CMB and GW constraints on Primordial Black Holes

Lecture II: GW constraints

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Less traveled path of dark matter, ICTS, Nov 11, 2020

Plan of this lecture

1- Basics of gravitational waves; the observational landscape

2- PBHs as source of GWs and resulting constraints

A- 2nd-order GWs from scalar perturbations

B- Formation of PBH binaries in the late Universe

C- Formation of PBH binaries in the early Universe

1- Basics of gravitational waves

space-time metric: $$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \qquad |h_{\mu\nu}| \ll 1$$

$h_{\mu\nu}$ has 10 indep. components. 6 linear combinations are gauge-invariant:

- 2 scalar potentials Φ, Ψ [generalization of Newtonian potential]
- 1 transverse ``vector" [2 independent components], and
- 1 transverse trace-free "tensor" [2 independent components] h_{ij}^{TT}

$$\sum_i h_{ii}^{\mathrm{TT}} = 0 = \nabla_i h_{ij}^{\mathrm{TT}}$$

h_{ij}^{TT} is the **gravitational-wave strain**.

It satisfies a wave equation. Propagates at speed of light.

- Quadrupole formula. Consider a (non-relativistic) matter source with a time-varying mass quadrupole moment $Q_{ij}(t)$ [$\sim M R^2$].

$$h_{ij}^{\rm TT}(t,r) = \frac{2}{r}\left[\ddot{Q}_{ij}(t-r)\right]^{\rm TT}$$

r: distance to the source

- Gravitational waves carry energy and momentum:

Energy density: $$\rho^{\rm GW} = \frac{1}{32\pi}\langle \dot{h}_{ij}^{\rm TT}\dot{h}_{ij}^{\rm TT}\rangle$$

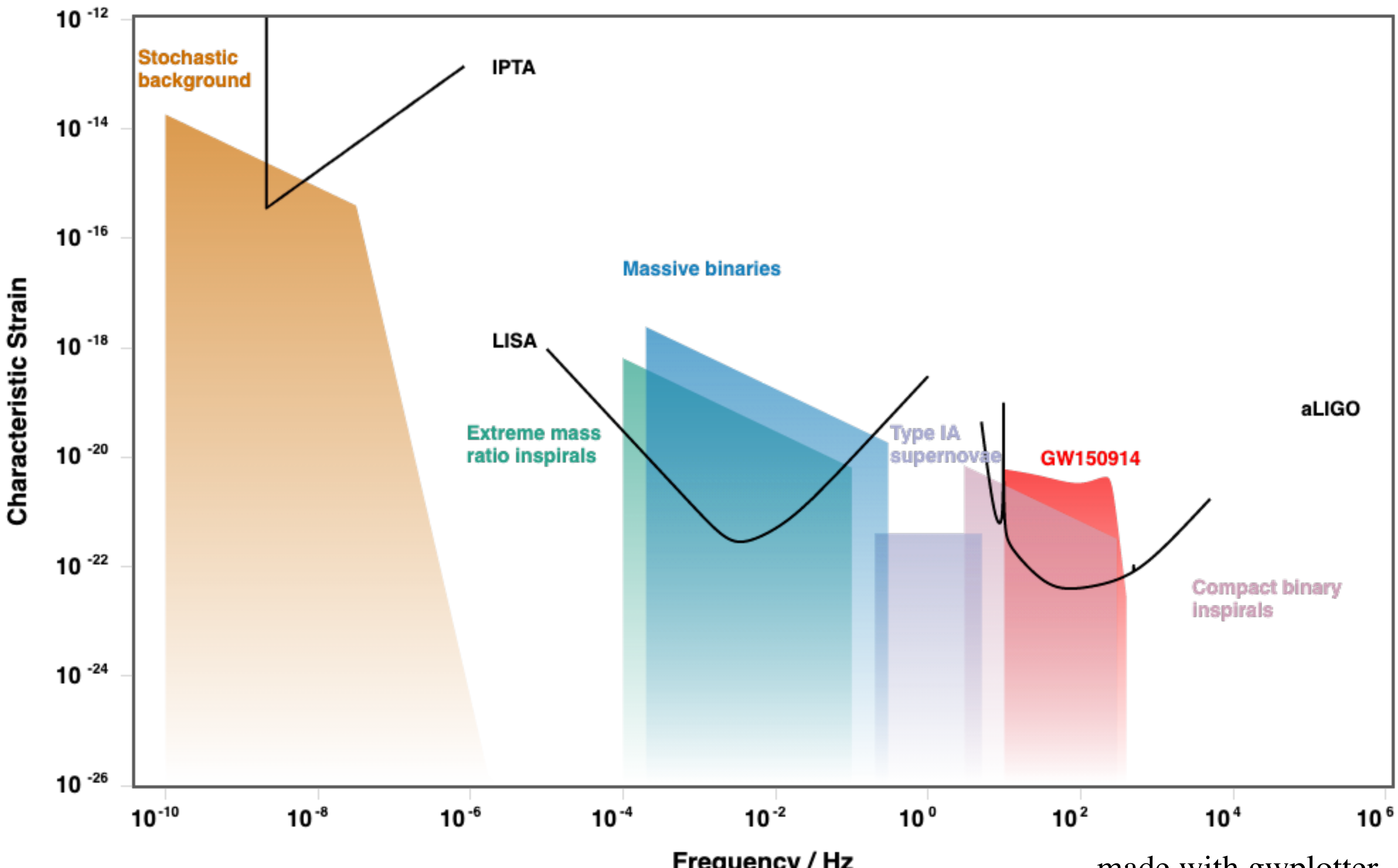
$\langle\ldots\rangle$ = average over several wavelengths

Energy flux/ momentum density: $$P_a^{\rm GW} = \frac{1}{32\pi}\langle \dot{h}_{ij}^{\rm TT}\nabla_a h_{ij}^{\rm TT}\rangle$$

➡ GW-power radiated by a time-varying quadrupole moment:

$$\left.\frac{dE}{dt}\right|_{\rm GW} = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle$$

The gravitational-wave landscape



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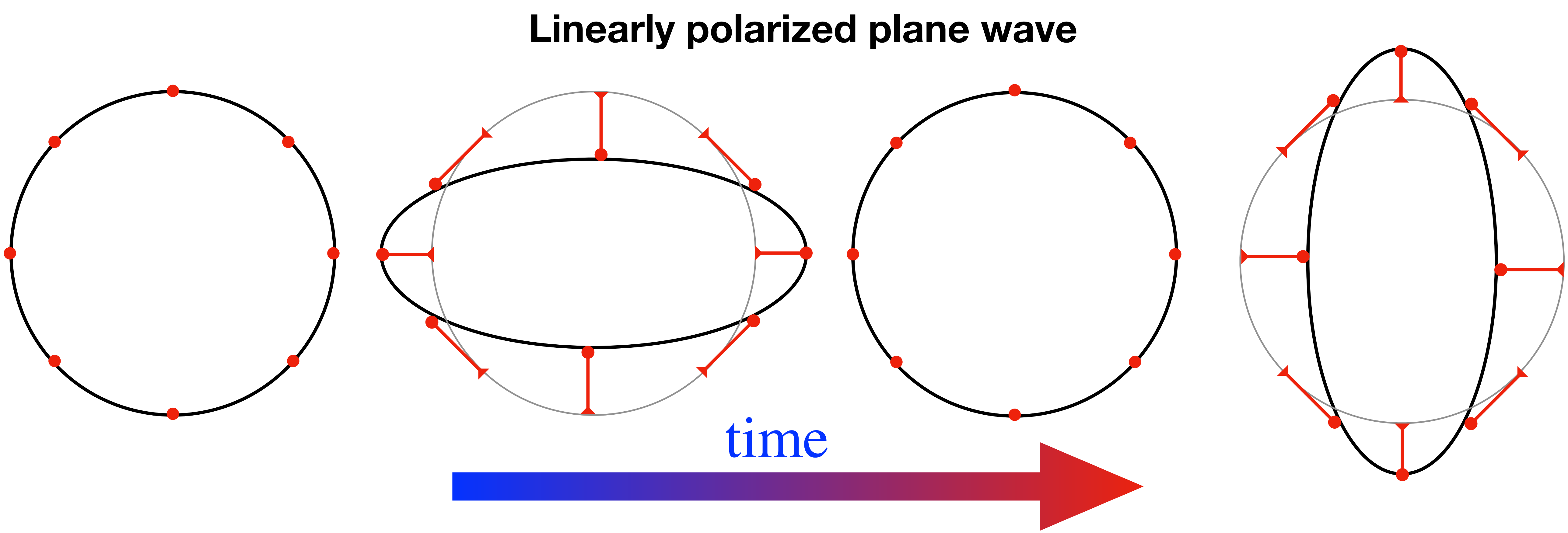
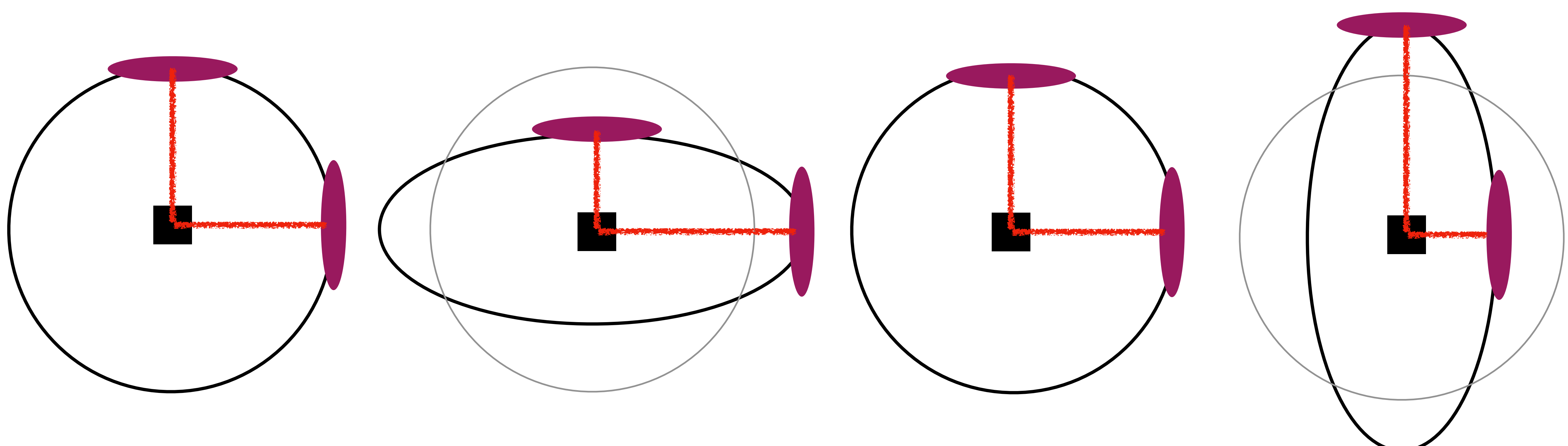
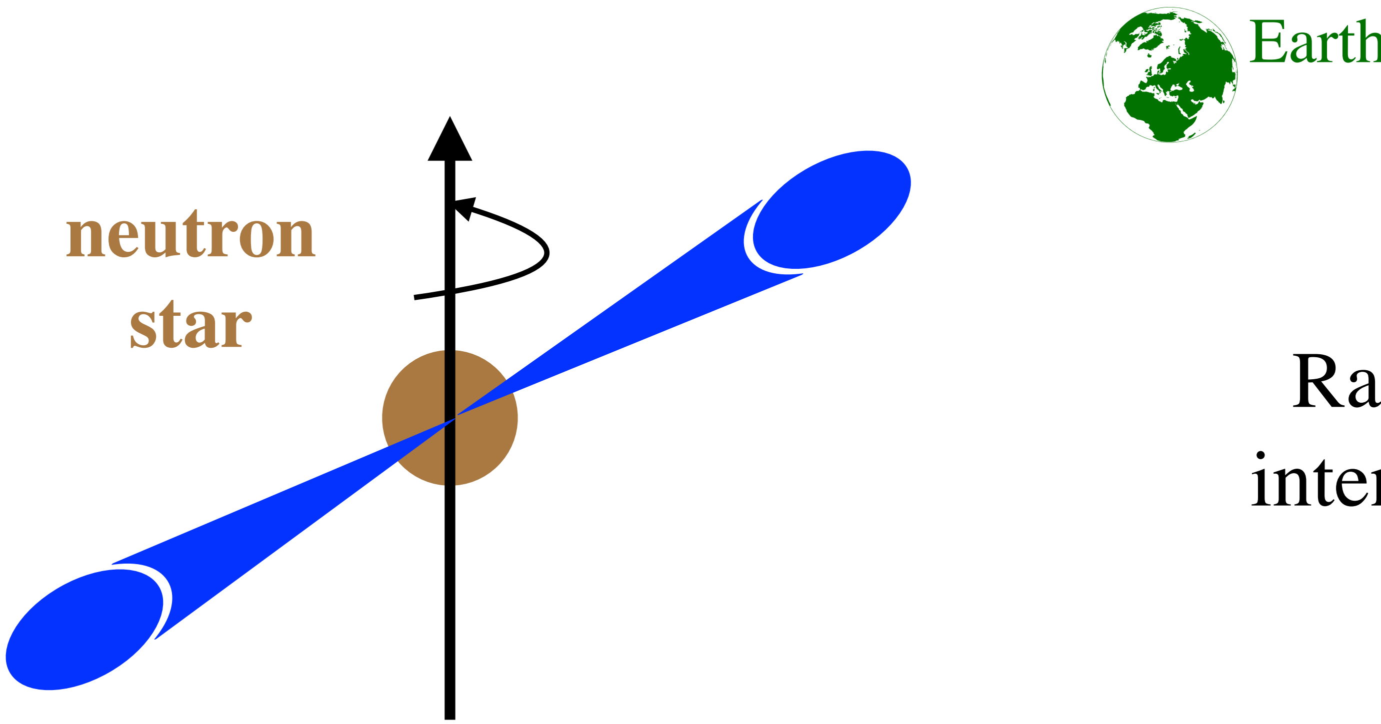
Geodesic deviation of test masses: $\Delta x_i(t) \approx \Delta x_i(0) + \frac{1}{2} h_{ij}(t) \Delta x_j(0)$

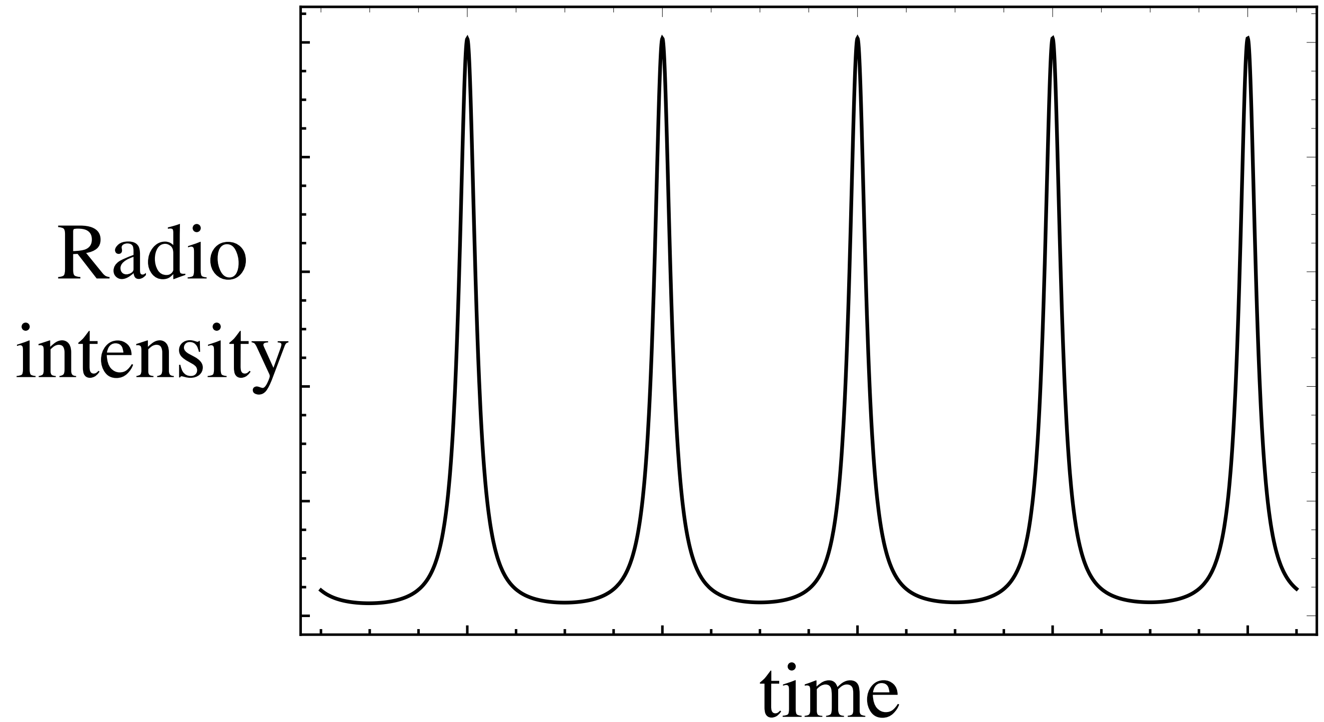


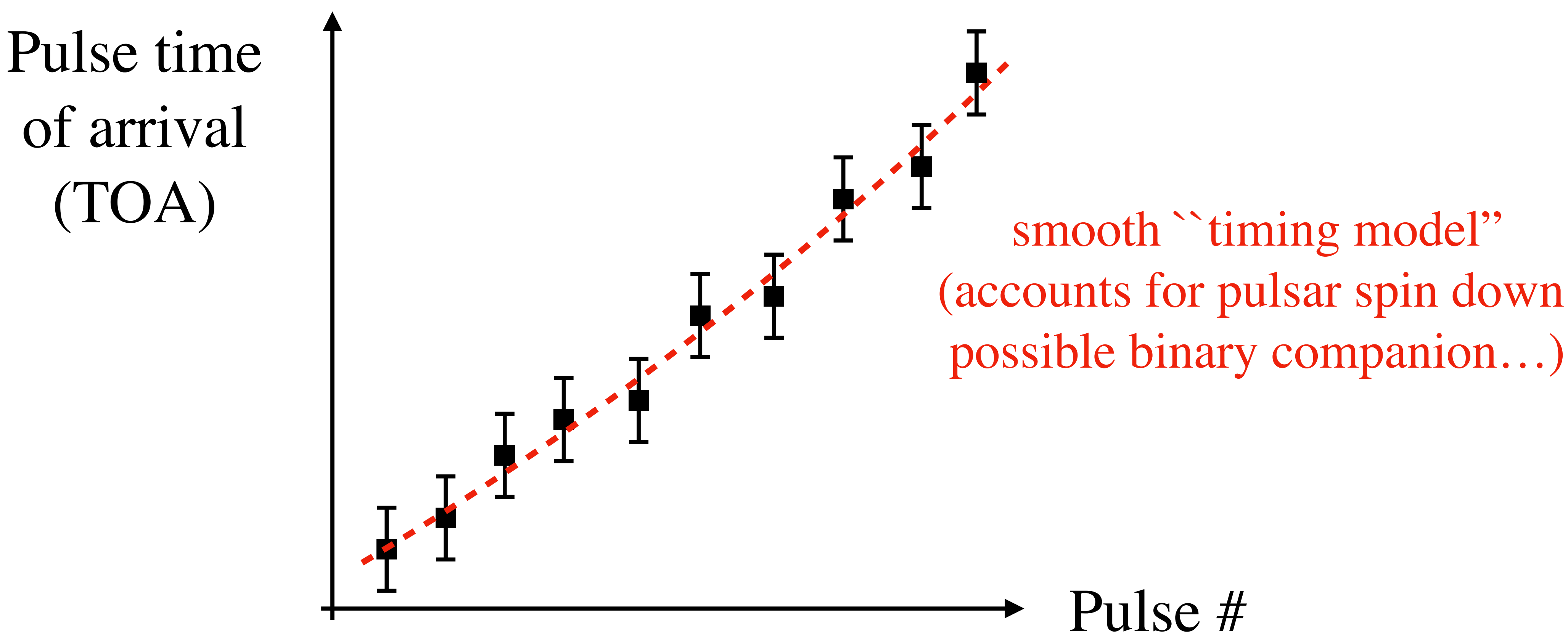
This is (heuristically) how LIGO (and LISA) work



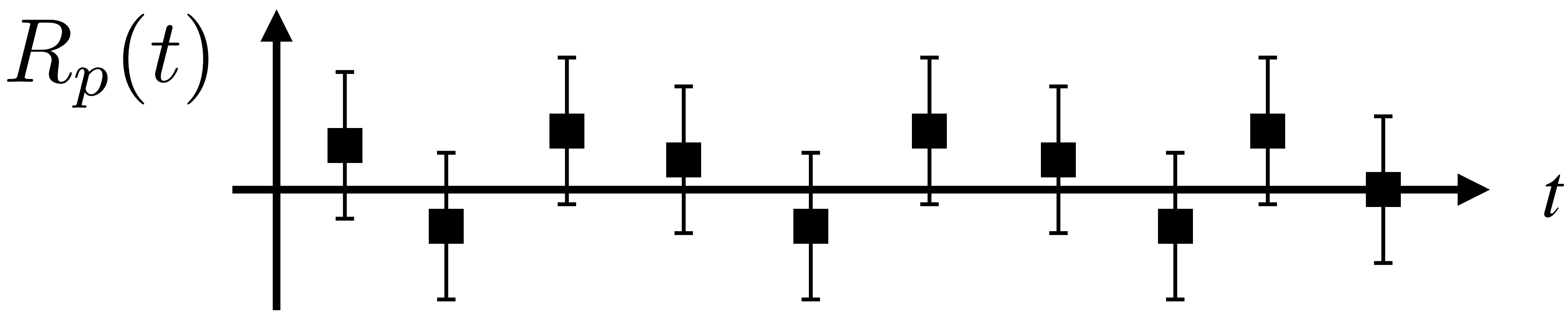
Basics of pulsar timing arrays



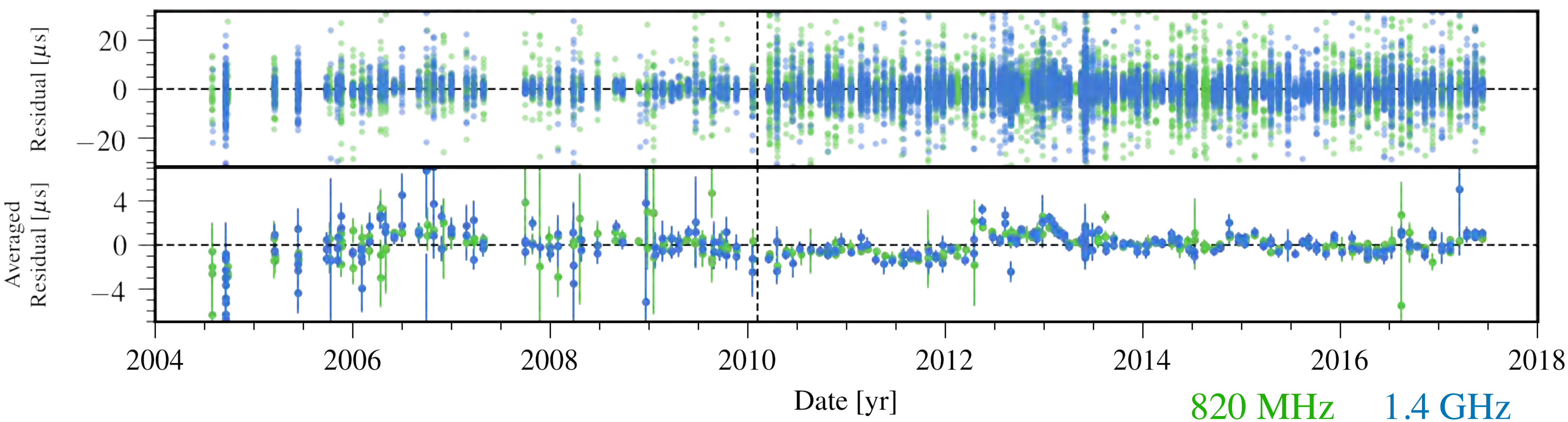




For each pulsar p: Time residual $R_p(t)$ = TOA - timing model(t)



example: J1012+5307 (NANOGrav 12.5-year data)



For each pulsar p:

$$R_p(t) = N_p(t) + R_p^{\mathrm{GW}}(t)$$

unknown intrinsic pulsar noise **GW-induced timing residual**

$$R_p^{\mathrm{GW}}(t) = \frac{1}{2}\hat{p}^a\hat{p}^b \int_{t-D_p}^{t} dt' \; h_{ab}(t', (t-t')\hat{p})$$

$\hat{p}$ = direction of pulsar p

In Fourier space:

$$R_p^{\mathrm{GW}}(f) = \frac{\hat{p}^a\hat{p}^b}{4\pi i f} \int d^2\hat{\Omega} \; \frac{h_{ab}(f,\hat{\Omega})}{(1+\hat{\Omega}\cdot\hat{p})}$$

$\hat{\Omega}$ = direction of GW propagation

The pulsar intrinsic noise is completely unknown (one cannot isolate pulsars in the lab!), but is uncorrelated between pulsars:

$$\langle N_p N_q \rangle = ??? \times \delta_{pq}$$

The only way to detect GWs through pulsar timing is through cross-correlations of different pulsars.

If GWs form a **stochastic background**, with isotropic energy flux:

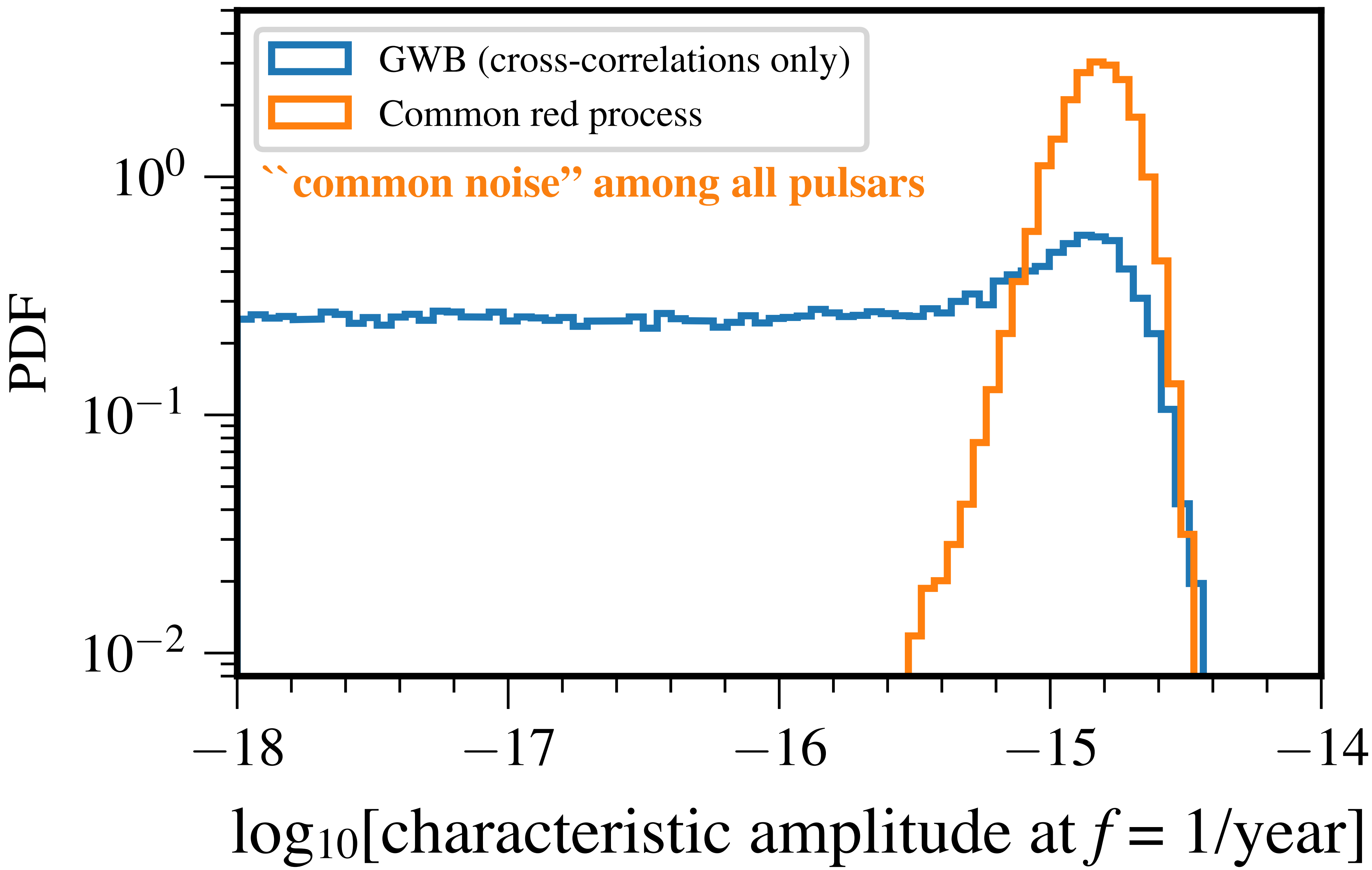
$$\langle R_p^{\rm GW}(f) R_q^{*\rm GW}(f)\rangle \propto h_c^2(f)\mathcal{H}(\theta_{pq})$$

characteristic GW strain

Hellings & Downs function of θ_{pq} = angle between p and q

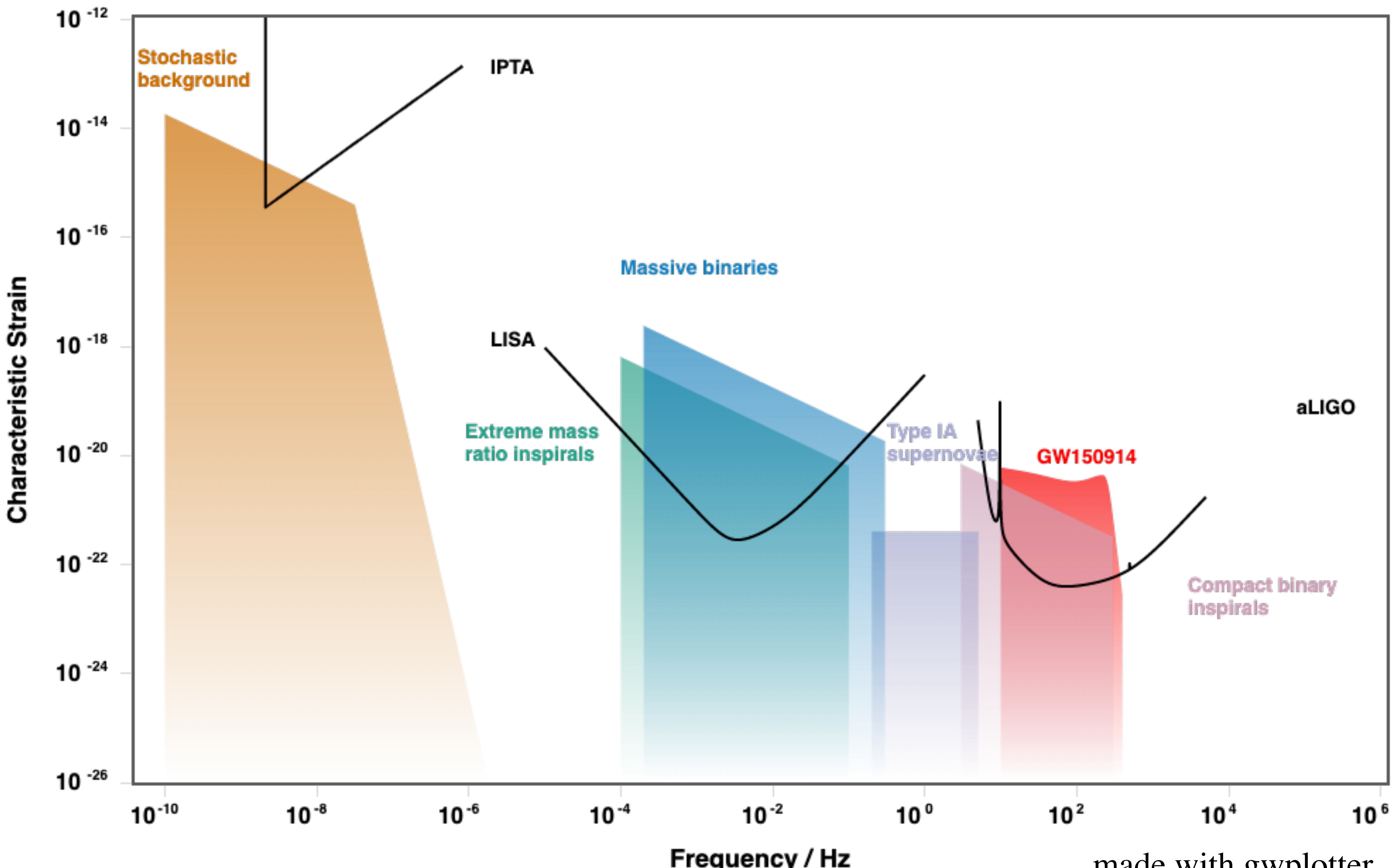
GWs also contribute to timing-residual auto-correlations, but these are completely degenerate with the unknown intrinsic noise of pulsars => auto-correlations can only be used to set upper limits.

NANOGrav 12.5-year results



Upper limit: $h_c(f = 1/\mathrm{yr}) \lesssim 3\text{e-}15$

Questions?



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2-A- GWs induced at second-order by small-scale density perturbations

Consider metric perturbation $$h_{\mu\nu} = \epsilon \; h^{(1)}_{\mu\nu} + \epsilon^2 \; h^{(2)}_{\mu\nu}$$

ε: bookeeping parameter

Suppose that the first-order perturbations are pure scalars:

$$h^{\mathrm{TT}(1)}_{ij} = 0$$

Expand the (non-linear) Einstein field equations to second-order

$$\frac{d^2 h^{\mathrm{TT}(2)}_{ij}}{d\eta^2} - \nabla^2 h^{\mathrm{TT}(2)}_{ij} + 2aH \frac{dh^{\mathrm{TT}(2)}_{ij}}{d\eta} = \mathcal{S}_{ij}$$

wave equation

effect of cosmological expansion

$$S_{ij} \sim \nabla_i \Phi^{(1)} \nabla_j \Phi^{(1)}$$

A *very* simple estimate

- Just after horizon entry $h_c \sim \Phi^2 \sim \zeta^2$
- GWs have energy density $\rho_{\rm GW} \propto (\dot{h}_c)^2 \sim \omega^2 h_c^2$
- GW frequencies decay as $\omega \propto 1/a \quad \Rightarrow \quad \rho_{\rm GW} \propto a^{-2} h_c^2$
- GWs behaves like radiation in the absence of sources =>

$$\rho_{\rm GW} \propto a^{-4} \quad \Rightarrow \quad h_c \propto 1/a$$

$$\Rightarrow \quad h_c(k;a) \sim \zeta^2(k)(a_{\rm entry}/a)$$

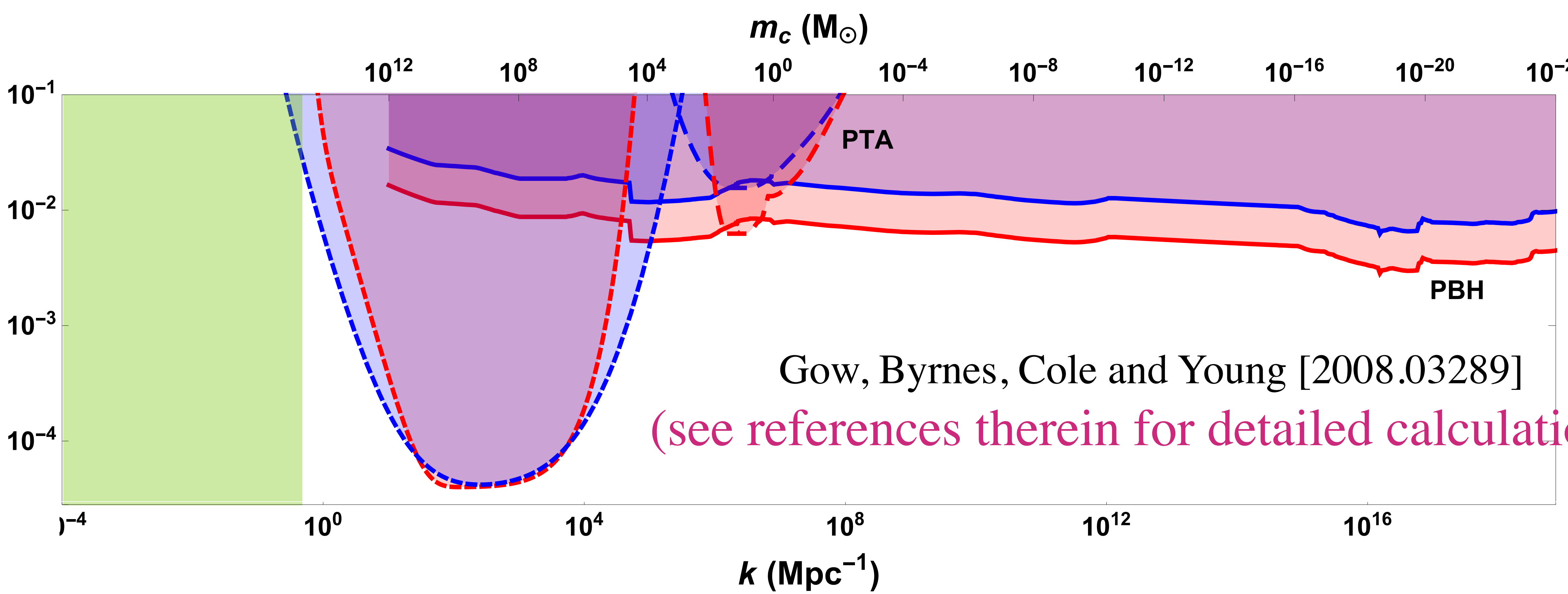
- For horizon entry (when $k = a\,H$) during radiation domination:

$$a_{\rm entry} \sim 10^{-3} \frac{0.01\ {\rm Mpc}^{-1}}{k} \sim 10^{-11} \frac{{\rm pc}^{-1}}{k}$$

$$h_c(k, a=1) \sim 10^{-11} \frac{\mathrm{pc}^{-1}}{k} \zeta^2(k)$$

PTAs constrain $h_c \lesssim$ 3e-15 at frequencies of ~1/yr, i.e. k~1/pc.

$$\Rightarrow \zeta(k \sim \mathrm{pc}^{-1}) \lesssim 10^{-2}$$



Gow, Byrnes, Cole and Young [2008.03289]

(see references therein for detailed calculations)

=> Currently PTA limits are *just consistent* with primordial fluctuations required to form PBHs. Future extended PTAs (e.g. built from SKA pulsars) should probe much lower amplitudes

2-B- Formation of PBH binaries in the late Universe (Bird et al. 2016)

Suppose 2 (equal) masses M approach each other on a nearly parabolic trajectory, with impact parameter b, relative velocity v.

At periapsis $r_p v_p = bv, \quad v_p^2 \approx \frac{M}{r_p} \quad \Rightarrow r_p \approx \frac{(bv)^2}{M}, \quad v_p \approx \frac{M}{bv}$

Timescale of passage at periapsis: $t_p \sim \frac{r_p}{v_p} \sim \frac{(bv)^3}{M^2}$

Mass quadrupole moment: $Q \sim M r_p^2$

Energy lost to GW radiation: $\Delta E^{\mathrm{GW}} \sim \dddot{Q}^2 t_p \sim \frac{M^8}{(bv)^7}$

=> masses become bound if $\Delta E^{\mathrm{GW}} \gtrsim M v^2 \quad \text{i.e.} \quad b \lesssim \frac{M}{v^{9/7}}$

=> Cross-section for capture through GW radiation:

$$\sigma(v) = \pi b_{\max}^2 \sim M^2 v^{-18/7}$$

Quinlan & Shapiro 1989

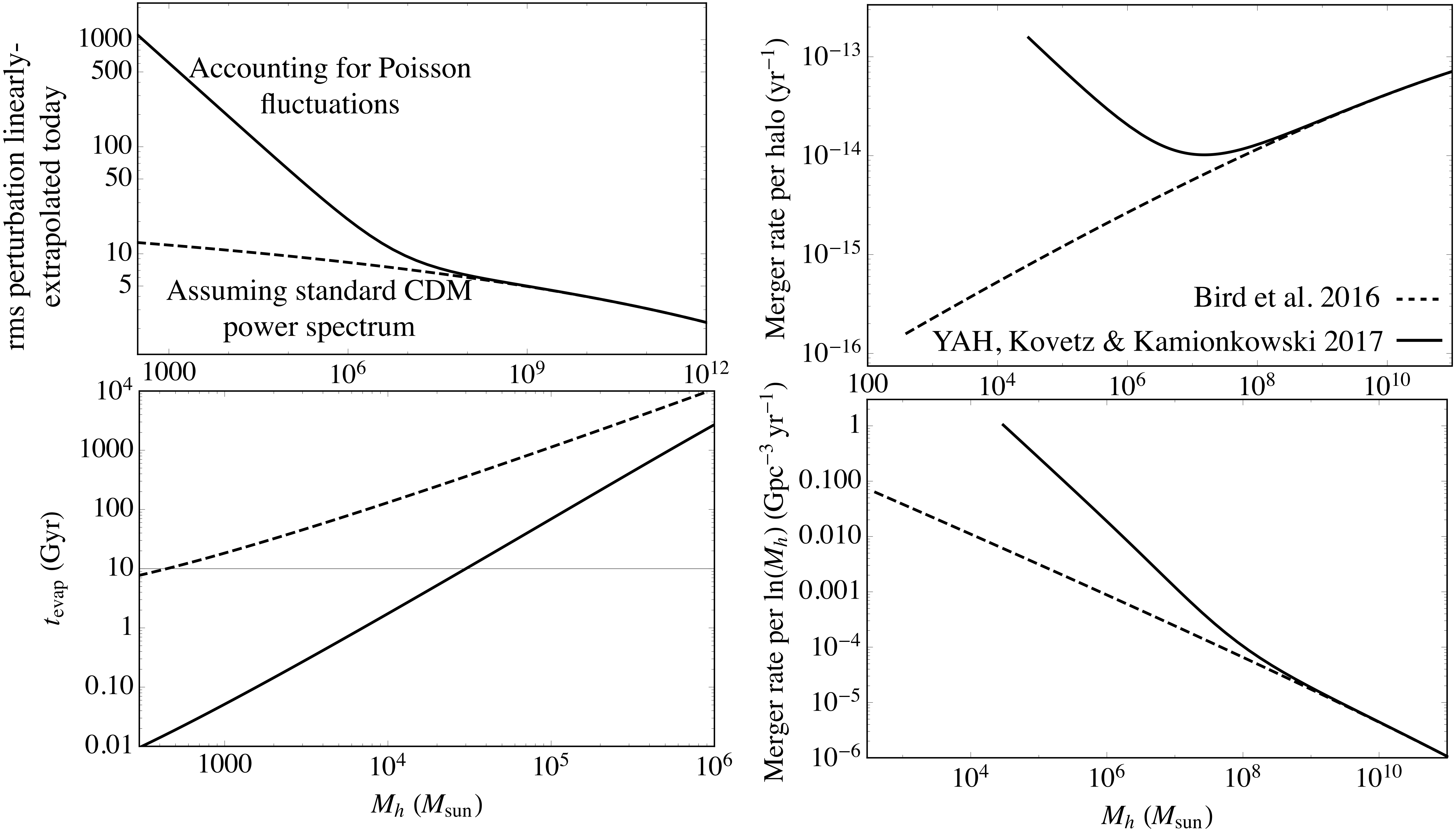
- Merger rate per halo of mass M_h, density ρ_h

$$\Gamma = \frac{1}{2}\int dV \langle \sigma v\rangle \left(\frac{\rho}{M}\right)^2 \sim \frac{GM_h}{c^2}\frac{G\rho_h}{c}(v_h/c)^{-11/7}$$

- Total merger rate per unit volume: $\mathcal{R} = \int dM_h \frac{dn_h}{dM_h}\Gamma(M_h)$

Integral diverges at M_h —> 0, but small halos ``evaporate'' within a Hubble time through 2-body relaxation (e.g. Binney & Tremaine)

- $\rho_h \sim 200 \times$ (mean density at redshift of halo collapse),
- approximate dn_h/dM_h by Press-Schechter mass function



In either case, one find, for ~30 M_{sun} PBHs making all of the DM

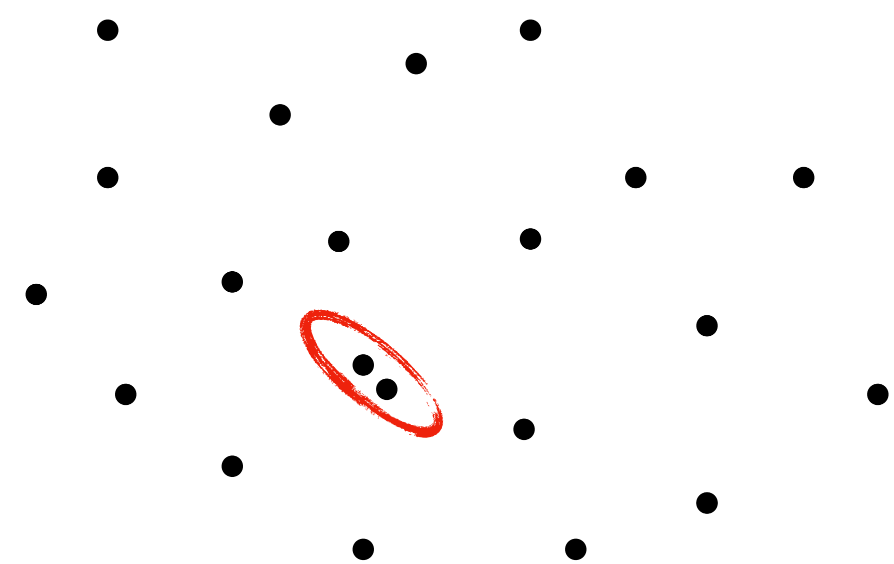
$$\mathcal{R} \sim 1\ \mathrm{Gpc}^{-3}\mathrm{yr}^{-1}$$

Somewhat lower than (but consistent with) merger rate inferred from LIGO

2-C- Formation of PBH binaries in the early Universe (Nakamura et al. 1997)

- On small scales PBHs are initially Poisson distributed [Ali-Haïmoud 2018, Ballesteros et al. 2018, Desjacques & Riotto 2018]

- PBHs pairs born close enough decouple from Hubble flow deep in the radiation era.



A: semi-major axis at decoupling

Scale factor of decoupling $a_{\rm dec}$ is such that

$$M \sim \frac{4\pi}{3}\rho_\gamma(a_{\rm dec})A^3 \qquad A = a_{\rm dec}X_{\rm com}$$

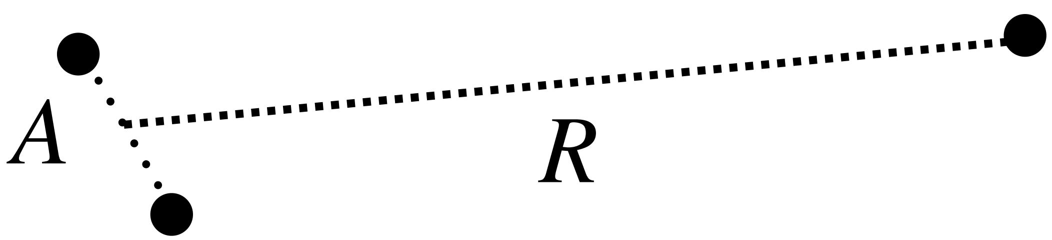
$$\Rightarrow A \sim \frac{4\pi\rho_\gamma^0}{3M}\,X_{\rm com}^4 \qquad P(X_{\rm com})$$

given by Poisson distribution of nearest neighbor

=> can compute the PDF of initial semi-major axis

- Once a pair decouples from the Hubble flow, it falls *almost head-on*

- It acquires some non-zero angular momentum dues to torquing by neighboring PBHs and large-scale tidal field



$$\frac{d\ell}{dt} \sim A \frac{M}{R^2} \Rightarrow \ell \sim A \frac{M}{R^2} t_{\rm dec}$$

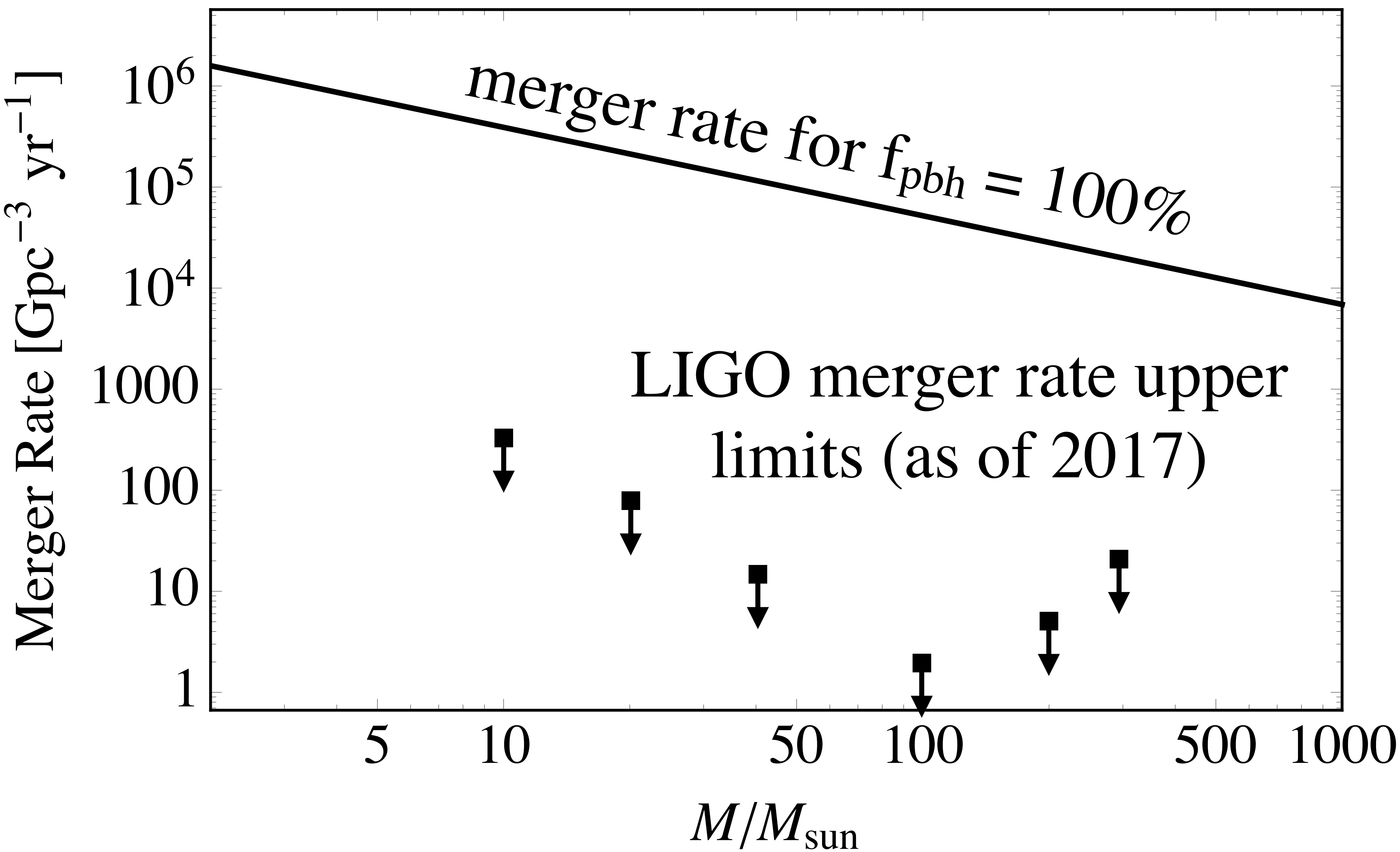
$$R \sim a_{\rm dec} X^{\rm 2nd}_{\rm com}$$

- Given Poisson distribution for $X^{\rm 2nd}$, can compute the PDF of initial angular momentum or equivalently initial eccentricity.

$$t_{\rm merge} = \frac{3}{170} \frac{A^4}{M^3} j^7, \quad j \equiv \sqrt{1 - e^2}$$ Peters 1964

- Given $P(A)$ and $P(e)$ can compute $P(t_{\rm merge})$ thus merger rate.

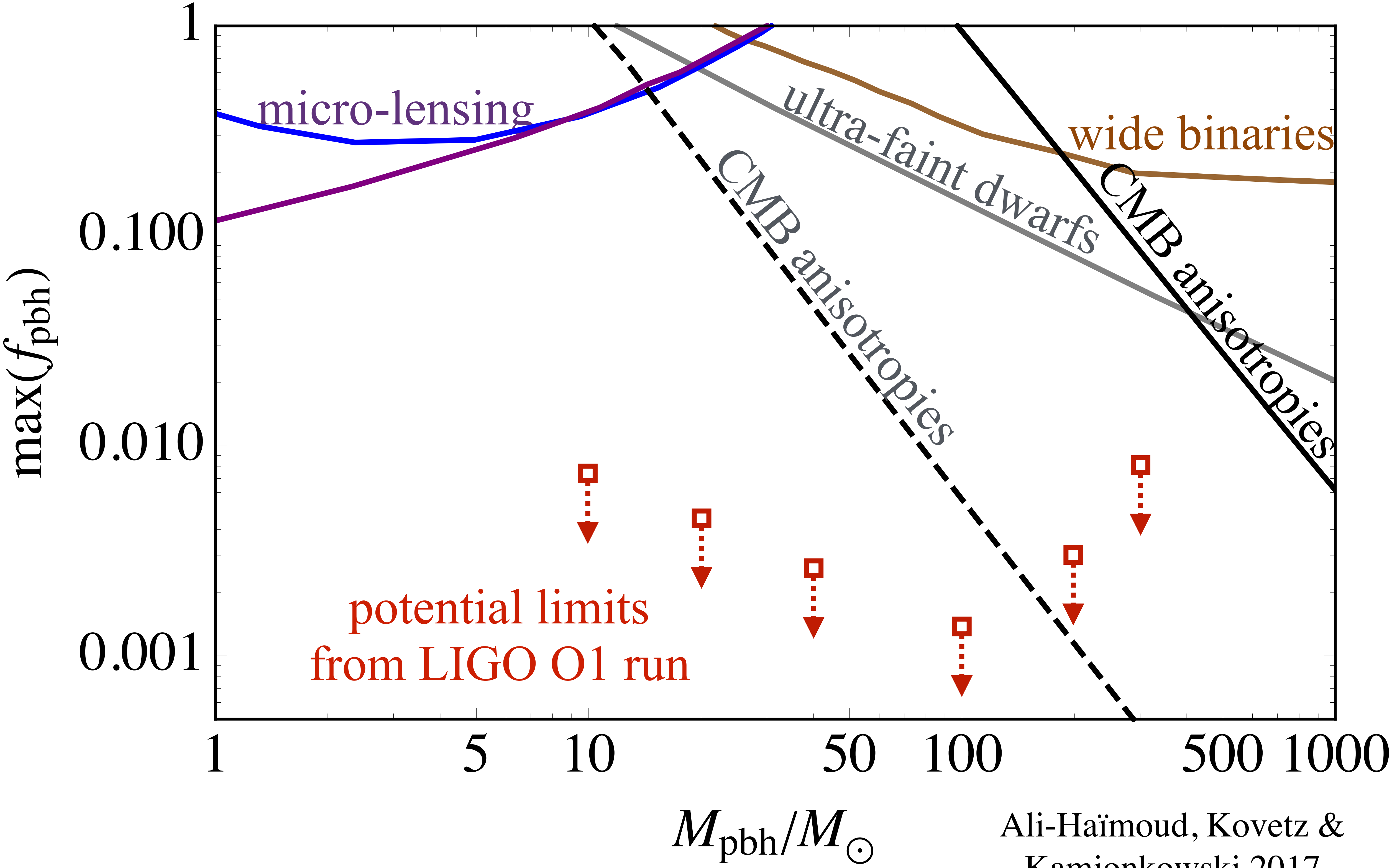
If PBH binaries are undisturbed between formation and merger (in particular if they do not get significantly torqued), merger rate today >> LIGO bounds if PBHs make all of the DM.



Ali-Haïmoud, Kovetz & Kamionkowski 2017

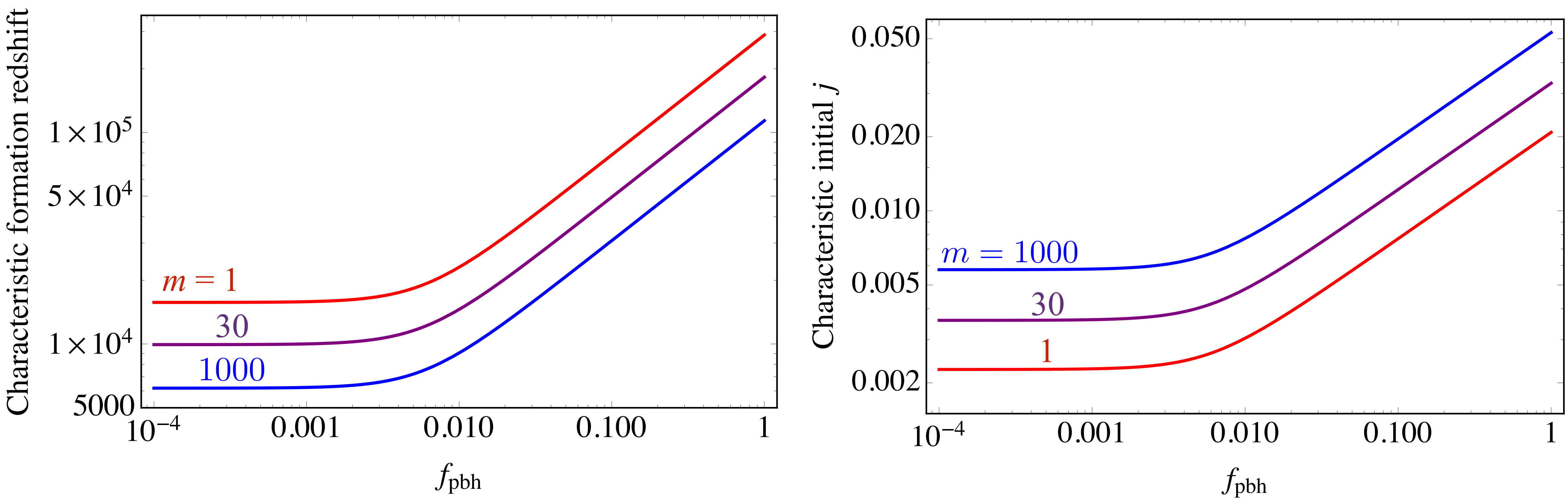
If PBH binaries are undisturbed between formation and merger,

then LIGO sets very stringent constraints on PBH abundance



Ali-Haïmoud, Kovetz & Kamionkowski 2017

PBH binaries typically start with a very small angular momentum.



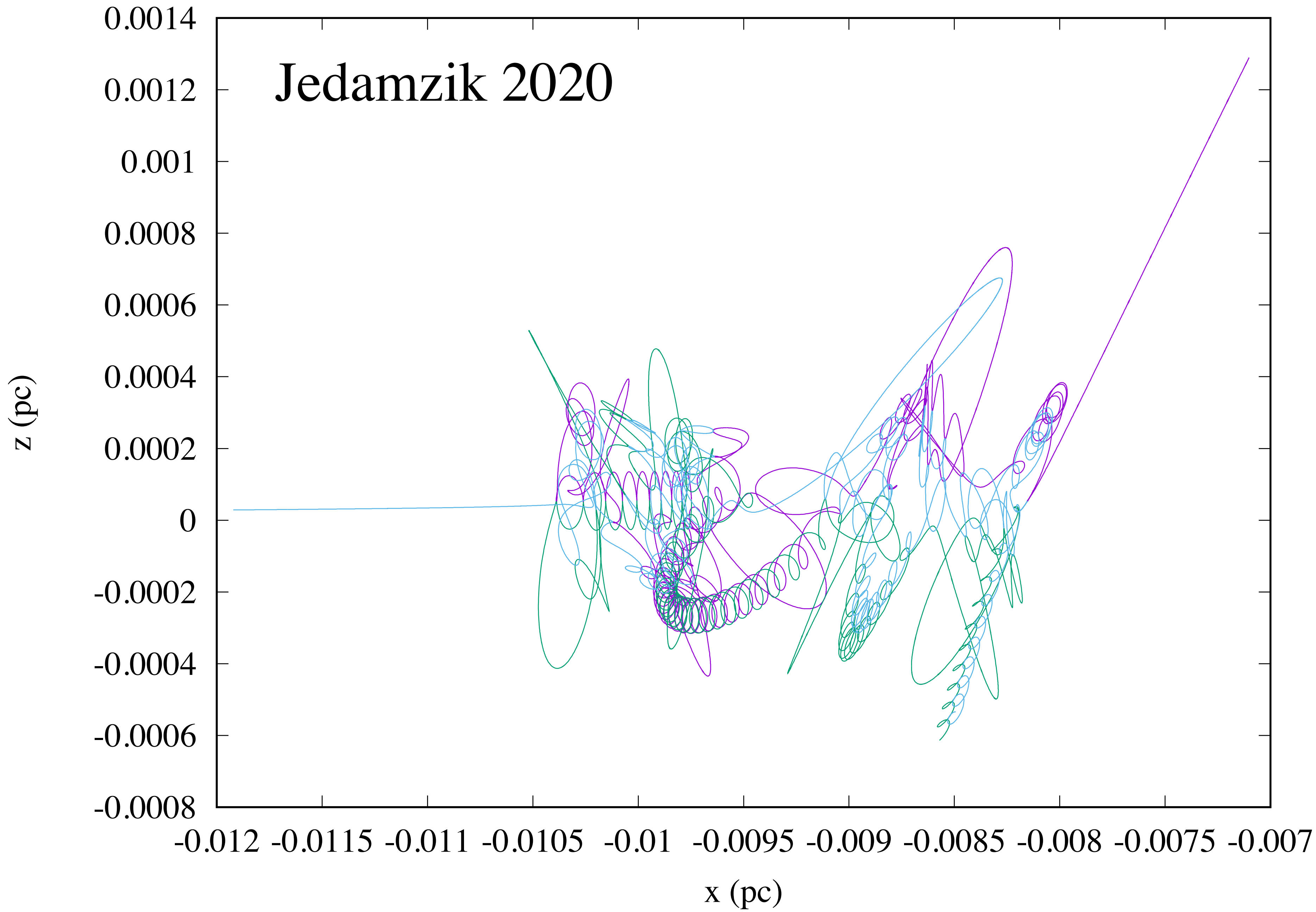
Since $t_{\rm merge} \sim j^7$, even small torques could change j by factors of a few, and $t_{\rm merge}$ (hence merger rate) by a lot!

In YAH, Kovetz & Kamionkowski 2017, we estimated analytically that PBH binaries should not be significantly torqued during subsequent structure formation.

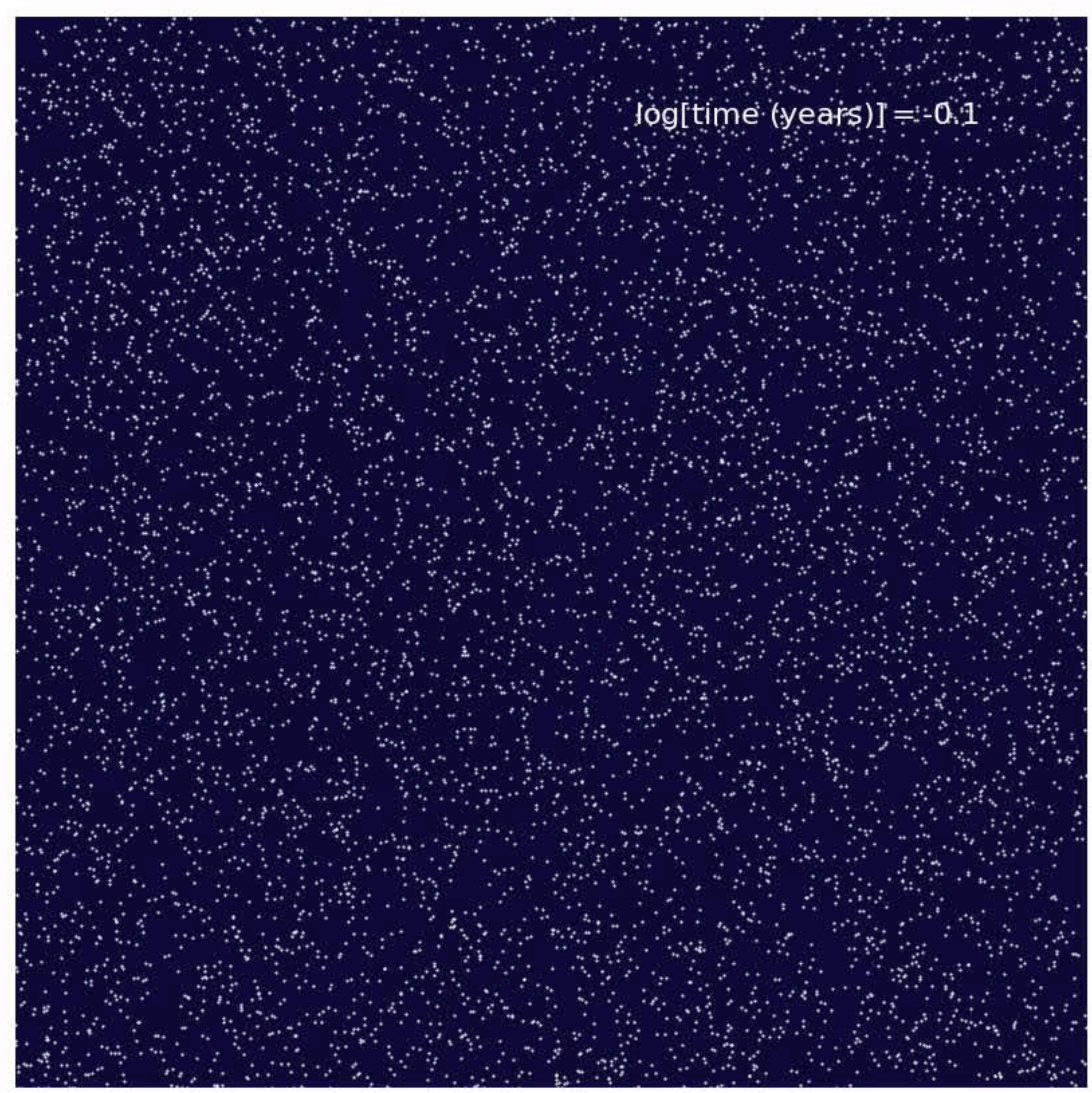
To be confirmed by simulations!

Two aspects to this problem

1- the ``microphysics”: what are the cross sections for different outcomes for 3-body interactions ?



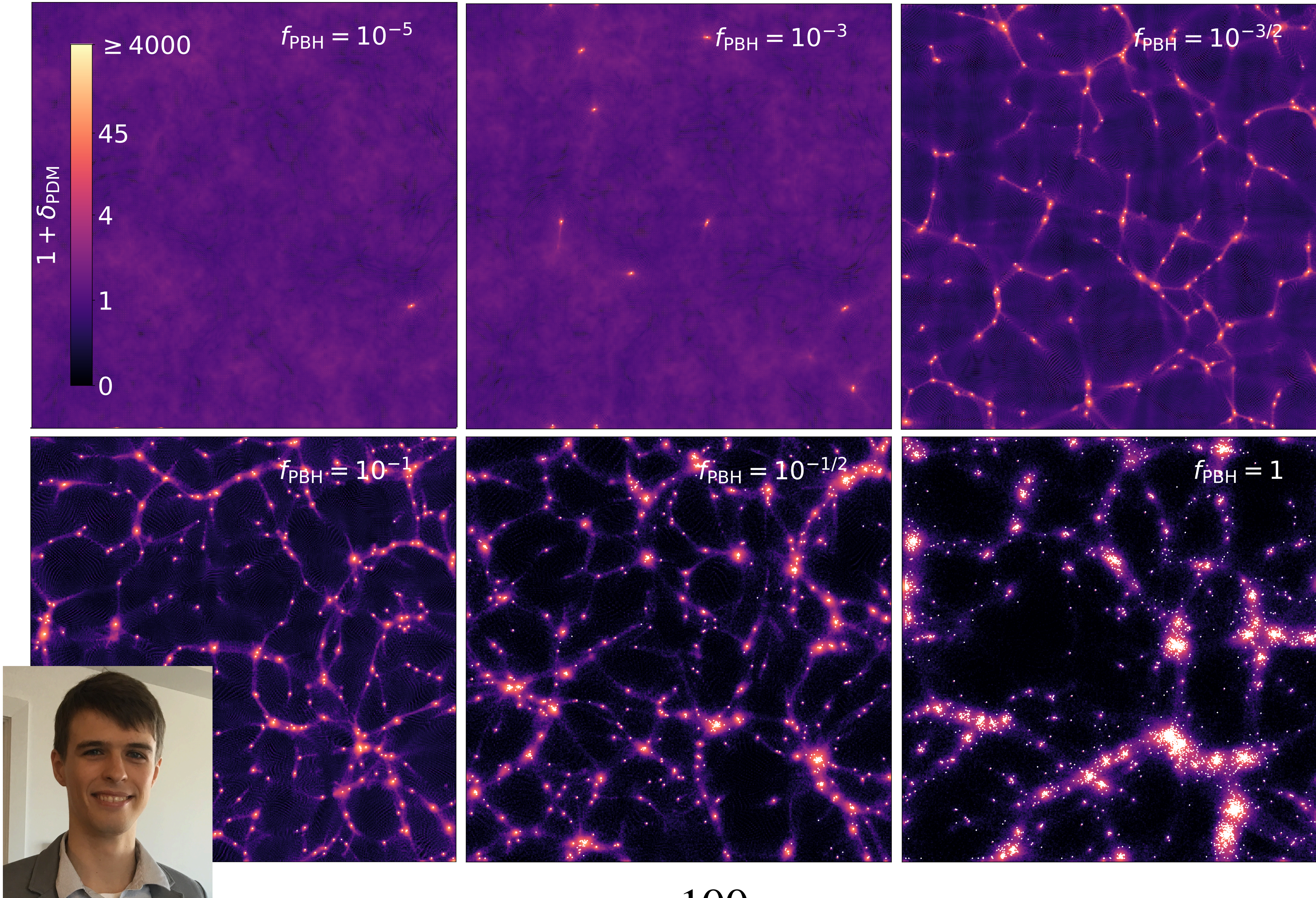
2- the ``macrophysics”: what are the properties of the first structures forming in a CDM + PBH universe?



$(30\ \mathrm{kpc}/h)^3$ box

Derek Inman

Inman & Ali-Haïmoud 2019



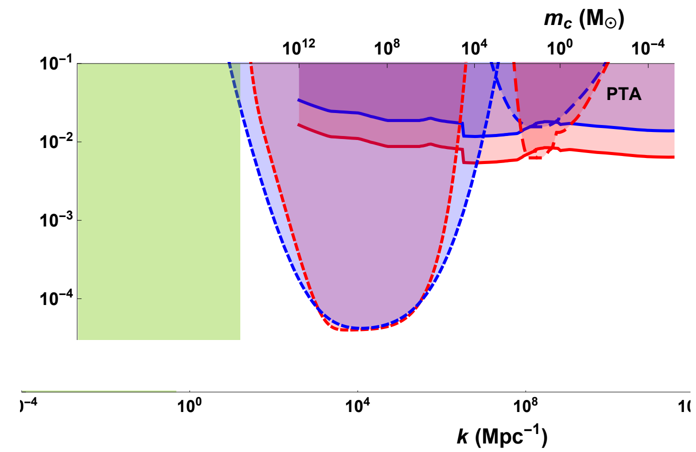
Derek Inman

$z = 100$

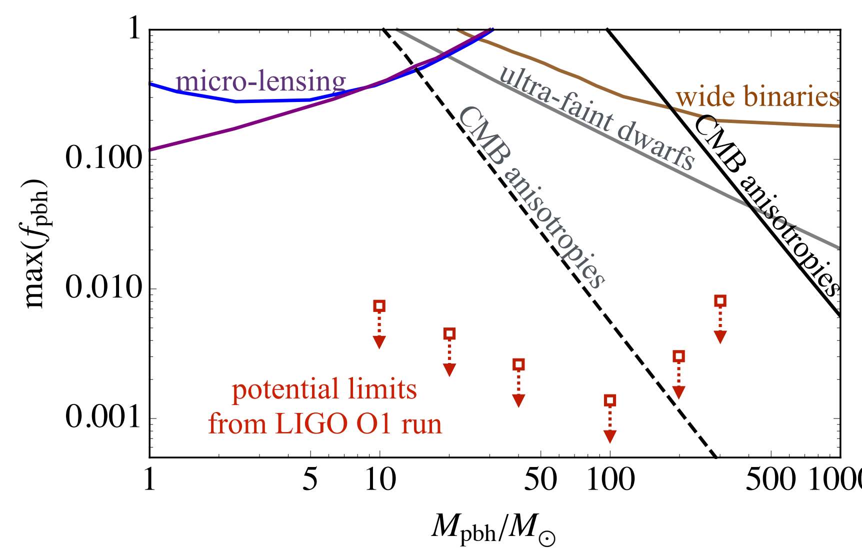
Inman & Ali-Haïmoud 2019

Summary of lecture 2

- PTAs indirectly probe the primordial power spectrum. For now, PTA limits do not put pressure on PBHs, but this could change with next-generation PTAs.



- LIGO could potentially place very stringent limits on the PBH abundance between ~1 and ~10^3 solar masses.



- Depends on whether early-Universe PBH binaries are disturbed during formation of the first structures. To be checked numerically.