Generalized k- Chromatic polynomials and root multiplicities of BKM Lie superalgebras

(Joint work with Dr. G. Arunkumar)

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Outline of the talk

- Chromatic polynomials
- Generalized $\mathbf{k}$ Chromatic polynomials
- BKM Lie superalgebras
- Denominator identity of BKM Lie superalgebras
- Quasi Dynkin diagram
- Relationship between Chromatic polynomial of Quasi Dynkin diagram and root multiplicities of free roots

Chromatic polynomials

Preliminaries

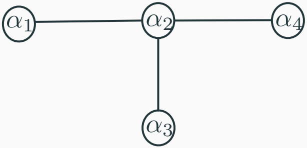
- Let I be any countable set and G be a graph with vertex set $V = \{\alpha_i : i \in I\}$ and edge set $E(G)$.
- Let $\{1, 2, \cdots, q\}$ be a set of q-distinct colors and $\mathcal{P}(\{1, 2, \cdots, q\})$ denotes its power set.

Definition

The number of ways a Graph G can be properly colored using q colors is a polynomial in variable q, called **the Chromatic polynomial** of the graph G, denoted by $\chi(G, q)$.

Example

Consider the following graph



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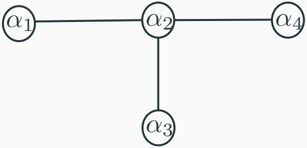

Using two colors: green and red, it can be properly colored using the following 2 ways.

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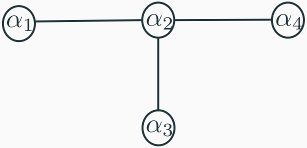

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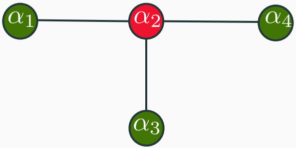
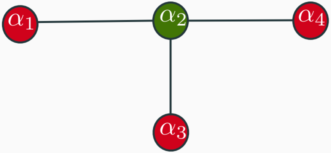




The Chromatic polynomial of the above graph is $\chi(G, q) = q(q-1)^3$.

Generalized k-Chromatic polynomials

For a tuple $\mathbf{k} = (k_i : i \in I)$ of non-negative integers, we define $\mathrm{supp}(\mathbf{k}) = \{i \in I : k_i \neq 0\}$.

Definition

Let $\mathbf{k} = (k_i : i \in I)$ s.t. $|\mathrm{supp}(\mathbf{k})| < \infty$. A map $\tau : V \longrightarrow \mathcal{P}(\{1, 2, \cdots, q\})$ is said to be **proper vertex $\mathbf{k}$-multicoloring** of a graph G if the following conditions are satisfied:

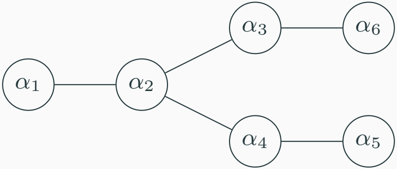
- $|\tau(\alpha_i)| = k_i$ for all $i \in I$,
- $\tau(\alpha_i) \cap \tau(\alpha_j) = \phi$ if $(\alpha_i, \alpha_j) \in E(G)$.

Definition

The number of ways in which a graph G can be proper $\mathbf{k}$-multicolored using q colors is a polynomial in q called **the generalized $\mathbf{k}$-Chromatic polynomial**, denoted by $\pi^G_{\mathbf{k}}(q)$.

Example

Let $\mathbf{k} = (2, 1, 3, 2, 1, 2)$ and G be the following supergraph:



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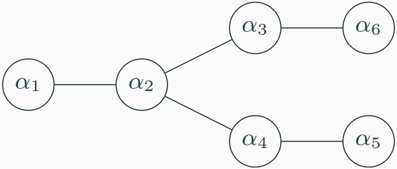

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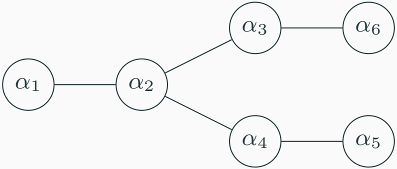

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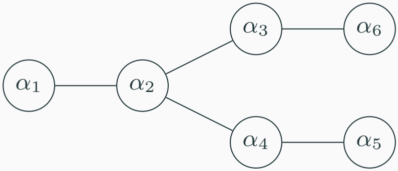

- α_1 node can be colored in $\binom{q}{2}$-ways
- α_2 node can be colored in $\binom{q-2}{1}$-ways
- α_3,α_4, α_5 and α_6 nodes can be colored in $\binom{q-1}{3}$, $\binom{q-1}{2}$, $\binom{q-2}{1}$ and $\binom{q-3}{2}$ ways respectively.

$$\pi_{\mathbf{k}}^{G}(q) = \binom{q}{2}\binom{q-2}{1}\binom{q-1}{3}\binom{q-1}{2}\binom{q-2}{1}\binom{q-3}{2}.$$

Generalised $\mathbf{k}$- Chromatic polynomial

Let $P_\ell(\mathbf{k}, G)$ be the set of all ordered ℓ-tuples $(P_1, P_2, \ldots, P_\ell)$ such that

(i) each P_j is a non-empty independent subset of V, i.e., no two vertices have an edge between them; and

(ii) for all $j \in I$, α_j occurs exactly k_j times in the disjoint union $P_1 \sqcup \ldots \sqcup P_\ell$.

Then

$$\pi_{\mathbf{k}}^G(q) = \sum_{\ell \geq 0} |P_\ell(\mathbf{k}, G)| \binom{q}{\ell}.$$

Borcherds Kac Moody Lie superalgebras

BKM supermatrix

Let I be a countable (possibly infinite) set. Fix a set $\Psi \subseteq I$. A real matrix $(A = (a_{ij})_{i,j \in I}, \Psi)$ is said to be a **BKM supermatrix** if the following conditions are satisfied: For $i, j \in I$ we have

1. $a_{ii} = 2$ or $a_{ii} \leq 0$.
2. $a_{ij} \leq 0$ if $i \neq j$.
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.
4. $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$.
5. $a_{ij} \in 2\mathbb{Z}$ if $a_{ii} = 2$ and $i \in \Psi$.

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   We are interested in symmetrizable BKM supermatrix.
6. A is symmetrizable, i.e., DA is symmetric for some diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ with positive entries.

Notations

- $I^{re} = \{i \in I : a_{ii} = 2\}, I^{im} = I \setminus I^{re}$
- $\Psi^{re} = \Psi \cap I^{re}$
- $\Psi_0 = \{i \in \Psi : a_{ii} = 0\}$

Definition

The **BKM Lie superalgebra** associated with (A, Ψ) is the Lie superalgebra $\mathfrak{L}(A, \Psi)$ generated by $e_i, f_i, h_i, i \in I$ with the following defining relations:

1. $[h_i, h_j] = 0$, $[e_i, f_j] = \delta_{ij} h_i$ for $i, j \in I$,
2. $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$ for $i, j \in I$,
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5. $(\text{ad } e_i)^{1-a_{ij}} e_j = 0 = (\text{ad } f_i)^{1-a_{ij}} f_j$ if $i \in I^{re}$ and $i \neq j$,
6. $(\text{ad } e_i)^{1-\frac{a_{ij}}{2}} e_j = 0 = (\text{ad } f_i)^{1-\frac{a_{ij}}{2}} f_j$ if $i \in \Psi^{re}$ and $i \neq j$,
7. $(\text{ad } e_i)^{1-\frac{a_{ij}}{2}} e_j = 0 = (\text{ad } f_i)^{1-\frac{a_{ij}}{2}} f_j$ if $i \in \Psi_0$ and $i = j$,

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8. $[e_i, e_j] = 0 = [f_i, f_j]$ if $a_{ij} = 0$.

Idea

- $\mathfrak{L}$= Free Lie superalgebra generated by $\{e_i, f_i, h_i : i \in I\}$.

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- $\mathfrak{L}(A, \Psi) = \frac{\widetilde{\mathfrak{L}}}{\text{Serre relations}}$
- Our interest is in root spaces independent of Serre relations. We call such root spaces as free root spaces.

Notations

- Δ:= Set of roots, $\Pi := \{\alpha_i : i \in I\}$= Set of simple roots.
- $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i, \quad Q_+ := \bigoplus_{i\in I} \mathbb{Z}_+\alpha_i.$
- $I = I_0 \sqcup I_1$ where $I_1 = \Psi, I_0 = I \setminus \Psi$

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Definition

Let $\alpha = \sum_{i\in I} k_i\alpha_i \in Q_+$,

- **weight** of α is defined as $\mathbf{k} = (k_i : i \in I)$.
- α is said to be **free** if $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$.

Denominator identity of BKM Lie superalgebras

Denominator identity

Let Ω be the set of all $\gamma \in Q_+$ such that

1. $\gamma = \sum_{j=1}^{r} \alpha_{i_j} + \sum_{k=1}^{s} l_{i_k} \beta_{i_k}$ where the α_{i_j} (resp. β_{i_k}) are distinct even (resp. odd) imaginary simple roots,
2. $(\alpha_{i_j}, \alpha_{i_k}) = (\beta_{i_j}, \beta_{i_k}) = 0$ for $j \neq k$; $(\alpha_{i_j}, \beta_{i_k}) = 0$ for all j, k;
3. if $l_{i_k} \geq 2$, then $(\beta_{i_k}, \beta_{i_k}) = 0$.

The following denominator identity of BKM superalgebras is proved in [1][Wak01, Section 2.5]:

$$U := \sum_{w \in W} \sum_{\gamma \in \Omega} \epsilon(w)\epsilon(\gamma) e^{w(\rho-\gamma)-\rho} = \frac{\prod_{\alpha \in \Delta_+^0} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_+^1} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}}$$

where $\mathrm{mult}(\alpha) = \dim \mathfrak{g}_\alpha$, $\epsilon(w) = (-1)^{l(w)}$ and $\epsilon(\gamma) = (-1)^{\mathrm{ht}\,\gamma}$.

[1] Minoru Wakimoto; Infinite-dimensional Lie algebras. *American Mathematical Society, Providence, RI*, Translations of Mathematical Monographs, 195, 2001.

Quasi Dynkin diagram

Quasi Dynkin diagram

Let (G, Ψ) be a graph with

- $V = \{\alpha_i : i \in I\}$ be a $\mathbb{Z}_2$-graded vertex set and
- $E(G) = \{(\alpha_i, \alpha_j) : a_{ij} \neq 0, i, j \in I\}$ be an edge set,

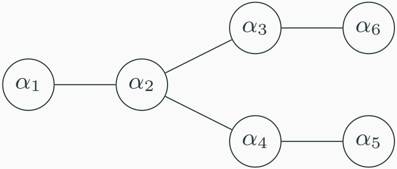
then the resulting super-graph is called Quasi Dynkin diagram of $\mathfrak{L}$ where $\Psi \subset I$ parametrizes the odd vertices of G.

Example

Go to slide

Let $I = \{1, 2, 3, 4, 5, 6\}, \Psi = \{3, 5\}$. Consider the BKM supermatrix

- $A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -4 & -1 & 0 & 0 \\ 0 & -4 & -4 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -3 \end{bmatrix}.$
- The quasi-Dynkin diagram G of $\mathfrak{L}$ is as follows:



- Let $\alpha = 3\alpha_3 + 3\alpha_6 \in \Delta_+^1$, i.e. $\mathbf{k} = (0, 0, 3, 0, 0, 3)$ then $\pi_{\mathbf{k}}^G(q) = \binom{q}{3}\binom{q-3}{3}$.

Bond Lattice

Fix a tuple $\mathbf{k} = (k_i)_{i \in I}$ such that $k_i \leq 1$ for $i \in I^{re} \cup \Psi_0$ and $|\operatorname{supp}(\mathbf{k})| < \infty$. Set $\eta(\mathbf{k}) = \sum k_i \alpha_i$.

Definition

Let $L_G(\mathbf{k})$ be the **weighted bond lattice** of G, which is the set of $\mathbf{J} = \{J_1, \ldots, J_\ell\}$ satisfying the following properties:

1. $\mathbf{J}$ is a multiset, i.e., $J_i = J_j$ for some $i \neq j$.
2. each J_i is a multiset, and the subgraph spanned by the underlying set of J_i is a connected subgraph of G for each $1 \leq i \leq \ell$.
3. For all $i \in I$, α_i occurs exactly k_i times in the total disjoint union $J_1 \sqcup \ldots \sqcup J_\ell$.

Isomorphism of bond lattices

For $\mathbf{J} \in L_G(\mathbf{k})$,

- $D(J_i, \mathbf{J})$ = multiplicity of J_i in $\mathbf{J}$,
- $\beta(J_i) = \sum_{\alpha \in J_i} \alpha$ and $\dim \mathfrak{L}_{\beta(J_i)} := \mathrm{mult}(\beta(J_i))$,
- $\mathbf{J}_0 = \{J_i \in \mathbf{J} : \beta(J_i) \in \Delta_+^0\}$ and $\mathbf{J}_1 = \mathbf{J} \setminus \mathbf{J}_0$.

Lemma ([2][AKV18])

Let $\mathcal{R}$ be the collection of multisets $\gamma = \{\beta_1, \ldots, \beta_r\}$ such that each $\beta_i \in \Delta_+$ and $\beta_1 + \ldots + \beta_r = \eta(\mathbf{k})$. The map $\phi : L_G(\mathbf{k}) \longrightarrow \mathcal{R}$ defined by

$$\{J_1, \ldots, J_\ell\} \mapsto \{\beta(J_1), \ldots, \beta(J_\ell)\}$$

is a bijection.

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[2] G. Arunkumar, and Deniz Kus and R. Venkatesh; Root multiplicities for Borcherds algebras and graph coloring. *J. Algebra*, 499: 538-569, 2018.

Relationship between Chromatic polynomials and root multiplicities

Relation between root multiplicities and Chromatic polynomials

Theorem ([3][VV15])

Let G be the simple graph of Kac Moody Lie algebra $\mathfrak{g}$. Assume $\mathbf{k} = (k_i)$; $k_i = 1$ for finitely many i and 0 otherwise; $\ell = |\operatorname{supp}(\mathbf{k})|$. Then

$$\chi(G, q) = \sum_{\mathbf{J} \in L_G} (-1)^{\ell - |\mathbf{J}|} \operatorname{mult}(\beta(\mathbf{J})) q^{|\mathbf{J}|}$$

where L_G is the bond lattice of weight $\mathbf{k}$ of the graph G.

[3] R. Venkatesh and S Viswanath; Chromatic polynomials of graphs from Kac Moody Lie algebras. *J. Algebraic Combin.*, 41(4): 1133-1142, 2015.

Relation between root multiplicities and generalized $\mathbf{k}$ Chromatic polynomials

Theorem ([4][AKV18])

Let G be the quasi Dynkin diagram of a Borcherds Lie algebra $\mathfrak{g}$.
Assume $\mathbf{k} = (k_i : i \in I) \in \mathbb{Z}_+^I$ is such that $k_i \leq 1$ for $i \in I^{re}$. Then

$$\pi_{\mathbf{k}}^{G}(q) = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))} \sum_{\mathbf{J} \in L_G(\mathbf{k})} (-1)^{|\mathbf{J}|} \prod_{J \in \mathbf{J}} \binom{q \ \mathrm{mult}(\beta(J))}{D(J, \mathbf{J})}$$

where $L_G(\mathbf{k})$ is the bond lattice of weight $\mathbf{k}$ of the graph G.

[4] G. Arunkumar, and Deniz Kus and R. Venkatesh; Root multiplicities for Borcherds algebras and graph coloring. *J. Algebra*, 499: 538-569, 2018.

Relation between root multiplicities and generalized $\mathbf{k}$ Chromatic polynomials

The following result is one of the main results from [5][RA21]

Theorem (_, G.Arunkumar)

Let G be the quasi Dynkin diagram of a BKM superalgebra $\mathfrak{L}$. Assume $\mathbf{k} = (k_i : i \in I) \in \mathbb{Z}_+^I$ is such that $k_i \leq 1$ for $i \in I^{re} \cup \Psi_0$. Then

$$\pi_{\mathbf{k}}^G(q) = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))} \sum_{\mathbf{J} \in L_G(\mathbf{k})} (-1)^{|\mathbf{J}|+|\mathbf{J}_1|} \prod_{J \in \mathbf{J}_0} \binom{q \text{ mult}(\beta(J))}{D(J, \mathbf{J})} \prod_{J \in \mathbf{J}_1} \binom{-q \text{ mult}(\beta(J))}{D(J, \mathbf{J})}$$

where $L_G(\mathbf{k})$ is the bond lattice of weight $\mathbf{k}$ of the graph G.

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[5] Shushma Rani and G. Arunkumar; A study on free roots of Borcherds-Kac-Moody Lie Superalgebras. *arXiv e-prints*, arXiv:2103.12332, March 2021.

Corollary

Let $\eta(\mathbf{k}) = \sum_{i \in I} k_i \alpha_i \in \Delta^+$ *such that* $k_i \leq 1$ *for all* $i \in I^{re} \sqcup \Psi_0$*. Then*

$$\mathrm{mult}(\eta(\mathbf{k})) = \begin{cases} \sum\limits_{\ell | \mathbf{k}} \frac{\mu(\ell)}{\ell} \, |\pi^G_{\mathbf{k}/\ell}(q)[q]|, & \textit{if } \eta(\mathbf{k}) \in \Delta^0_+ \\ \sum\limits_{\ell | \mathbf{k}} \frac{(-1)^{\ell+1}\mu(\ell)}{\ell} \, |\pi^G_{\mathbf{k}/\ell}(q)[q]|, & \textit{if } \eta(\mathbf{k}) \in \Delta^1_+ \end{cases}$$

where $|\pi^G_{\mathbf{k}}(q)[q]|$ *denotes the absolute value of the coefficient of* q *in* $\pi^G_{\mathbf{k}}(q)$ *and* μ *is the Möbius function. If* k_i*'s are relatively prime (in particular if for some* $i \in I$, $k_i = 1$*), we have,*

$$\mathrm{mult}(\eta(\mathbf{k})) = |\pi^G_{\mathbf{k}}(q)[q]| \quad \textit{for any } \eta(\mathbf{k}) \in \Delta_+.$$

Example of main result

Example

Consider the BKM superalgebra $\mathfrak{L}$ and the root space $\eta(\mathbf{k}) = 3\alpha_3 + 3\alpha_6 \in \Delta_+^1$ from Example 14. The $\mathbf{k}$-chromatic polynomial of the quasi Dynkin diagram G of $\mathfrak{L}$ is equal to

$$\pi_{\mathbf{k}}^G(q) = \binom{q}{3}\binom{q-3}{3} = \frac{1}{3!3!} q(q-1)(q-2)(q-3)(q-4)(q-5).$$

$$\begin{aligned}
\operatorname{mult}(\eta(\mathbf{k})) &= \sum_{\ell | \mathbf{k}} \frac{(-1)^{\ell+1}\mu(\ell)}{\ell} \, |\pi_{\mathbf{k}/\ell}^G(q)[q]| \\
&= |\pi_{\mathbf{k}}^G(q)[q]| + \frac{\mu(3)}{3} |\pi_{\mathbf{k}'}^G(q)[q]| \text{ where } \mathbf{k}' = (0,0,1,0,0,1) \\
&= \frac{10}{3} - \frac{1}{3} = 3
\end{aligned}$$

G. Arunkumar, Deniz Kus, and R. Venkatesh.
Root multiplicities for Borcherds algebras and graph coloring.
J. Algebra, 499:538–569, 2018.

Shushma Rani and G. Arunkumar.
A study on free roots of Borcherds-Kac-Moody Lie Superalgebras.
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R. Venkatesh and Sankaran Viswanath.
Chromatic polynomials of graphs from Kac-Moody algebras.
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Minoru Wakimoto.
Infinite-dimensional Lie algebras, volume 195 of *Translations of Mathematical Monographs*.
American Mathematical Society, Providence, RI, 2001.
Translated from the 1999 Japanese original by Kenji Iohara, Iwanami Series in Modern Mathematics.

Thank You!