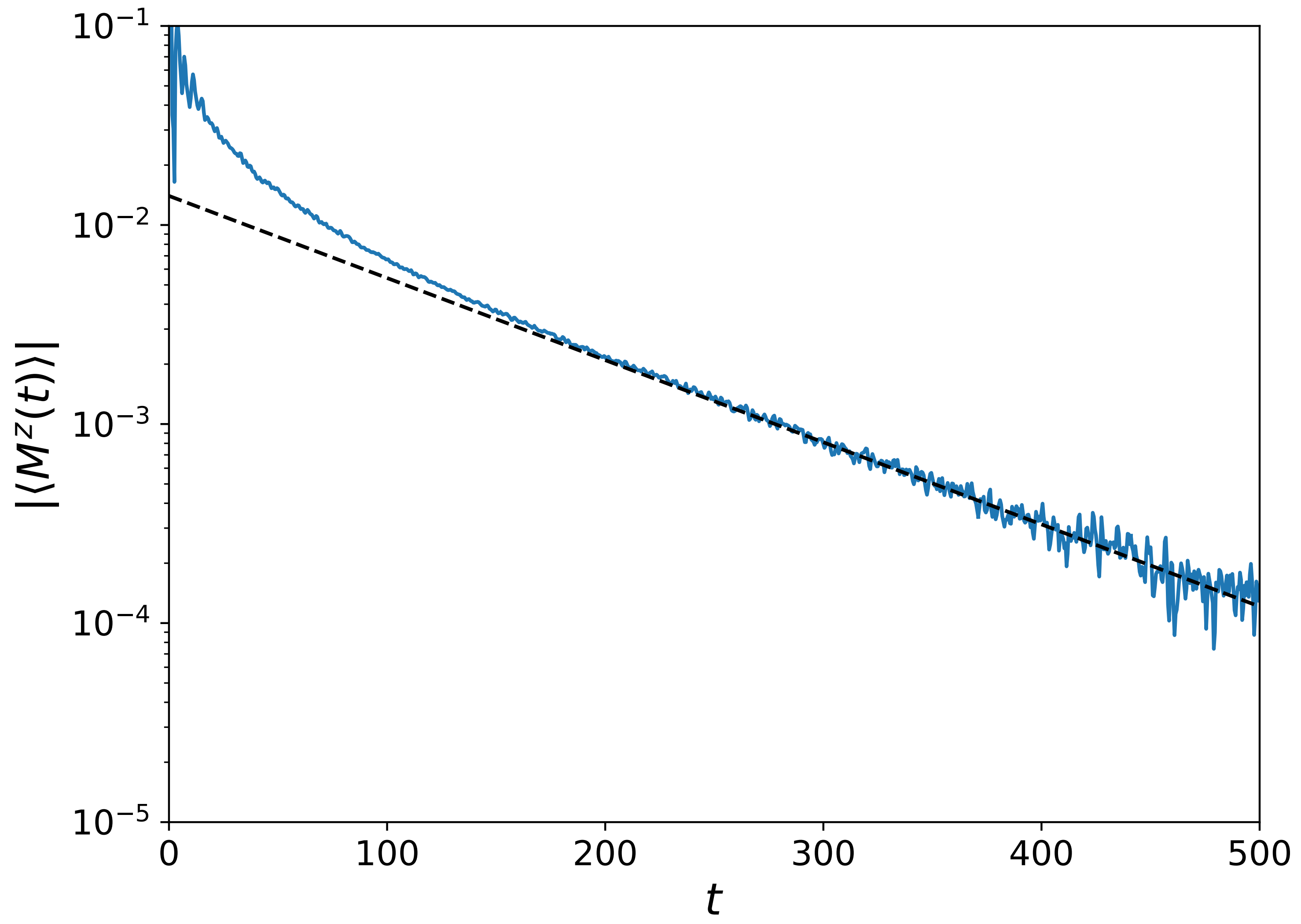
projected Liouvillian gap as a RP resonance

solid line: numerical solution of the Schrödinger equation for $L = 28$

initial state: all-down state

dashed line: exponential decay predicted by the projected Liouvillian gap $\exp\left(-\bar{g}_P t\right)$

$\bar{g}_P$ is estimated by using numerical data for $L = 9$-13



conclusion

arXiv: 2311.10304

- The intrinsic decay rate of the isolated system (quantum Ruelle-Pollicott resonance) is obtained from an open-system analysis in the weak dissipation limit

- In bulk-dissipated Floquet systems, the Liouvillian gap in the thermodynamic limit is discontinuous at $\gamma = 0$, and a nonzero limit $\bar{g} = \lim_{\gamma \to +0} \lim_{L \to \infty} g$ corresponds to the leading Ruelle-Pollicott resonance

- In bulk-dissipated static systems, the leading Ruelle-Pollicott resonance is identified not with the Liouvillian gap but with the projected Liouvillian gap

- **take-home message**: open-system analysis deepens our understanding of isolated systems