Unique Factorization of Tensor products for Parabolic Verma Modules

V. Sathish Kumar, IMSc Chennai, HBNI Mumbai
Joint work with K.N. Raghavan, R. Venkatesh and Sankaran Viswanath.

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Problem

Suppose

$$M_1 \otimes M_2 \otimes \cdots \otimes M_r \cong N_1 \otimes N_2 \otimes \cdots \otimes N_s$$

Question 1: Is $r = s$?

Question 2: If $r = s$ what more can be said about the families $\{M_i\}$ and $\{N_j\}$?

Question 3: What can be said about their characters?

Theorem (CS Rajan)

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $M_1, \cdots, M_r, N_1, \cdots, N_s$ be finite dimensional non-trivial simple modules of $\mathfrak{g}$ such that

$$M_1 \otimes M_2 \otimes \cdots \otimes M_r \cong N_1 \otimes N_2 \otimes \cdots \otimes N_s$$

Then,

1. $r = s$
2. $M_i \cong N_{\sigma(i)}$ *for some permutation* σ

History

Theorem (Venkatesh, Viswanath)

Let $\mathfrak{g}$ be an indecomposable symmetrizable Kac-Moody Lie algebra and $M_1, \cdots, M_r, N_1, \cdots, N_s$ be irreducible integrable modules of $\mathfrak{g}$ with dimension ≥ 2 such that

$$M_1 \otimes M_2 \otimes \cdots \otimes M_r \cong N_1 \otimes N_2 \otimes \cdots \otimes N_s$$

Then,

1. $r = s$
2. $M_i \cong N_{\sigma(i)} \otimes Z_i$ *for some permutation σ and one dimensional modules Z_i*

The second conclusion translates to the fact that the highest weights are determined up to their action on the simple coroots.

In other words,

Theorem (Venkatesh, Viswanath)

Let $\mathfrak{g}$ be an indecomposable symmetrizable Kac-Moody Lie algebra and $V(\lambda_1), \cdots, V(\lambda_r), V(\mu_1), \cdots, V(\mu_s)$ be irreducible integrable modules of $\mathfrak{g}$ with dimension ≥ 2 such that

$$V(\lambda_1) \otimes \cdots \otimes V(\lambda_r) \cong V(\mu_1) \otimes \cdots \otimes V(\mu_s)$$

Then,

1. $r = s$
2. *For some permutation σ we have $\lambda_i(\alpha^\vee) = \mu_{\sigma(i)}(\alpha^\vee)$ for all simple coroots α.*

Symmetrizable Kac Moody Algebras

Root space decomposition

- A - symmetrizable GCM
- $\mathfrak{g} = \mathfrak{g}(A)$ - Kac Moody algebra corresponding to A
- $\mathfrak{h}$ - Cartan subalgebra of $\mathfrak{g}$
- $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $[h, x] = \alpha(h)x$ for all $x \in \mathfrak{g}_\alpha$
- $\mathfrak{g}$ is generated by $\{e_{\alpha_i}, f_{\alpha_i} \in \mathfrak{g}_{\pm\alpha_i} \mid 1 \leq i \leq n\}$ along with $\mathfrak{h}$
- $\Pi := \{\alpha_1, \cdots, \alpha_n\}$ - simple roots
- $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ - simple coroots

Weyl Group

For $1 \leq i \leq n$ the reflection $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ is defined by

$$s_i(\nu) = \nu - \nu(\alpha_i^\vee)\alpha_i \qquad \forall \nu \in \mathfrak{h}^*$$

The Weyl group of $\mathfrak{g}$ is

$$W := \langle s_1, s_2, \cdots, s_n \rangle \subset GL(\mathfrak{h}^*)$$

Parabolic Weyl Group

For $J \subset \{1, 2, \cdots, n\}$, the parabolic subgroup of the Weyl group is

$$W_J := \langle \{s_j : j \in J\} \rangle \subset W$$

- Parabolic Weyl groups are Coxeter groups with generators $\{s_j : j \in J\}$
- For $w \in W_J$,

$$I(w) := \{s_{j_1}, s_{j_2} \cdots, s_{j_k}\}$$

  where, $s_{j_1} s_{j_2} \cdots s_{j_k}$ is a reduced expression for w

Parabolic Verma Modules

- The *integrability* of $\lambda \in \mathfrak{h}^*$ is

$$J_\lambda := \{i \in [n] \mid \lambda(\alpha_i^\vee) \text{ is a non-negative integer}\}$$

- $\mathcal{P} := \{(\lambda, I) \mid \lambda \in \mathfrak{h}^* \text{ and } I \subset J_\lambda\}$
- For $\lambda \in \mathfrak{h}^*$, the associated Verma module is

$$M(\lambda) := \frac{\mathfrak{U}(\mathfrak{g})}{\langle \{e_{\alpha_i},\ h - \lambda(h) \mid 1 \leq i \leq n,\ h \in \mathfrak{h}\} \rangle}$$

- For $(\lambda, I) \in \mathcal{P}$, the *Parabolic Verma Module* is

$$M(\lambda, I) := \frac{M(\lambda)}{\sum_{\alpha \in I} U(\mathfrak{g}) f_\alpha^{\lambda(\alpha^\vee)+1} m_\lambda}$$

  where, m_λ is the cyclic generator (or highest weight vector) of $M(\lambda)$, the Verma module corresponding to λ.

- The characters of Parabolic Verma modules are given by

$$\mathrm{ch}_{\mathfrak{h}}\, M(\lambda, I) = \frac{\sum_{w \in W_I} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}.$$

Main theorem 1

Theorem (_, Raghavan, Venkatesh, Viswanath)

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a symmetrizable Kac-Moody algebra. Suppose that

$$\otimes_{k=1}^{r} M(\lambda_k, I_k) \cong \otimes_{k=1}^{r} M(\mu_k, J_k)$$

where $\forall k$,

1. (λ_k, I_k), $(\mu_k, J_k) \in \mathcal{P}$.
2. I_k *and* J_k *are connected.*

Then, $\sum_{k=1}^{r} \lambda_k = \sum_{k=1}^{r} \mu_k$ *and there exists* $\sigma \in \mathfrak{S}_r$ *such that*

1. $I_k = J_{\sigma k}$ *and*
2. $\lambda_k(\alpha_i^\vee) = \mu_{\sigma k}(\alpha_i^\vee)$ *for all* $i \in I_k$.

Diagram Automorphisms

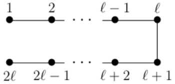
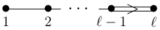
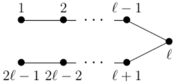
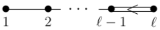
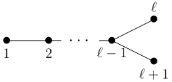
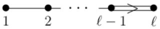
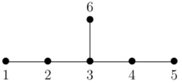
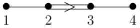
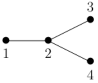
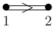
- A map $\omega \in \mathfrak{S}_n$ is an automorphism of a GCM $A = (a_{ij})$ if

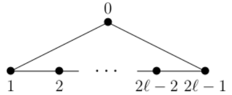
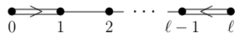
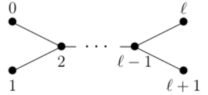
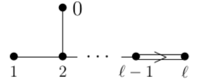
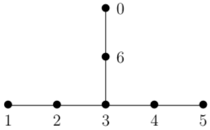
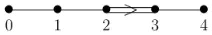
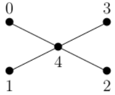
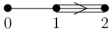
$$a_{ij} = a_{\sigma(i)\sigma(j)} \quad (1 \leq i, j \leq n)$$

- One can associate a graph to A with vertex set $\{1, 2, \cdots, n\}$ and edges depending on the entries of A.
- Automorphisms of a GCM and graph automorphisms of the (Dynkin) graphs of the GCM are one and the same.
- Any automorphism σ of the GCM A induces a (unique) automorphism (of the same order) of the Lie algebra $\mathfrak{g}(A)$ satisfying:
  1. $e_i \rightarrow e_{\sigma(i)}$
  2. $f_i \rightarrow f_{\sigma(i)}$
  3. $\sigma(\mathfrak{h}) = \mathfrak{h}$
  4. $(h, h') = (\sigma(h), \sigma(h'))$ for all $h, h' \in \mathfrak{h}$

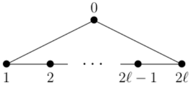
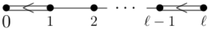
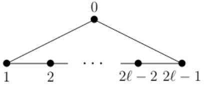
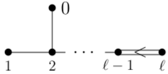
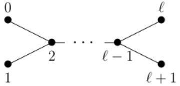
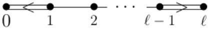
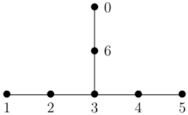
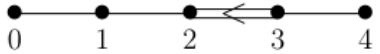
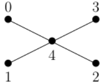
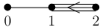
Fixed point subalgebras

- $\omega \in$ Diagram Automorphisms($\mathfrak{g}$)
- $\mathfrak{g}^\omega := \{x \in \mathfrak{g} | \omega(x) = x\}$ is a Lie algebra

$\mathfrak{g}$	$G(\mathfrak{g})$	$o(\omega)$	$\mathfrak{g}^\omega$	$G(\mathfrak{g}^\omega)$
$A_{2\ell}$		2	B_ℓ	
$A_{2\ell-1}$		2	C_ℓ	
$D_{\ell+1}$		2	B_ℓ	
E_6		2	F_4	
D_4		3	G_2	

$\mathfrak{g}$	$G(\mathfrak{g})$	$o(\omega)$	$\mathfrak{g}^\omega$	$G(\mathfrak{g}^\omega)$
$A_{2\ell-1}^{(1)}$		2	$C_\ell^{(1)}$	
$D_{\ell+1}^{(1)}$		2	$B_\ell^{(1)}$	
$E_6^{(1)}$		2	$F_4^{(1)}$	
$D_4^{(1)}$		3	$G_2^{(1)}$	

Fixed point subalgebras

<table>
<tr><th>$\mathfrak{g}$</th><th>$G(\mathfrak{g})$</th><th>$o(\omega)$</th><th>$\mathfrak{g}^\omega$</th><th>$G(\mathfrak{g}^\omega)$</th></tr>
<tr><td>$A_{2\ell}^{(1)}$</td><td></td><td>2</td><td>$A_{2\ell}^{(2)}$</td><td></td></tr>
<tr><td>$A_{2\ell-1}^{(1)}$</td><td></td><td>2</td><td>$A_{2\ell-1}^{(2)}$</td><td></td></tr>
<tr><td>$D_{\ell+1}^{(1)}$</td><td></td><td>2</td><td>$D_{\ell+1}^{(2)}$</td><td></td></tr>
<tr><td>$E_6^{(1)}$</td><td></td><td>2</td><td>$E_6^{(2)}$</td><td></td></tr>
<tr><td>$D_4^{(1)}$</td><td></td><td>3</td><td>$D_4^{(3)}$</td><td></td></tr>
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Main theorem 2

Theorem (_, Raghavan, Venkatesh, Viswanath)

Let $A_{n\times n}$ be a GCM of finite, affine, or hyperbolic type. Let Γ be a group of diagram automorphisms of $\mathfrak{g}$. Let $\mathfrak{g}^\Gamma$ denote the fixed point subalgebra. Suppose that

$$\otimes_{k=1}^{r} Res_{\mathfrak{g}^\Gamma} M(\lambda_k, I_k) = \otimes_{k=1}^{r} Res_{\mathfrak{g}^\Gamma} M(\mu_k, J_k)$$

where $\forall k$,

1. $(\lambda_k, I_k), (\mu_k, J_k) \in \mathcal{P}$
2. λ_k, $\mu_k \in \mathfrak{h}^*$ *is* Γ*-invariant*
3. I_k, J_k *are connected subsets of* $[n]$ *and are* Γ*-stable*

Then, $\sum_{k=1}^{r} \lambda_k = \sum_{k=1}^{r} \mu_k$ *and* $\exists \sigma \in \mathfrak{S}_r$ *such that* $I_k = J_{\sigma k}$ *and* $\lambda_k(\alpha_i^\vee) = \mu_{\sigma k}(\alpha_i^\vee)$ *for all* $i \in I_k$.

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Thank you for your attention!