ICTS Bengaluru: Quantum Trajectories January 2025

Slides based on
A tutorial introduction to quantum stochastic master equations based on the qubit/photon system,
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Outline

1. Schrödinger ($\hbar = 1$): wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\mathrm{H}|\psi\rangle, \quad \mathrm{H} = \mathrm{H}_0 + u\mathrm{H}_1 = \mathrm{H}^\dagger, \quad \frac{d}{dt}\rho = -i[\mathrm{H}, \rho].$$

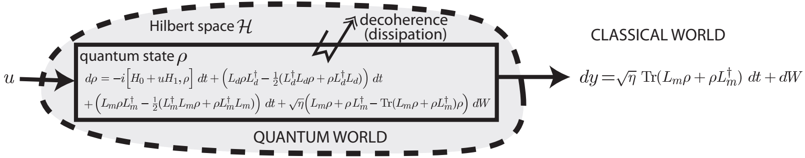
2. Origin of dissipation: collapse of the wave packet induced by the measurement of $\mathrm{O} = \mathrm{O}^\dagger$ with spectral decomp. $\sum_y \lambda_y \mathrm{P}_y$:
   - measurement outcome y with proba. $\mathbb{P}_y = \langle\psi|\mathrm{P}_y|\psi\rangle = \mathrm{Tr}\left(\rho \mathrm{P}_y\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome y:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\mathrm{P}_y|\psi\rangle}{\sqrt{\langle\psi|\mathrm{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\mathrm{P}_y \rho \mathrm{P}_y}{\mathrm{Tr}\left(\rho \mathrm{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, C):
   - Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$
   - Hamiltonian $\mathrm{H} = \mathrm{H}_s \otimes \mathrm{I}_c + \mathrm{H}_{sc} + \mathrm{I}_s \otimes \mathrm{H}_c$
   - observable on sub-system C only: $\mathrm{O} = \mathrm{I}_s \otimes \mathrm{O}_c$.

[1] S. Haroche and J.M. Raimond (2006). *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts.

Diffusive stochastic master equation[2]



$t \mapsto \rho_t$ continuous time function (not differentiable), solution of

$$d\rho_t = -i\left[H_0 + uH_1, \rho_t\right] dt + \left(\sum_{\nu=d,m} L_\nu \rho_t L_\nu^\dagger - \frac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu)\right) dt + \ldots$$

$$\ldots + \sqrt{\eta}\left(L_m\rho_t + \rho_t L_m^\dagger - \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\rho_t\right) dW_t,$$

where $\eta \in [0, 1]$ and the same Wiener process W_t is shared by the state dynamics and the output map

$$dy_t = \sqrt{\eta}\ \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\ dt + dW_t.$$

[2] A. Barchielli and M. Gregoratti. *Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case*. Springer Verlag, 2009.

$t \mapsto \rho_t$ piece wise smooth time function, solution of

$$d\rho_t = \left(-i[\mathsf{H},\rho_t] + \mathsf{V}\rho_t\mathsf{V}^\dagger - \tfrac{1}{2}(\mathsf{V}^\dagger\mathsf{V}\rho_t + \rho_t\mathsf{V}^\dagger\mathsf{V})\right)\, dt$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathsf{V}\rho_t\mathsf{V}^\dagger}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)} - \rho_t\right)\left(dy_t - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\right) dt\right)$$

where $\bar\theta \geq 0$ (dark count rate) and $\bar\eta \in [0,1]$ (detection efficiency) and where the counting detector outcome $dy_t \in \{0,1\}$ with

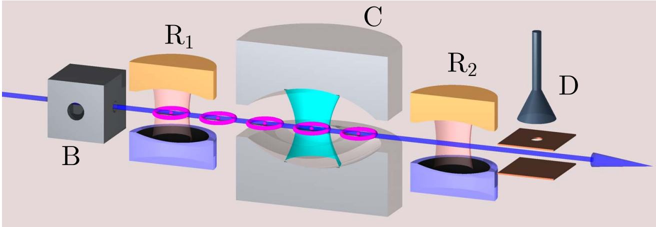
- $dy_t = 0$ with probability $1 - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\right) dt$ and then

$$\rho_{t+dt} = \rho_t + \Big(-i[\mathsf{H},\rho_t] + \mathsf{V}\rho_t\mathsf{V}^\dagger - \tfrac{1}{2}(\mathsf{V}^\dagger\mathsf{V}\rho_t + \rho_t\mathsf{V}^\dagger\mathsf{V})$$
$$+ \bar\eta\Big(\ \mathrm{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\rho_t - \mathsf{V}\rho_t\mathsf{V}^\dagger\Big)\Big)\, dt$$

- $dy_t = 1$ with probability $\left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)\right) dt$, and then

$$\rho_{t+dt} = \frac{\bar\theta\rho_t + \bar\eta\mathsf{V}\rho_t\mathsf{V}^\dagger}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathsf{V}\rho_t\mathsf{V}^\dagger\right)}.$$

[3] see, e.g., J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Phys. Rev. Lett.*, 68(5):580–583, February 1992.



- Dispersive qubit/photon interaction: $\mathsf{H}_{int} = -\chi\big(|e\rangle\langle e| - |g\rangle\langle g|\big) \otimes \mathsf{n}$ (with χ a constant parameter) yields $e^{-iT\mathsf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given with $\theta = \chi T$ by

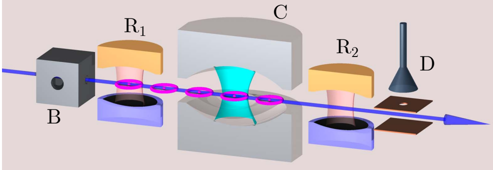
$$\mathsf{U}_\theta = |g\rangle\langle g| \otimes e^{-i\theta\mathsf{n}} + |e\rangle\langle e| \otimes e^{i\theta\mathsf{n}}.$$

- resonant qubit/photon interaction: $\mathsf{H}_{int} = i\frac{\omega}{2}\Big(|g\rangle\langle e| \otimes \mathsf{a}^\dagger - |e\rangle\langle g| \otimes \mathsf{a}\Big)$ (with ω a constant parameter) yields $e^{-iT\mathsf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given with $\theta = \omega T/2$ by

$$\mathsf{U}_\theta = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\mathsf{n}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\mathsf{n}+\mathsf{I}}) + |g\rangle\langle e| \otimes \frac{\sin(\theta\sqrt{\mathsf{n}})}{\sqrt{\mathsf{n}}}\mathsf{a}^\dagger - |e\rangle\langle g| \otimes \mathsf{a}\frac{\sin(\theta\sqrt{\mathsf{n}})}{\sqrt{\mathsf{n}}}.$$

[4] LKB for Laboratoire Kastler Brossel.

Outline



$$U = \left(\left(\left(\frac{|g\rangle - |e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle e|\right) \otimes I\right)$$
$$\left(|g\rangle\langle g| \otimes e^{-i\theta \mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta \mathbf{n}}\right)$$
$$\left(\left(\left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{-|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle e|\right) \otimes I\right)$$

applied on $|\Psi\rangle = |g\rangle \otimes |\psi\rangle$ yields

$$U\ (|g\rangle|\psi\rangle) = |g\rangle\ \cos(\theta \mathrm{n})|\psi\rangle + |e\rangle\ i\sin(\theta \mathrm{n})|\psi\rangle.$$

Markov process induced by the passage of qubit number k:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\cos(\theta \mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta \mathbf{n})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta \mathrm{n})|\psi_k\rangle\ ; \\ \frac{i\sin(\theta \mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta \mathbf{n})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta \mathrm{n})|\psi_k\rangle\ ; \end{cases}$$

where $y_k \in \{g, e\}$ classical signal produced by measurement of qubit k.

The density operator formulation ($\rho \equiv |\psi\rangle\langle\psi|$):

$$\rho_{k+1} = \begin{cases} \frac{M_g \rho_k M_g^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) ; \\ \frac{M_e \rho_k M_e^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) ; \end{cases}$$

with measurement Kraus operators $M_g = \cos(\theta N)$ and $M_e = \sin(\theta N)$. Notice that $M_g^\dagger M_g + M_e^\dagger M_e = I$.

For θ/π irrational, almost sure convergence towards a Fock state $|\bar{n}\rangle\langle\bar{n}|$ for some $\bar{n}$ based on the Lyapunov function (super-martingale)

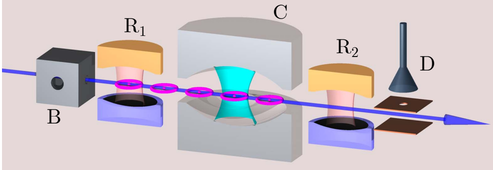
$$V(\rho) = \sum_{0 \le n_1 < n_2} \sqrt{\langle n_1|\rho|n_1\rangle\langle n_2|\rho|n_2\rangle}$$

that converges in average towards 0 since

$$\mathbb{E}\left(V(\rho_{k+1}) \,\middle|\, \rho_k\right) \le \left(\max_{0 \le n_1 < n_2} |\cos(\theta(n_1 \pm n_2))|\right) V(\rho_k).$$

Probability that a realisation converges towards $|\bar{n}\rangle\langle\bar{n}|$ given by its initial population $\langle\bar{n}|\rho_0|\bar{n}\rangle$

Outline



Wave function $|\Psi\rangle$ of the composite qubit/photon system just before D:

$$\Bigg(|g\rangle\langle g| \cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e| \cos(\theta\sqrt{\mathbf{n}+\mathbf{I}})$$

$$+ |g\rangle\langle e| \frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}} \mathbf{a}^\dagger - |e\rangle\langle g| \mathbf{a} \frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}} \Bigg) |g\rangle|\psi\rangle$$

$$= |g\rangle \ \cos(\theta\sqrt{\mathbf{n}})|\psi\rangle - |e\rangle \ \mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi\rangle$$

Resulting Markov process associated to the measurement of the observable $\sigma_{\mathbf{z}} = |e\rangle\langle e| - |g\rangle\langle g|$ with classical output signal $y \in \{g, e\}$:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\cos(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle \ ; \\ -\frac{\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle \ ; \end{cases}$$

Density operator formulation;

$$\rho_{k+1} = \begin{cases} \frac{\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger}{\mathrm{Tr}\left(\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger\right); \\ \frac{\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger}{\mathrm{Tr}\left(\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger\right); \end{cases}$$

with measurement Kraus operators $\mathsf{M}_g = \cos(\theta\sqrt{\mathsf{n}})$ and $\mathsf{M}_e = \mathsf{a}\frac{\sin(\theta\sqrt{\mathsf{n}})}{\sqrt{\mathsf{n}}}$. Notice that, once again, $\mathsf{M}_g^\dagger \mathsf{M}_g + \mathsf{M}_e^\dagger \mathsf{M}_e = \mathsf{I}$.

For $\theta\sqrt{n}/\pi$ irrational for all n, almost surely towards vacuum state $|0\rangle\langle 0|$. Results from the following the Lyapunov function (super-martingale)

$$V(\rho) = \mathrm{Tr}\left(\mathsf{n}\rho\right)$$

since

$$\mathbb{E}\left(V(\rho_{k+1}) \,\middle|\, \rho_k\right) = V(\rho_k) - \mathrm{Tr}\left(\sin^2(\theta\sqrt{\mathsf{n}})\rho_k\right).$$

Outline

With measurement imperfections, use Bayes rule by taking as quantum state, the expectation value of ρ_{k+1} knowing ρ_k and the information provides by the imperfect measurement outcome.
Assume detector D broken. From

$$\rho_{k+1} = \begin{cases} \frac{\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger}{\mathrm{Tr}\left(\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger\right) ; \\ \frac{\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger}{\mathrm{Tr}\left(\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger\right) ; \end{cases}$$

we get the quantum channel:

$$\rho_{k+1} = \mathcal{K}(\rho_k) \triangleq \mathbb{E}\left(\rho_{k+1} \;\middle|\; \rho_k\right) = \mathsf{M}_g \rho_k \mathsf{M}_g^\dagger + \mathsf{M}_e \rho_k \mathsf{M}_e^\dagger.$$

When the qubit detector D, producing the classical measurement signal $y_k \in \{g, e\}$, has errors characterized by the error rate $\eta_e \in (0,1)$ (resp. $\eta_g \in (0,1)$) the probability of detector outcome g (resp. e) knowing that the perfect outcome is e (resp. g), Bayes law gives directly

$$\rho_{k+1} = \begin{cases} \mathbb{E}\left(\rho_{k+1} \,\middle|\, y_k = g, \rho_k\right) = \frac{(1-\eta_g)\mathrm{M}_g\rho_k\mathrm{M}_g^\dagger + \eta_e\mathrm{M}_e\rho_k\mathrm{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)\mathrm{M}_g\rho_k\mathrm{M}_g^\dagger + \eta_e\mathrm{M}_e\rho_k\mathrm{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = g|\rho_k) = \ \mathrm{Tr}\left((1-\eta_g)\mathrm{M}_g\rho_k\mathrm{M}_g^\dagger + \eta_e\mathrm{M}_e\rho_k\mathrm{M}_e^\dagger\right), \\ \\ \mathbb{E}\left(\rho_{k+1} \,\middle|\, y_k = e, \rho_k\right) = \frac{\eta_g\mathrm{M}_g\rho_k\mathrm{M}_g^\dagger + (1-\eta_e)\mathrm{M}_e\rho_k\mathrm{M}_e^\dagger}{\mathrm{Tr}\left(\eta_g\mathrm{M}_g\rho_k\mathrm{M}_g^\dagger + (1-\eta_e)\mathrm{M}_e\rho_k\mathrm{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = e|\rho_k) = \ \mathrm{Tr}\left(\eta_g\mathrm{M}_g\rho_k\mathrm{M}_g^\dagger + (1-\eta_e)\mathrm{M}_e\rho_k\mathrm{M}_e^\dagger\right) \end{cases}$$

Notice that a broken detector corresponds to $\eta_e = \eta_g = 1/2$ and one recovers the above quantum channel.

General structure of discrete-time SME based on a quantum channel with the following Kraus decomposition (which is not unique)

$$\mathcal{K}(\rho) = \sum_{\mu} \mathrm{M}_\mu \rho \mathrm{M}_\mu^\dagger \quad \text{where} \sum_{\mu} \mathrm{M}_\mu^\dagger \mathrm{M}_\mu = \mathrm{I}$$

and a left stochastic matrix $(\eta_{y,\mu})$ where y corresponds to the different imperfect measurement outcomes. With $\mathcal{K}_y(\rho) = \sum_\mu \eta_{y,\mu} \mathrm{M}_\mu \rho \mathrm{M}_\mu^\dagger$, ones gets the following SME:

$$\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)} \quad \text{where } y_k = y \text{ with probability } \mathrm{Tr}\left(\mathcal{K}_y(\rho_k)\right)$$

Notice that $\mathcal{K} = \sum_y \mathcal{K}_y$ since η is left stochastic.

Here the Hilbert space $\mathcal{H}$ is arbitrary and can be of infinite dimension, the Kraus operator M_μ are bounded operator on $\mathcal{H}$ and ρ is a density operator on $\mathcal{H}$ (Hermitian, trace-class with trace one, non-negative). When the index y or μ are continuous, discrete sums are replaced by integrals and probabilities by probability densities.

Outline

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Probe photon in the coherent state $|i\frac{\alpha}{\sqrt{2}}\rangle$ with $\alpha > 0$. Just before D the composite qubit/photon wave function $|\Psi\rangle$ reads:

$$\left(|g\rangle\langle g|e^{-i\theta \mathrm{n}} + |e\rangle\langle e|e^{i\theta \mathrm{n}}\right)|\psi\rangle|i\tfrac{\alpha}{\sqrt{2}}\rangle = \langle g|\psi\rangle\,|g\rangle\ |ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle + \langle e|\psi\rangle\,|e\rangle\ |ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle.$$

Measurement outcome $y \in \mathbb{R}$ corresponding to observable

$$\mathrm{Q} = \frac{\mathrm{a}+\mathrm{a}^\dagger}{\sqrt{2}} \equiv \int_{-\infty}^{+\infty} q|q\rangle\langle q|dq \text{ where } \langle q|\,q'\rangle = \delta(q-q').$$

Since $|ie^{\pm i\theta}\frac{\alpha}{\sqrt{2}}\rangle = \frac{1}{\pi^{1/4}}\int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta} e^{-\frac{(q\pm\alpha\sin\theta)^2}{2}}|q\rangle dq$, we have

$$\begin{aligned}&\langle g|\psi\rangle\,|g\rangle\ |ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle + \langle e|\psi\rangle\,|e\rangle\ |ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle \\ &= \frac{1}{\pi^{1/4}}\int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta}\left(e^{-\frac{(q-\alpha\sin\theta)^2}{2}}\langle g|\psi\rangle\,|g\rangle + e^{-\frac{(q+\alpha\sin\theta)^2}{2}}\langle e|\psi\rangle\,|e\rangle\right)|q\rangle dq.\end{aligned}$$

Thus

$$|\psi_{k+1}\rangle = e^{iy_k\alpha\cos\theta}\frac{e^{-\frac{(y_k-\alpha\sin\theta)^2}{2}}\langle g|\psi_k\rangle\,|g\rangle + e^{-\frac{(y_k+\alpha\sin\theta)^2}{2}}\langle e|\psi_k\rangle\,|e\rangle}{\sqrt{e^{-(y_k-\alpha\sin\theta)^2}|\langle g|\psi_k\rangle|^2 + e^{-(y_k+\alpha\sin\theta)^2}|\langle e|\psi_k\rangle|^2}}$$

where $y_k \in [y, y+dy]$ with prob. $\frac{e^{-(y-\alpha\sin\theta)^2}|\langle g|\psi_k\rangle|^2 + e^{-(y+\alpha\sin\theta)^2}|\langle e|\psi_k\rangle|^2}{\sqrt{\pi}}dy$.

Density operator formulation

$$\rho_{k+1} = \frac{M_{y_k}\rho_k M_{y_k}^\dagger}{\mathrm{Tr}\left(M_{y_k}\rho_k M_{y_k}^\dagger\right)} \quad \text{where } y_k \in [y, y+dy] \text{ with probability } \mathrm{Tr}\left(M_y\rho_k M_y^\dagger\right) dy$$

and measurement Kraus operators

$$M_y = \tfrac{1}{\pi^{1/4}} e^{-\frac{(y-\alpha\sin\theta)^2}{2}} |g\rangle\langle g| + \tfrac{1}{\pi^{1/4}} e^{-\frac{(y+\alpha\sin\theta)^2}{2}} |e\rangle\langle e|.$$

Notice that

$$\mathrm{Tr}\left(M_y\rho M_y^\dagger\right) = \frac{1}{\sqrt{\pi}} e^{-(y-\alpha\sin\theta)^2}\langle g|\rho|g\rangle + \frac{1}{\sqrt{\pi}} e^{-(y+\alpha\sin\theta)^2}\langle e|\rho|e\rangle$$

and $\int_{-\infty}^{+\infty} M_y^\dagger M_y \, dy = |g\rangle\langle g| + |e\rangle\langle e| = I$.
For $\alpha \neq 0$, almost sure convergence towards $|g\rangle$ or $|e\rangle$ deduced from Lyapunov function

$$V(\rho) = \sqrt{\langle g|\rho|g\rangle\langle e|\rho|e\rangle} \text{ with } \mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k\right) = e^{-\alpha^2\sin^2\theta}\; V(\rho_k).$$

Detection imperfections: probability density of y knowing perfect detection q is a Gaussian given by $\frac{1}{\sqrt{\pi\sigma}}e^{-\frac{(y-q)^2}{\sigma}}$ for some error parameter $\sigma > 0$. Then the above Markov process becomes

$$\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)}$$

where

$$\mathcal{K}_y(\rho) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\sigma}}e^{-\frac{(y-q)^2}{\sigma}} \mathrm{M}_q \rho \mathrm{M}_q^\dagger \, dq$$

Standard computations using

$$\mathrm{M}_q = \frac{1}{\pi^{1/4}}e^{-\frac{(q-\alpha\sin\theta)^2}{2}} |g\rangle\langle g| + \frac{1}{\pi^{1/4}}e^{-\frac{(q+\alpha\sin\theta)^2}{2}} |e\rangle\langle e|$$

show that

$$\mathcal{K}_y(\rho) = \frac{1}{\sqrt{\pi(1+\sigma)}}\Bigg(e^{-\frac{(y-\alpha\sin\theta)^2}{1+\sigma}}\langle g|\rho|g\rangle|g\rangle\langle g| + e^{-\frac{(y+\alpha\sin\theta)^2}{1+\sigma}}\langle e|\rho|e\rangle|e\rangle\langle e| \\ + e^{-\frac{y^2}{1+\sigma}-(\alpha\sin\theta)^2}\left(\langle e|\rho|g\rangle|e\rangle\langle g| + \langle g|\rho|e\rangle|g\rangle\langle e|\right)\Bigg).$$

Continuous-time diffusive limit (1)

Density operator formulation (perfect detection)

$$\rho_{k+1} = \frac{\mathsf{M}_{y_k}\rho_k \mathsf{M}_{y_k}^\dagger}{\mathrm{Tr}\left(\mathsf{M}_{y_k}\rho_k \mathsf{M}_{y_k}^\dagger\right)} \quad \text{where } y_k \in [y, y+dy] \text{ with probability } \mathrm{Tr}\left(\mathsf{M}_y \rho_k \mathsf{M}_y^\dagger\right) dy$$

and measurement Kraus operators

$$\mathsf{M}_y = \tfrac{1}{\pi^{1/4}} e^{-\frac{(y-\alpha\sin\theta)^2}{2}} |g\rangle\langle g| + \tfrac{1}{\pi^{1/4}} e^{-\frac{(y+\alpha\sin\theta)^2}{2}} |e\rangle\langle e|.$$

Since

$$\mathbb{E}\left(y_k \;\middle|\; \rho_k = \rho\right) \triangleq \overline{y} = -\alpha\sin\theta \;\; \mathrm{Tr}\left(\sigma_z\rho\right), \; \mathbb{E}\left(y_k^2 \;\middle|\; \rho_k = \rho\right) \triangleq \overline{y^2} = 1/2 + (\alpha\sin\theta)^2.$$

When $0 < \alpha\sin\theta = \epsilon \ll 1$, we have up-to third order terms versus ϵy,

$$\frac{\mathsf{M}_y\rho \mathsf{M}_y^\dagger}{\mathrm{Tr}\left(\mathsf{M}_y\rho \mathsf{M}_y^\dagger\right)} = \frac{(\cosh(\epsilon y) - \sinh(\epsilon y)\sigma_z)\rho(\cosh(\epsilon y) - \sinh(\epsilon y)\sigma_z)}{\cosh(2\epsilon y) - \sinh(2\epsilon y)\;\mathrm{Tr}\left(\sigma_z\rho\right)}$$

$$\approx \frac{\rho - \epsilon y(\sigma_z\rho + \rho\sigma_z) + (\epsilon y)^2(\rho + \sigma_z\rho\sigma_z)}{1 - 2\epsilon y\;\mathrm{Tr}\left(\sigma_z\rho\right) + 2(\epsilon y)^2}$$

$$= \rho + (\epsilon y)^2\left(\sigma_z\rho\sigma_z - \rho\right) + \left(\sigma_z\rho + \rho\sigma_z - 2\;\mathrm{Tr}\left(\sigma_z\rho\right)\rho\right)\left(-\epsilon y - 2(\epsilon y)^2\;\mathrm{Tr}\left(\sigma_z\rho\right)\right).$$

Replacing $\epsilon^2 y^2$ by its expectation value one gets, up to third order in ϵy and ϵ:

$$\frac{M_y \rho M_y^\dagger}{\mathrm{Tr}\left(M_y \rho M_y^\dagger\right)} \approx \rho + \frac{\epsilon^2}{2}\Big(\sigma_z \rho \sigma_z - \rho\Big) + \Big(\sigma_z \rho + \rho \sigma_z - 2\ \mathrm{Tr}\left(\sigma_z \rho\right) \rho\Big)\Big(-\epsilon y - \epsilon^2\ \mathrm{Tr}\left(\sigma_z \rho\right)\Big).$$

Set $\epsilon^2 = 2dt$ and $\epsilon y = -2\ \mathrm{Tr}\left(\sigma_z \rho\right) dt - dW$. Since by construction

$$\mathbb{E}\left(\epsilon y_k \;\middle|\; \rho_k = \rho\right) = -\epsilon^2\ \mathrm{Tr}\left(\sigma_z \rho\right) \text{ and } \mathbb{E}\left((\epsilon y_k)^2 \;\middle|\; \rho_k = \rho\right) = \epsilon^2 + \epsilon^4$$

one has $\mathbb{E}\left(dW \;\middle|\; \rho\right) = 0$ and $\mathbb{E}\left(dW^2 \;\middle|\; \rho\right) = dt$ up to order 4 versus ϵ. Thus for dt very small, we recover the following diffusive SME[5]

$$\rho_{t+dt} = \rho_t + dt\Big(\sigma_z \rho_t \sigma_z - \rho\Big) + \Big(\sigma_z \rho_t + \rho_t \sigma_z - 2\ \mathrm{Tr}\left(\sigma_z \rho_t\right) \rho\Big)\Big(dy_t - 2\ \mathrm{Tr}\left(\sigma_z \rho_t\right) dt\Big)$$

with $dy_t = 2\ \mathrm{Tr}\left(\sigma_z \rho_t\right) dt + dW_t$ replacing $-\epsilon y$ and $dy_t^2 = dW_t^2 = dt$ (Ito rules).

[5] Convergence in distribution when $dt \mapsto 0^+$: tightness property

$$\forall T > 0, \exists M > 0, \forall dt > 0, \forall k, k_1, k_2 \in \{0, \dots, [T/dt]\}, \mathbb{E}\left(\|\rho_{k_1} - \rho_k\|^2\|\ \|\rho_{k_2} - \rho_k\|^2 \;\middle|\; \rho_0\right) \leq M(k_1 - k_2)\, dt,$$

and (Markov generator) convergence of $\dfrac{\mathbb{E}\left(f(\rho_{k+1} \;\middle|\; \rho_k = \rho\right) - f(\rho)}{dt}$ towards $\mathbb{E}\left(df_t \;\middle|\; \rho_t = \rho\right)/dt$ for any C^2 real function f.

With measurement errors parameterized by $\sigma > 0$, the partial Kraus map

$$\mathcal{K}_y(\rho) = \frac{1}{\sqrt{\pi(1+\sigma)}}\Bigg(e^{-\frac{(y-\epsilon)^2}{1+\sigma}} \langle g|\rho|g\rangle |g\rangle\langle g| + e^{-\frac{(y+\epsilon)^2}{1+\sigma}} \langle e|\rho|e\rangle |e\rangle\langle e|$$
$$+ e^{-\frac{y^2}{1+\sigma}-\epsilon^2}\Big(\langle e|\rho|g\rangle |e\rangle\langle g| + \langle g|\rho|e\rangle |g\rangle\langle e|\Big)\Bigg)$$

yields $\mathbb{E}\left(y_k \mid \rho_k\right) \triangleq \bar{y} = -\epsilon \operatorname{Tr}(\sigma_z \rho)$ and $\mathbb{E}\left(y_k^2 \mid \rho_k\right) \triangleq \overline{y^2} = (1+\sigma)/2 + \epsilon^2$.
Similar approximations with $\epsilon^2 = 2dt$ and dt very small, yield an SME with detection efficiency $\eta = \frac{1}{1+\sigma}$:

$$\rho_{t+dt} = \rho_t + dt\Big(\sigma_z \rho_t \sigma_z - \rho\Big) + \sqrt{\eta}\Big(\sigma_z \rho_t + \rho_t \sigma_z - 2 \operatorname{Tr}(\sigma_z \rho_t)\, \rho\Big) dW_t$$

with $dy_t = \sqrt{\eta} \operatorname{Tr}(\sigma_z \rho_t + \rho_t \sigma_z) + dW_t \sim -\epsilon y / \sqrt{1+\sigma}$.
Convergence towards either $|g\rangle$ or $|e\rangle$ (QND measurement of the qubit) based on Lyapunov fonction $V(\rho) = \sqrt{1 - \operatorname{Tr}(\sigma_z \rho)^2}$ and Ito rules:

$$dV = -\frac{z dz}{\sqrt{1-z^2}} - \frac{dz^2}{2(1-z^2)^{3/2}} = -\frac{z dz}{\sqrt{1-z^2}} - 2\eta^2 V dt$$

where $z = \operatorname{Tr}(\sigma_z \rho)$, $dz = 2\eta(1-z^2) dW$ and $dz^2 = 4\eta^2(1-z^2)^2 dt$. Since $\mathbb{E}\left(dz \mid z\right) = 0$, $\bar{V}_t = \mathbb{E}\left(V(z_t) \mid z_0\right)$ solution of $\frac{d}{dt}\bar{V}_t = -2\eta^2 \bar{V}_t$.

Diffusive SME[6]

General form of diffusive SME with Ito formulation:

$$d\rho_t = \left(-i[\mathrm{H},\rho_t] + \sum_\nu \mathrm{L}_\nu \rho_t \mathrm{L}_\nu^\dagger - \frac{1}{2}(\mathrm{L}_\nu^\dagger \mathrm{L}_\nu \rho_t + \rho_t \mathrm{L}_\nu^\dagger \mathrm{L}_\nu)\right) dt + \sum_\nu \sqrt{\eta_\nu}\left(\mathrm{L}_\nu \rho_t + \rho_t \mathrm{L}_\nu^\dagger - \mathrm{Tr}\left((\mathrm{L}_\nu + \mathrm{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t},$$

$$dy_{\nu,t} = \sqrt{\eta_\nu}\,\mathrm{Tr}\left(\mathrm{L}_\nu \rho_t + \rho_t \mathrm{L}_\nu^\dagger\right) dt + dW_{\nu,t}$$

with efficiencies $\eta_\nu \in [0,1]$ and $dW_{\nu,t}$ being independent Wiener processes.
Equivalent formulation with Ito rules:

$$\rho_{t+dt} = \frac{\mathrm{M}_{dy_t}\rho_t \mathrm{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\mathrm{L}_\nu \rho_t \mathrm{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\mathrm{M}_{dy_t}\rho_t \mathrm{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\mathrm{L}_\nu \rho_t \mathrm{L}_\nu^\dagger dt\right)}$$

with $\mathrm{M}_{dy_t} = \mathrm{I} + \left(-i\mathrm{H} - \frac{1}{2}\sum_\nu \mathrm{L}_\nu^\dagger \mathrm{L}_\nu\right) dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \mathrm{L}_\nu$. Moreover $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ follows the following probability density knowing ρ_t:

$$\mathbb{P}\Big(\left(s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu]\right)_\nu \mid \rho_t\Big) = \mathrm{Tr}\left(\mathrm{M}_{s\sqrt{dt}}\,\rho_t \mathrm{M}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu)\mathrm{L}_\nu \rho_t \mathrm{L}_\nu^\dagger dt\right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

[6] A. Barchielli and M. Gregoratti. *Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case*. Springer Verlag, 2009.

Linearity/positivity/trace preserving numerical integration scheme for

$$d\rho_t = \left(-i[\mathsf{H},\rho_t] + \sum_\nu \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger - \frac{1}{2}(\mathsf{L}_\nu^\dagger \mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger \mathsf{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(\mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger - \ \mathrm{Tr}\left((\mathsf{L}_\nu + \mathsf{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t},$$

$$dy_{\nu,t} = \sqrt{\eta_\nu}\ \mathrm{Tr}\left(\mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger\right) dt + dW_{\nu,t}$$

With $\mathsf{M}_0 = \mathsf{I} + \big(-i\mathsf{H} - \frac{1}{2}\sum_\nu \mathsf{L}_\nu^\dagger \mathsf{L}_\nu\big)dt$, $\quad \mathsf{S} = \mathsf{M}_0^\dagger \mathsf{M}_0 + \left(\sum_\nu \mathsf{L}_\nu^\dagger \mathsf{L}_\nu\right) dt$ set

$$\widetilde{\mathsf{M}}_0 = \mathsf{M}_0 \mathsf{S}^{-1/2}, \quad \widetilde{\mathsf{L}}_\nu = \mathsf{L}_\nu \mathsf{S}^{-1/2}.$$

Sampling of $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ according to the following probability law:

$$\mathbb{P}\Big(\left(s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu]\right)_\nu \mid \rho_t\Big) = \ \mathrm{Tr}\left(\widetilde{\mathsf{M}}_{s\sqrt{dt}} \rho_t \widetilde{\mathsf{M}}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu)\widetilde{\mathsf{L}}_\nu \rho_t \widetilde{\mathsf{L}}_\nu^\dagger dt\right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

where $\widetilde{\mathsf{M}}_{dy_t} = \widetilde{\mathsf{M}}_0 + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \widetilde{\mathsf{L}}_\nu$. Exact Kraus-map formulation:

$$\rho_{t+dt} = \frac{\widetilde{\mathsf{M}}_{dy_t} \rho_t \widetilde{\mathsf{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\widetilde{\mathsf{L}}_\nu \rho_t \widetilde{\mathsf{L}}_\nu^\dagger dt}{\mathrm{Tr}\left(\widetilde{\mathsf{M}}_{dy_t} \rho_t \widetilde{\mathsf{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\widetilde{\mathsf{L}}_\nu \rho_t \widetilde{\mathsf{L}}_\nu^\dagger dt\right)}.$$

[7] A. Jordan, A. Chantasri, PR, and B.Huard. Anatomy of fluorescence: quantum trajectory statistics from continuously measuring spontaneous emission. *Quantum Studies: Mathematics and Foundations*, 3(3):237–263, 2016.

Outline

Probe photon is in the vacuum state $|0\rangle$. Composite qubit/photon wave function $|\Psi\rangle$ before D:

$$\Bigg(|g\rangle\langle g|\cos(\theta\sqrt{\mathrm{n}}) + |e\rangle\langle e|\cos(\theta\sqrt{\mathrm{n}+\mathrm{I}}) \\ + |g\rangle\langle e|\frac{\sin(\theta\sqrt{\mathrm{n}})}{\sqrt{\mathrm{n}}}\mathrm{a}^\dagger - |e\rangle\langle g|\mathrm{a}\frac{\sin(\theta\sqrt{\mathrm{n}})}{\sqrt{\mathrm{n}}}\Bigg)|\psi\rangle|0\rangle \\ = \big(\langle g|\psi\rangle\,|g\rangle + \cos\theta\,\langle e|\psi\rangle\,|e\rangle\big)|0\rangle + \sin\theta\,\langle e|\psi\rangle\,|g\rangle|1\rangle.$$

With measurement observable $\mathrm{n} = \sum_{n\geq 0} n|n\rangle\langle n|$, outcome $y \in \{0,1\}$ reads (density operator formulation)

$$\rho_{k+1} = \begin{cases} \frac{\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger\right)} & \text{if } y_k = 0 \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger\right); \\ \frac{\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger\right)} & \text{if } y_k = 1 \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger\right); \end{cases}$$

measurement Kraus operators $\mathrm{M}_0 = |g\rangle\langle g| + \cos\theta|e\rangle\langle e|$ and $\mathrm{M}_1 = \sin\theta|g\rangle\langle e|$. Almost convergence analysis when $\cos^2(\theta) < 1$ towards $|g\rangle$ via the Lyapunov function (super martingale)

$$V(\rho) = \mathrm{Tr}\left(|e\rangle\langle e|\rho\right) \text{ since } \mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k\right) = \cos^2\theta\; V(\rho_k).$$

Towards jump SME (1)

Since $\mathrm{Tr}\left(\mathrm{M}_0\rho\mathrm{M}_0^\dagger\right) = 1 - \sin^2\theta\ \mathrm{Tr}\left(\sigma_-\rho\sigma_+\right)$ and $\mathrm{Tr}\left(\mathrm{M}_1\rho\mathrm{M}_1^\dagger\right) = \sin^2\theta\ \mathrm{Tr}\left(\sigma_-\rho\sigma_+\right)$, one gets with $\sin^2\theta = dt$ and $y \sim dN$, an SME driven by Poisson process $dN_t \in \{0,1\}$ of expectation value $\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right) dt$ knowing ρ_t:

$$
\begin{aligned}
d\rho_t = &\left(\sigma_-\rho_t\sigma_+ - \tfrac{1}{2}(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_-)\right)\, dt \\
&+ \left(\frac{\sigma_-\rho_t\sigma_+}{\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)} - \rho_t\right)\left(dN_t - \left(\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right) dt\right).
\end{aligned}
$$

At each time-step, one has the following choice:

- with probabilty $1 -\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right) dt$, $dN_t = N_{t+dt} - N_t = 0$ and
$$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger\right)}$$
with $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}\sigma_+\sigma_-$.
- with probability $\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right) dt$, $dN_t = N_{t+dt} - N_t = 1$ and
$$\rho_{t+dt} = \frac{\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$
with $\mathrm{M}_1 = \sqrt{dt}\ \sigma_-$.

Towards jump SME (2)

With left stochastic matrix $\begin{pmatrix} 1-\bar\theta dt & 1-\bar\eta \\ \bar\theta dt & \bar\eta \end{pmatrix}$ including shot noise of rate $\bar\theta \geq 0$ and detection efficiency $\bar\eta \in [0,1]$:

- $dN_t = N_{t+dt} - N_t = 0$ and

$$\rho_{t+dt} = \frac{(1-\bar\theta dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left((1-\bar\theta dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$

$$= \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)} + O(dt^2).$$

  with probability

$$1-\left(\bar\theta+\bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt = \ \mathrm{Tr}\left((1-\bar\theta dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right) + O(dt^2)$$

  and where $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}\sigma_+\sigma_-$ and $\mathrm{M}_1 = \sqrt{dt}\ \sigma_-$.

- $dN_t = N_{t+dt} - N_t = 1$ and

$$\rho_{t+dt} = \frac{\bar\theta\, dt\, \mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar\eta\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\bar\theta\, dt\, \mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar\eta\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)} = \frac{\bar\theta\rho_t + \bar\eta\sigma_-\rho_t\sigma_+}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)} + O(dt)$$

  with probability

$$\left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt = \ \mathrm{Tr}\left(\bar\theta\, dt\, \mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar\eta\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right) + O(dt^2)$$

Towards jump SME (3)

Jump SME with shot noise rate $\bar{\theta}$ and detection efficiency $\bar{\eta}$

$$d\rho_t = \left(\sigma_- \rho_t \sigma_+ - \tfrac{1}{2}(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_-)\right)\, dt + \left(\frac{\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+}{\mathrm{Tr}\left(\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+\right)} - \rho_t\right)\left(dN_t - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right) dt\right).$$

corresponds to the following choices

- $dN_t = N_{t+dt} - N_t = 0$

$$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$

  with probability $1 - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt$,

- $dN_t = N_{t+dt} - N_t = 1$ and

$$\rho_{t+dt} = \frac{\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+}{\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)}$$

  with probability $1 - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt$,

where $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}(\sigma_+\sigma_- + \mathrm{I})$ and $\mathrm{M}_1 = \sqrt{dt}\ \sigma_-$.

Jump SME in continuous-time[8] (1)

General structure of a Jump SME in continuous time with counting process N_t with increment expectation value knowing ρ_t given by $\langle dN_t \rangle = \left(\bar\theta + \bar\eta \operatorname{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, with $\bar\theta \geq 0$ (shot-noise rate) and $\bar\eta \in [0,1]$ (detection efficiency):

$$d\rho_t = \left(-i[\mathrm{H},\rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})\right) dt$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt\right).$$

Here H and V are operators on an underlying Hilbert space $\mathcal{H}$, H being Hermitian. At each time-step between t and $t+dt$, one has the following recipe

- $dN_t = 0$ with probability $1 - \left(\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$

$$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{V}\rho_t\mathrm{V}^\dagger dt}{\operatorname{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{V}\rho_t\mathrm{V}^\dagger dt\right)}$$

  where $\mathrm{M}_0 = I - \left(iH + \frac{1}{2}\mathrm{V}^\dagger\mathrm{V}\right) dt$.
- $dN_t = 1$ with probability $\left(\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$,

$$\rho_{t+dt} = \frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta \operatorname{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)}.$$

[8] J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Phys. Rev. Lett.*, 68(5):580–583, 1992.

$$d\rho_t = \left(-i[\mathrm{H}, \rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})\right)\, dt$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right)dt\right).$$

Take a discretization step $dt > 0$ and set $\mathrm{M}_0 = I - \left(iH + \frac{1}{2}\mathrm{V}^\dagger\mathrm{V}\right)dt$, $\widetilde{\mathrm{M}}_0 = \mathrm{M}_0\mathrm{S}^{-1/2}$ and $\widetilde{\mathrm{V}} = \mathrm{V}\mathrm{S}^{-1/2}$ with $\mathrm{S} = \mathrm{M}_0^\dagger\mathrm{M}_0 + \mathrm{V}^\dagger\mathrm{V}dt$. Use the following numerical CP scheme:

- $dN_t = 0$ with probability $\mathrm{Tr}\left(e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)$

$$\rho_{t+dt} = \frac{e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger}{\mathrm{Tr}\left(e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)}.$$

- $dN_t = 1$ with probability $\mathrm{Tr}\left(\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)$

$$\rho_{t+dt} = \frac{\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger}{\mathrm{Tr}\left(\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)}.$$

Probabilities are preserved exactly: for any ρ_t, $\bar\theta \geq 0$, $\bar\eta \in [0,1]$

$$\mathrm{Tr}\left(e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right) + \mathrm{Tr}\left(\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right) \equiv 1$$