# Fourier analysis and number theory

Emanuel Carneiro

ICTP - International Centre for Theoretical Physics
Trieste, Italy

Modern Trends in Harmonic Analysis, July 2023

# Prelude

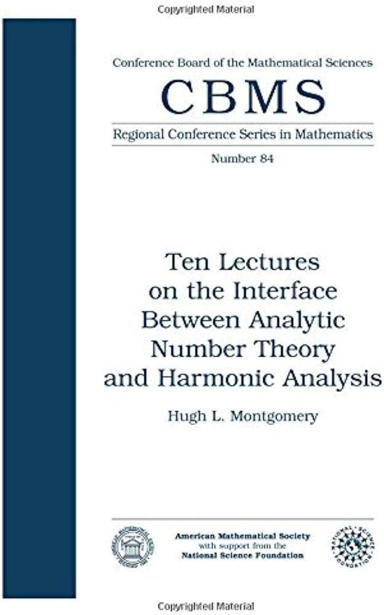
- Harmonic analysis is concerned with understanding oscillation.

# Prelude

- Harmonic analysis is concerned with understanding oscillation.
- Relations to many other fields such as: approximation theory, number theory, complex analysis, probability, PDEs, ...

# Prelude

- Harmonic analysis is concerned with understanding oscillation.
- Relations to many other fields such as: approximation theory, number theory, complex analysis, probability, PDEs, ...
- I am particularly interested in studying extremal phenomena.


Copyrighted Material
Conference Board of the Mathematical Sciences
CBMS
Regional Conference Series in Mathematics
Number 84
Ten Lectures
on the Interface
Between Analytic
Number Theory
and Harmonic Analysis
Hugh L. Montgomery
American Mathematical Society
with support from the
National Science Foundation
Copyrighted Material

# Basic plan

- This is a conversation in analysis and number theory.

# Basic plan

- This is a conversation in analysis and number theory.
- We shall discuss three problems in number theory
  - Prime gaps;
  - Least quadratic non-residue;
  - Least prime in an arithmetic progression;

  and some related Fourier optimization problems.

# Basic plan

- This is a conversation in analysis and number theory.
- We shall discuss three problems in number theory
  - Prime gaps;
  - Least quadratic non-residue;
  - Least prime in an arithmetic progression;

  and some related Fourier optimization problems.
- Try to keep the focus on the big picture (on how we arrive at the Fourier optimization wonderland).

# Fourier optimization framework

- Step I. Design “a" Fourier optimization problem connected to the number theory problem (proof of concept).

# Fourier optimization framework

- Step I. Design “a" Fourier optimization problem connected to the number theory problem (proof of concept).

- Step II. Solve the Fourier optimization problem (or at least try to find a good approximation for the solution).

# Fourier optimization framework

- Step I. Design “a" Fourier optimization problem connected to the number theory problem (proof of concept).
- Step II. Solve the Fourier optimization problem (or at least try to find a good approximation for the solution).
- Step III. Evolve towards designing what should be “the correct" Fourier optimization problem. Return to Step II.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.
- Hilbert-type inequalities / large-sieve inequalities.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.
- Hilbert-type inequalities / large-sieve inequalities.
- Sphere packings.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.
- Hilbert-type inequalities / large-sieve inequalities.
- Sphere packings.
- Sign Fourier uncertainty.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.
- Hilbert-type inequalities / large-sieve inequalities.
- Sphere packings.
- Sign Fourier uncertainty.
- Bounds for the Riemann zeta-function on the critical strip.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.
- Hilbert-type inequalities / large-sieve inequalities.
- Sphere packings.
- Sign Fourier uncertainty.
- Bounds for the Riemann zeta-function on the critical strip.
- Bounds for Montgomery's pair correlation conjecture.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.
- Hilbert-type inequalities / large-sieve inequalities.
- Sphere packings.
- Sign Fourier uncertainty.
- Bounds for the Riemann zeta-function on the critical strip.
- Bounds for Montgomery's pair correlation conjecture.
- Bounds for low-lying zeros and vanishing of $L$-functions.

# Part of the history

Examples of applications of Fourier analysis in number theory include:

- Erdös-Turán discrepancy inequalities.
- Hilbert-type inequalities / large-sieve inequalities.
- Sphere packings.
- Sign Fourier uncertainty.
- Bounds for the Riemann zeta-function on the critical strip.
- Bounds for Montgomery's pair correlation conjecture.
- Bounds for low-lying zeros and vanishing of $L$-functions.
- and so on...

Part I - Prime gaps

# Why is this going to be useful?

Tonight...

# Why is this going to be useful?

## Tonight...

Our hero

# Why is this going to be useful?

## Tonight...

Matt

# Why is this going to be useful?

## Tonight...

Mistery girl

Matt

# Why is this going to be useful?
## Tonight...

Mistery girl

Matt

Matt: How about I buy you a beer?

# Why is this going to be useful?

## Tonight...

Matt: How about I buy you a beer?

# Why is this going to be useful?

## Tonight...


Matt: How about I buy you a beer?

Girl: Sure, but only if you can prove to me that under RH there is always a prime in the interval $[x\ ,\ x + \frac{93}{100}\sqrt{x}\log x]$, for $x$ large.

# Why is this going to be useful?

## Tonight...


Matt: How about I buy you a beer?

Girl: Sure, but only if you can prove to me that under RH there is always a prime in the interval $[x\,,\ x + \frac{93}{100}\sqrt{x}\log x]$, for $x$ large.

Matt: I guess it is my lucky day!

# A classical problem

1. For $f : \mathbb{R} \to \mathbb{R}$, our normalization for Fourier transform is

$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i x t} f(x)\,\mathrm{d}x.$$

# A classical problem

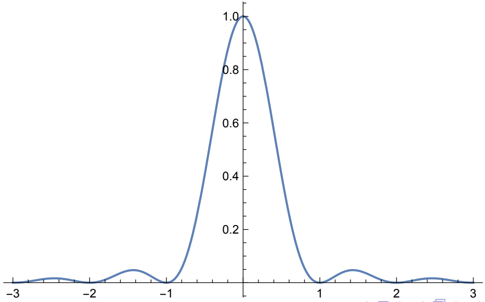
1. For $f : \mathbb{R} \to \mathbb{R}$, our normalization for Fourier transform is
$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i x t} f(x)\, \mathrm{d}x.$$
2. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-negative function, with $F(0) = 1$ and $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?

# A classical problem

1. For $f : \mathbb{R} \to \mathbb{R}$, our normalization for Fourier transform is
$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i x t} f(x)\,\mathrm{d}x.$$
2. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-negative function, with $F(0) = 1$ and $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?
3. Answer =1.
$F(x) = (\sin(\pi x)/(\pi x))^2$

# A classical problem

1. For $f : \mathbb{R} \to \mathbb{R}$, our normalization for Fourier transform is
$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-2\pi ixt} f(x)\,\mathrm{d}x.$$
2. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-negative function, with $F(0) = 1$ and $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?
3. Answer =1.
$F(x) = (\sin(\pi x)/(\pi x))^2$

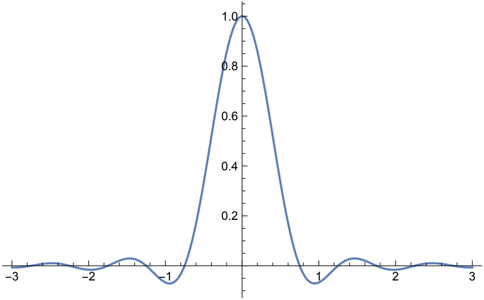


# An 'innocent' variant

1. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $F(0) = 1$ and $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?

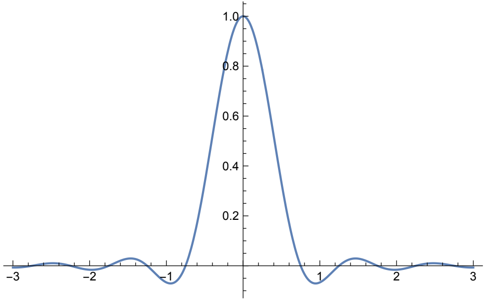
# An ‘innocent’ variant

1. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $F(0) = 1$ and $\text{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?
2. $H(x) = \frac{\cos(2\pi x)}{1-16x^2}$ yields $\|H\|_{L^1(\mathbb{R})} = 0.9259...$



# An 'innocent' variant

1. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $F(0) = 1$ and $\text{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?
2. $H(x) = \frac{\cos(2\pi x)}{1-16x^2}$ yields $\|H\|_{L^1(\mathbb{R})} = 0.9259...$

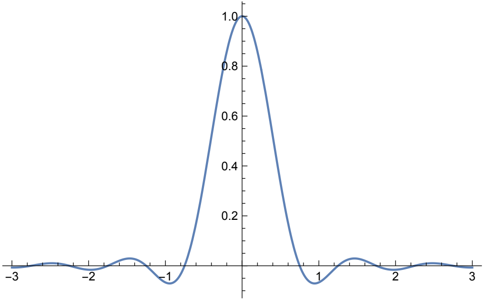



3. Best up-to-date (Hörmander and Bernhardsson '93)

$$0.9243360302\ldots < C < 0.9243360304\ldots$$

# An ‘innocent’ variant

1. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $F(0) = 1$ and $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$. What is the minimal value of $\|F\|_{L^1(\mathbb{R})}$?
2. $H(x) = \frac{\cos(2\pi x)}{1-16x^2}$ yields $\|H\|_{L^1(\mathbb{R})} = 0.9259...$




3. Best up-to-date (Hörmander and Bernhardsson ’93)

$$0.9243360302\ldots < C < 0.9243360304\ldots$$

4. There exists a unique extremizer.

# Prime gaps

1. Bertrand's postulate (1845): is there always a prime in the interval $[x, 2x]$?

# Prime gaps

1. Bertrand's postulate (1845): is there always a prime in the interval $[x, 2x]$?
2. Chebyshev (1852): Yes. There is always a prime in $[x, x + \frac{x}{\log x}]$ for $x$ large.

# Prime gaps

1. Bertrand's postulate (1845): is there always a prime in the interval $[x, 2x]$?
2. Chebyshev (1852): Yes. There is always a prime in $[x, x + \frac{x}{\log x}]$ for $x$ large.
3. Hoheisel (1930): There is always a prime in $[x, x + x^{\theta}]$ for some $0 < \theta < 1$, and $x$ large.

# Prime gaps

1. Bertrand's postulate (1845): is there always a prime in the interval $[x, 2x]$?
2. Chebyshev (1852): Yes. There is always a prime in $[x, x + \frac{x}{\log x}]$ for $x$ large.
3. Hoheisel (1930): There is always a prime in $[x, x + x^{\theta}]$ for some $0 < \theta < 1$, and $x$ large.
4. Baker - Harman - Pintz (2001): There is always a prime in $[x, x + x^{0.525}]$ for $x$ large.

# Prime gaps on RH

## Cramér's bounds (1920)

1. $$p_{n+1} - p_n = O(\sqrt{p_n} \log p_n),$$

   i.e. every interval $[x, x + c\sqrt{x} \log x]$, for some $c > 0$, contains a prime when $x$ is large.

# Prime gaps on RH

## Cramér's bounds (1920)

1. $$p_{n+1} - p_n = O(\sqrt{p_n} \log p_n),$$

   i.e. every interval $[x, x + c\sqrt{x} \log x]$, for some $c > 0$, contains a prime when $x$ is large.

2. Historic progress:
   - Goldston '83: $c = 4$.
   - Ramaré and Saouter '03: $c = 8/5$
   - Dudek '15: $c = 1 + o(1)$.

# Prime gaps on RH

## Cramér's bounds (1920)

1. $$p_{n+1} - p_n = O(\sqrt{p_n} \log p_n),$$

   i.e. every interval $[x, x + c\sqrt{x} \log x]$, for some $c > 0$, contains a prime when $x$ is large.

2. Historic progress:
   - Goldston '83: $c = 4$.
   - Ramaré and Saouter '03: $c = 8/5$
   - Dudek '15: $c = 1 + o(1)$.

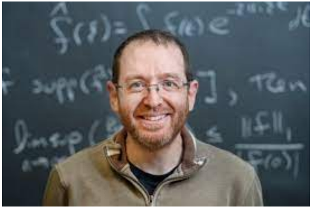
3. Non-asymptotic version: Dudek, Grenié, Molteni '16: for $x \geq 4$,

   $$[x, x + c\sqrt{x} \log x]$$

   contains a prime. Here $c = 1 + \frac{4}{\log x}$.
   - Ramaré and Saouter '03 ($c = 8/5$)

# Our team

- M. Milinovich (Mississippi) and K. Soundararajan (Stanford)



# Improved estimates

joint with M. Milinovich and K. Soundararajan '19

## Theorem (Asymptotic version)

*Assume RH. For $x$ large, every interval*

$$\left[x, x + \frac{21}{25}\sqrt{x}\log x\right]$$

*contains a prime.*

# Improved estimates

joint with M. Milinovich and K. Soundararajan '19

**Theorem (Asymptotic version)**

*Assume RH. For $x$ large, every interval*

$$\left[x, x + \frac{21}{25}\sqrt{x}\log x\right]$$

*contains a prime.*

**Theorem (Non-asymptotic version)**

*Assume RH. For $x \geq 4$, every interval*

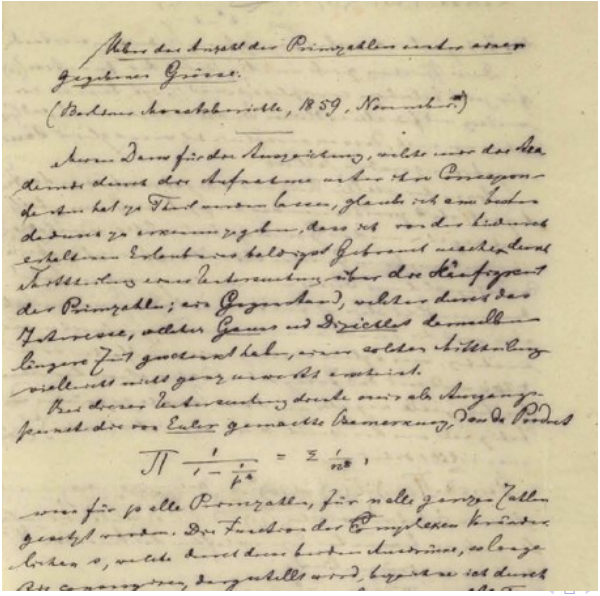
$$\left[x, x + \frac{22}{25}\sqrt{x}\log x\right]$$

*contains a prime.*

# Strategy

1. Explicit formula connecting zeros of $\zeta(s)$ and primes.
2. Fourier optimization problems.
3. Brun-Titchmarsh inequality.

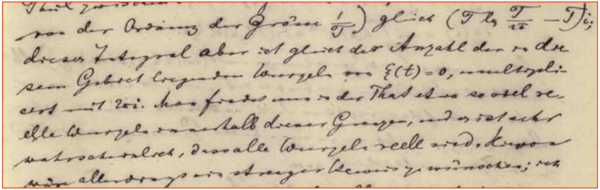
# Original manuscript - I



Riemann's 1859 manuscript

(Source: American Institute of Mathematics).

# Original manuscript - II

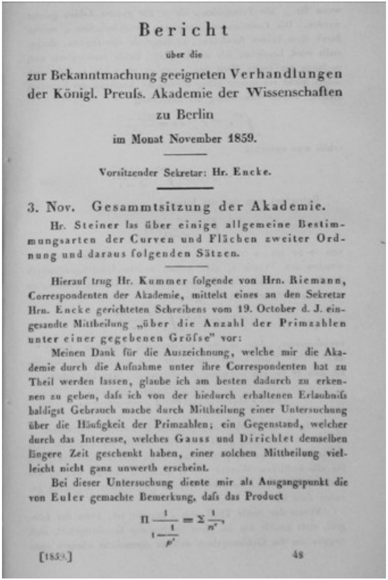


(Source: American Institute of Mathematics).

"Man findet nun in der That etwa so viel reele Wurzeln innerhalb dieser Grenzen, und es is sehr wahrscheinlich, dass alle Wurzeln reele sind."

"One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real."

# Original manuscript - III



Bericht

über die

zur Bekanntmachung geeigneten Verhandlungen der Königl. Preufs. Akademie der Wissenschaften zu Berlin

im Monat November 1859.

Vorsitzender Sekretar: Hr. Encke.

3. Nov. Gesammtsitzung der Akademie.

Hr. Steiner las über einige allgemeine Bestimmungsarten der Curven und Flächen zweiter Ordnung und daraus folgenden Sätzen.

Hierauf trug Hr. Kummer folgende von Hrn. Riemann, Correspondenten der Akademie, mittelst eines an den Sekretar Hrn. Encke gerichteten Schreibens vom 19. October d. J. eingesandte Mittheilung „über die Anzahl der Primzahlen unter einer gegebenen Gröfse" vor:

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dafs ich von der hiedurch erhaltenen Erlaubnifs baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dafs das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

[1859.] 48

First expression of the Riemann hypothesis in *Monatsberichte der Berliner Akademie*, November, 1859.

(Source: American Institute of Mathematics).

For $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\ldots$$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

For $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\ldots$$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Then $\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$, with $\Lambda(n) = \log p$, if $n = p^k$, $p$ prime.

For $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\ldots$$

$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Then $\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$, with $\Lambda(n) = \log p$, if $n = p^k$, $p$ prime.

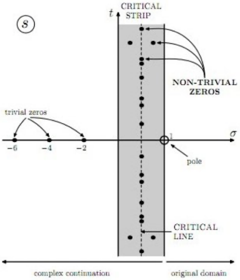
Let $\xi(s) = \frac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Then $\xi(s) = \xi(1-s)$.

For $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \ldots\right)\ldots$$
$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Then $\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}$, with $\Lambda(n) = \log p$, if $n = p^k$, $p$ prime.

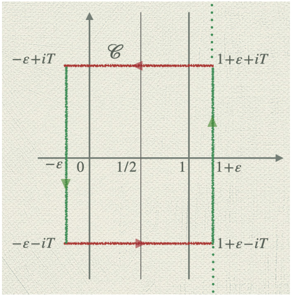
Let $\xi(s) = \frac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Then $\xi(s) = \xi(1-s)$.



# Explicit formulas

- $\xi(s) = \frac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$. Then $\xi(s) = \xi(1-s)$.
- Now let $h$ be a good function and note that

$$\sum_{\rho\ ;\ \zeta(\rho)=0} h\left(\frac{\rho-\frac{1}{2}}{i}\right) = \frac{1}{2\pi i}\int_{\mathcal{C}} h\left(\frac{s-\frac{1}{2}}{i}\right)\frac{\xi'(s)}{\xi(s)}\,ds.$$



# Explicit formula

## Lemma (Guinand-Weil Explicit Formula)

*Let $h(s)$ be analytic in the strip $|\mathrm{Im}(s)| \leq 1/2 + \varepsilon$ for some $\varepsilon > 0$, and such that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ when $|\mathrm{Re}(s)| \to \infty$.*

$$\sum_{\rho} h(\gamma) = h\Big(\frac{1}{2i}\Big) + h\Big(-\frac{1}{2i}\Big) - \frac{1}{2\pi}\widehat{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\,\mathrm{d}u - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\Big(\widehat{h}\Big(\frac{\log n}{2\pi}\Big) + \widehat{h}\Big(\frac{-\log n}{2\pi}\Big)\Big),$$

*where $\rho = \frac{1}{2} + i\gamma$ are the non-trivial zeros of $\zeta(s)$ and $\Lambda(n)$ is defined to be $\log p$ if $n = p^k$, $p$ a prime and $k \geq 1$, and zero otherwise.*

# Explicit formula

## Lemma (Guinand-Weil Explicit Formula)

*Let $h(s)$ be analytic in the strip $|\mathrm{Im}(s)| \leq 1/2 + \varepsilon$ for some $\varepsilon > 0$, and such that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ when $|\mathrm{Re}(s)| \to \infty$.*

$$\sum_{\rho} h(\gamma) = h\Big(\frac{1}{2i}\Big) + h\Big(-\frac{1}{2i}\Big) - \frac{1}{2\pi}\widehat{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\,\mathrm{d}u - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\Big(\widehat{h}\Big(\frac{\log n}{2\pi}\Big) + \widehat{h}\Big(\frac{-\log n}{2\pi}\Big)\Big),$$

*where $\rho = \frac{1}{2} + i\gamma$ are the non-trivial zeros of $\zeta(s)$ and $\Lambda(n)$ is defined to be $\log p$ if $n = p^k$, $p$ a prime and $k \geq 1$, and zero otherwise.*

Idea: to use this with $\widehat{h}\Big(\pm\frac{\log\cdot}{2\pi}\Big)$ localized in an interval without primes.

# Setup

For this let $f$ be a smooth function such that $\text{supp}(\widehat{f}) \subset [-1, 1]$, let $0 < \Delta \leq 1$, let $1 < a$, and set

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z)a^{iz}$$

# Setup

For this let $f$ be a smooth function such that $\mathrm{supp}(\widehat{f}) \subset [-1, 1]$, let $0 < \Delta \leq 1$, let $1 < a$, and set

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z)a^{iz}$$

Assume that for a certain $c > 0$ there is an infinite sequence of $x \to \infty$ such that $[x, x + c\sqrt{x}\log x]$ contains no primes. Choose

$$[x, x + c\sqrt{x}\log x] = \left[a\, e^{-2\pi\Delta}, a\, e^{2\pi\Delta}\right]$$

# Setup

For this let $f$ be a smooth function such that $\mathrm{supp}(\widehat{f}) \subset [-1,1]$, let $0 < \Delta \leq 1$, let $1 < a$, and set

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z) a^{iz}$$

Assume that for a certain $c > 0$ there is an infinite sequence of $x \to \infty$ such that $[x, x + c\sqrt{x}\log x]$ contains no primes. Choose

$$[x, x + c\sqrt{x}\log x] = \left[a\, e^{-2\pi\Delta}, a\, e^{2\pi\Delta}\right]$$

Then

$$4\pi\Delta = \log\left(1 + c\frac{\log x}{\sqrt{x}}\right) = c\frac{\log x}{\sqrt{x}} + O\left(\frac{\log^2 x}{x}\right)$$

and

$$a = x\left(1 + c\frac{\log x}{\sqrt{x}}\right)^{1/2} = x + O(\sqrt{x}\log x).$$

# Setup

For this let $f$ be a smooth function such that $\mathrm{supp}(\widehat{f}) \subset [-1,1]$, let $0 < \Delta \leq 1$, let $1 < a$, and set

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z) a^{iz}$$

Assume that for a certain $c > 0$ there is an infinite sequence of $x \to \infty$ such that $[x, x + c\sqrt{x}\log x]$ contains no primes. Choose

$$[x, x + c\sqrt{x}\log x] = \left[a\, e^{-2\pi\Delta}, a\, e^{2\pi\Delta}\right]$$

Then

$$4\pi\Delta = \log\left(1 + c\frac{\log x}{\sqrt{x}}\right) = c\frac{\log x}{\sqrt{x}} + O\left(\frac{\log^2 x}{x}\right)$$

and

$$a = x\left(1 + c\frac{\log x}{\sqrt{x}}\right)^{1/2} = x + O(\sqrt{x}\log x).$$

Idea: Perform an asymptotic analysis as $x \to \infty$.

# Setup II

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z)a^{iz}$$

Then

$$\sum_{\rho} h(\gamma) = h\Big(\frac{1}{2i}\Big) + h\Big(-\frac{1}{2i}\Big) - \frac{1}{2\pi}\widehat{h}(0)\log\pi$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iu}{2}\Big)\,\mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\Big(\widehat{h}\Big(\frac{\log n}{2\pi}\Big) + \widehat{h}\Big(\frac{-\log n}{2\pi}\Big)\Big),$$

# Setup II

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z) a^{iz}$$

Then

$$\sum_{\rho} g(\gamma) a^{i\gamma} = g\Big(\frac{1}{2i}\Big) a^{1/2} + g\Big(-\frac{1}{2i}\Big) a^{-1/2} - \frac{1}{2\pi}\widehat{g}\Big(-\frac{\log a}{2\pi}\Big)\log \pi$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)\, a^{iu}\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\, \mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\Big(\widehat{g}\Big(\frac{\log(n/a)}{2\pi}\Big) + \widehat{g}\Big(\frac{-\log na}{2\pi}\Big)\Big),$$

# Setup II

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z) a^{iz}$$

Then

$$\sum_{\rho} g(\gamma) a^{i\gamma} = g\Big(\frac{1}{2i}\Big) a^{1/2} + g\Big(-\frac{1}{2i}\Big) a^{-1/2} - \frac{1}{2\pi} \widehat{g}\Big(-\frac{\log a}{2\pi}\Big) \log \pi$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u)\, a^{iu}\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\, \mathrm{d}u$$

$$- \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \Big(\widehat{g}\Big(\frac{\log(n/a)}{2\pi}\Big) + \widehat{g}\Big(\frac{-\log na}{2\pi}\Big)\Big),$$

# Main competition

Matters are reduced to

$$\left| g\left(\frac{1}{2i}\right) a^{1/2} + g\left(-\frac{1}{2i}\right) a^{-1/2} \right| \leq \sum_{\gamma} |g(\gamma)| + O(1).$$

# Main competition

Matters are reduced to

$$\left| g\left(\frac{1}{2i}\right) a^{1/2} + g\left(-\frac{1}{2i}\right) a^{-1/2} \right| \leq \sum_{\gamma} |g(\gamma)| + O(1).$$

Observe that

$$\begin{aligned} g\left(\frac{1}{2i}\right) &= \Delta f\left(\frac{\Delta}{2i}\right) = \Delta \int_{-1}^{1} e^{\pi t \Delta}\, \widehat{f}(t)\,\mathrm{d}t \\ &= \Delta \int_{-1}^{1} \widehat{f}(t)\,\mathrm{d}t + \Delta \int_{-1}^{1} \left(e^{\pi t \Delta} - 1\right) \widehat{f}(t)\,\mathrm{d}t \\ &= \Delta f(0) + O(\Delta^2). \end{aligned}$$

# Main competition

Matters are reduced to

$$\left| g\left(\frac{1}{2i}\right) a^{1/2} + g\left(-\frac{1}{2i}\right) a^{-1/2} \right| \leq \sum_{\gamma} |g(\gamma)| + O(1).$$

Observe that

$$\begin{aligned} g\left(\frac{1}{2i}\right) &= \Delta f\left(\frac{\Delta}{2i}\right) = \Delta \int_{-1}^{1} e^{\pi t \Delta}\, \widehat{f}(t)\, \mathrm{d}t \\ &= \Delta \int_{-1}^{1} \widehat{f}(t)\, \mathrm{d}t + \Delta \int_{-1}^{1} \left(e^{\pi t \Delta} - 1\right) \widehat{f}(t)\, \mathrm{d}t \\ &= \Delta f(0) + O(\Delta^2). \end{aligned}$$

We may similarly estimate $g(-\frac{1}{2i})$ and, hence, the (LHS) above is

$$g\left(\frac{1}{2i}\right) a^{1/2} + g\left(-\frac{1}{2i}\right) a^{-1/2} = \Delta f(0)\left(a^{1/2} + a^{-1/2}\right) + O(\Delta^2 a^{1/2}).$$

# Sum over zeros

Let $N(x)$ denote the number of zeros with $0 < \gamma \leq x$. Using the fact that $N(x) = \frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} + O(\log x)$, we evaluate the sum $\sum_\gamma |g(\gamma)|$ using summation by parts to get

# Sum over zeros

Let $N(x)$ denote the number of zeros with $0 < \gamma \leq x$. Using the fact that $N(x) = \frac{x}{2\pi}\log\frac{x}{2\pi} - \frac{x}{2\pi} + O(\log x)$, we evaluate the sum $\sum_\gamma |g(\gamma)|$ using summation by parts to get

$$\sum_\gamma |g(\gamma)| = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g(x)|\, \log^+\frac{|x|}{2\pi}\,\mathrm{d}x + O\big(\|g\|_\infty + \|g'(x)\, \log^+|x|\|_1\big),$$

where $\log^+ x = \max\{\log x, 0\}$ for $x > 0$.

# Sum over zeros

Let $N(x)$ denote the number of zeros with $0 < \gamma \leq x$. Using the fact that $N(x) = \frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} + O(\log x)$, we evaluate the sum $\sum_\gamma |g(\gamma)|$ using summation by parts to get

$$\sum_\gamma |g(\gamma)| = \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(x)| \log^+ \frac{|x|}{2\pi} \, \mathrm{d}x + O\big(\|g\|_\infty + \|g'(x) \log^+|x|\|_1\big),$$

where $\log^+ x = \max\{\log x, 0\}$ for $x > 0$.

Recalling that $g(x) = \Delta f(\Delta x)$,

$$\begin{aligned}\sum_\gamma |g(\gamma)| &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(y)| \log^+|y/2\pi\Delta| \, \mathrm{d}y + O(1)\\ &= \frac{\log(1/2\pi\Delta)}{2\pi} \|f\|_1 + O(1).\end{aligned}$$

# Conclusion

We get

$$\Delta f(0)\, a^{1/2} + O(\Delta^2 a^{1/2}) \leq \frac{\log(1/2\pi\Delta)}{2\pi}\|f\|_1 + O(1).$$

# Conclusion

We get

$$\Delta f(0)\, a^{1/2} + O(\Delta^2 a^{1/2}) \leq \frac{\log(1/2\pi\Delta)}{2\pi}\|f\|_1 + O(1).$$

Since $4\pi\Delta = c\frac{\log x}{\sqrt{x}} + O\left(\frac{\log^2 x}{x}\right)$ and $a = x + O(\sqrt{x}\log x)$,

# Conclusion

We get

$$\Delta f(0)\, a^{1/2} + O(\Delta^2 a^{1/2}) \leq \frac{\log(1/2\pi\Delta)}{2\pi}\|f\|_1 + O(1).$$

Since $4\pi\Delta = c\frac{\log x}{\sqrt{x}} + O\left(\frac{\log^2 x}{x}\right)$ and $a = x + O(\sqrt{x}\log x)$,

$$\frac{c\, f(0)}{4\pi}\log x \leq \frac{\|f\|_1}{4\pi}\log x + O(1)$$

along this sequence of $x \to \infty$. This is only possible if

# Conclusion

We get

$$\Delta f(0)\, a^{1/2} + O(\Delta^2 a^{1/2}) \leq \frac{\log(1/2\pi\Delta)}{2\pi}\|f\|_1 + O(1).$$

Since $4\pi\Delta = c\frac{\log x}{\sqrt{x}} + O\left(\frac{\log^2 x}{x}\right)$ and $a = x + O(\sqrt{x}\log x)$,

$$\frac{c\, f(0)}{4\pi}\log x \leq \frac{\|f\|_1}{4\pi}\log x + O(1)$$

along this sequence of $x \to \infty$. This is only possible if

$$c \leq \frac{\|f\|_1}{f(0)} \leq 0.9259...$$

as we wanted to show.

# This leads to...

1. If $\operatorname{supp}(\widehat{F}) \subset [-1, 1]$ we would have

$$c \leq \frac{\|F\|_1}{F(0)}.$$

# This leads to...

1. If $\mathrm{supp}(\widehat{F}) \subset [-1, 1]$ we would have
$$c \leq \frac{\|F\|_1}{F(0)}.$$
2. One can actually do better by (over)estimating in $[-1, 1]^c$:
$$c \leq \frac{\|F\|_1}{\left(F(0) - \mathbf{B} \int_{[-1,1]^c} \left(\widehat{F}(t)\right)_+ \mathrm{d}t\right)}.$$
3. Here **B** is the Brun-Titchmarsh constant in our desired scale
$$\mathbf{B} := \limsup_{x \to \infty} \frac{\pi(x + \sqrt{x}) - \pi(x)}{\sqrt{x}/\log x}.$$

# This leads to...

1. If $\operatorname{supp}(\widehat{F}) \subset [-1, 1]$ we would have
$$c \leq \frac{\|F\|_1}{F(0)}.$$
2. One can actually do better by (over)estimating in $[-1, 1]^c$:
$$c \leq \frac{\|F\|_1}{\left(F(0) - \mathbf{B} \int_{[-1,1]^c} \left(\widehat{F}(t)\right)_+ \mathrm{d}t\right)}.$$
3. Here **B** is the Brun-Titchmarsh constant in our desired scale
$$\mathbf{B} := \limsup_{x \to \infty} \frac{\pi(x + \sqrt{x}) - \pi(x)}{\sqrt{x}/\log x}.$$
4. By the PNT (on the left) and work of Iwaniec (on the right):
$$1 \leq \mathbf{B} \leq \frac{36}{11}.$$