# Hautus conditions and perturbations for time reversible systems

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# Outline

- Introduction and main results
- Background on observability and Hautus test
- From the wave to the Schrödinger equation
- A frequency dependent Hautus test
- Linear perturbations
- The Von Karman system
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# Introduction and main results

# Control systems described by the Schrödinger or Euler-Bernoulli plate equations

$$(\Sigma_{\text{plate}}) \quad \begin{cases} \ddot{w}(t,x) + \Delta^2 w(t,x) & = u(t,x)\chi_{\mathcal{O}}(x) & (t \geqslant 0, x \in \Omega), \\ w(t,x) = 0, \ \Delta w(t,x) & = 0 & (t \geqslant 0, x \in \partial\Omega). \end{cases}$$

$$(\Sigma_{\text{schrod}}) \quad \begin{cases} \dot{z}(t,x) + i\Delta z(t,x) & = u(t,x)\chi_{\mathcal{O}}(x) & (t \geqslant 0, x \in \Omega), \\ z(t,x) & = 0 & (t \geqslant 0, x \in \partial\Omega), \end{cases}$$

**Known results:** The two above systems are exactly controllable in arbitrarilly small time if:

1) The system $(\Sigma_{\text{wave}})$ $\begin{cases} \ddot{v}(t,x) - \Delta v(t,x) & = u(t,x)\chi_{\mathcal{O}}(x) & (t \geqslant 0, x \in \Omega), \\ v(t,x) & = 0 & (t \geqslant 0, x \in \partial\Omega) \end{cases}$

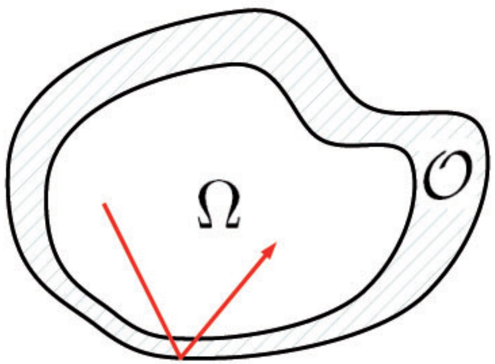
is exactly controllable (in some time), see Lebeau, 1992, ... .

2) $\Omega$ is a rectangle (Jaffard, 1990).

# The Bardos-Lebeau-Rauch condition

It has been shown in Bardos, Lebeau and Rauch (1992) that a necessary and sufficient condition on the control domain $\mathcal{O}$ in order to have the exact controllability in some time $\tau$ of $(\Sigma_{\text{wave}})$ is the following:

*(BLR)*: Any light ray traveling in $\Omega$ at unit speed and reflected according to geometric optics laws when it hits $\mathcal{O}$ will hit $\Omega$ in time $\leqslant \tau$.



# First perturbation results

$$(\tilde{\Sigma}_{\text{plate}}) \qquad \begin{cases} \ddot{w} + \Delta^2 w + \tilde{P} w & = u(t,x)\chi_{\mathcal{O}} \qquad (t \geqslant 0, x \in \Omega), \\ w(t,x) = 0, \ \Delta w(t,x) & = 0 \qquad (t \geqslant 0, x \in \partial\Omega), \end{cases}$$

where $\tilde{P}$ is a second order differential operator. Let $a \in \mathrm{L}^\infty(\Omega; i\mathbb{R})$.

$$(\tilde{\Sigma}_{\text{schrod}}) \qquad \begin{cases} \dot{z} + i\Delta z + az & = u\chi_{\mathcal{O}} \qquad (t \geqslant 0, x \in \Omega), \\ z(t,x) & = 0 \qquad (t \geqslant 0, x \in \partial\Omega), \end{cases}$$

**Known results:** The system $\tilde{\Sigma}_{\text{schrod}}$ is exactly controllable in arbitrarilly small time if:

1) The system $(\Sigma_{\text{wave}})$ $\begin{cases} \ddot{v}(t,x) - \Delta v(t,x) & = u(t,x)\chi_{\mathcal{O}}(x) \qquad (t \geqslant 0, x \in \Omega), \\ v(t,x) & = 0 \qquad (t \geqslant 0, x \in \partial\Omega), \end{cases}$

is exactly controllable (in some time), see ... folklore.

2) $\Omega$ is a rectangle (Burq and Zworski, 2011).

# Our (Bournissou, Ervedoza, M.T. (2024)) new (linear) perturbation result

**Theorem.** Assume that $a_{kl} \in W^{2,\infty}(\Omega;\mathbb{R})$, $a_{kl} = a_{lk}$ and

$$\tilde{P}w = \sum_{k,l=1}^{n} a_{kl} \frac{\partial^2 w}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k \frac{\partial w}{\partial x_k} + cw,$$

with $\sum_{l=1}^{n} \frac{\partial a_{kl}}{\partial x_l} = 0$. Moreover, assume that $(\Sigma_{\text{wave}})$ is exactly controllable (in some time). Then $\tilde{\Sigma}_{\text{plate}}$ is exactly controllable in arbitrarilly small time.

**Remark. Note that the generator of $\tilde{\Sigma}_{\text{plate}}$ is not skew-adjoint.**

# Local controllability for the von Karman system (I)

$$\begin{aligned}
&\ddot{w} + \Delta^2 w + [w, \Phi(w,w)] = f + u\chi_{\mathcal{O}}(x) \quad (t \in (0,\infty),\ x \in \Omega),\\
&w = \Delta w = 0 \qquad\qquad\qquad\qquad\qquad\qquad (x \in \partial\Omega,\ t \in (0,\infty)),\\
&w(0,x) = w_0(x), \quad \dot{w}(0,x) = w_1(x) \qquad\qquad (x \in \Omega),
\end{aligned}$$

where the Airy stress function $\Phi(v,w)$ is defined by

$$\begin{aligned}
&\Delta^2\Phi(v,w)(t,x) = [v,w](t,x) \qquad (t \in (0,\infty),\ x \in \Omega),\\
&w(t,x) = \frac{\partial w}{\partial \nu}(t,x) = 0 \qquad (t \in (0,\infty), x \in \partial\Omega),
\end{aligned}$$

and the bracket $[\cdot,\cdot] : H^2(\Omega) \times H^2(\Omega) \to L^1(\Omega)$ is defined by

$$[\psi,\varphi] = \frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\varphi}{\partial x_1^2} - 2\frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial^2\varphi}{\partial x_1\partial x_2} \qquad (\psi,\varphi \in H^2(\Omega)).$$

# Local controllability for the von Karman system (II) (Bournissou, Ervedoza, M.T. (2024))

**Theorem.** Assume that $(\Sigma_{\text{wave}})$ is exactly controllable (in some time) and that $\eta \in H^{3+\varepsilon}(\Omega)$ is an analytic (in $\omega$) stationary solution. Then for every $\tau > 0$, there exists $\alpha > 0$ such that for every

$$w_0 \in H^2(\Omega) \cap H_0^1(\Omega), \qquad w_1 \in L^2(\Omega),$$

with

$$\|w_0 - \eta\|_{H^2(\Omega)} + \|w_1\|_{L^2(\Omega)} \leqslant \alpha,$$

there exists $u \in L^2([0,\tau]; L^2(\Omega))$ such that

$$w(\tau, \cdot) = \eta, \qquad \dot{w}(\tau, \cdot) = 0.$$

# Background on exact observability and on the Hautus test

# Admissible observation operators

Let $X$ and $Y$ be Hilbert spaces, $A : \mathcal{D}(A) \to X$ et $C \in \mathcal{L}(\mathcal{D}(A), Y)$.

$$\dot{w}(t) = Aw(t), \quad y(t) = Cw(t).$$

Assume that $A$ generates a $C^0$ semigroup, denoted $\mathbb{T}$, in $X$.

**Definition 1.** $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is an *admissible observation operator for* $\mathbb{T}$ if there exist $\tau > 0$, $k_\tau > 0$ such that

$$k_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_0\|_Y^2 \mathrm{d}t \leqslant \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A).$$

# Observability types

**Definition 2.** Let $\tau > 0$ and let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$.

- The pair $(A, C)$ is *exactly observable in time* $\tau$ if there exists $K_\tau > 0$ such that

$$K_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_0\|_Y^2 \mathrm{d}t \geqslant \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A).$$

- The pair $(A, C)$ is *approximately observable in time* $\tau$ if the only $z_0 \in X$ such that $C\mathbb{T}_t z_0 = 0$ for every $t \in [0, \tau]$ is $z_0 = 0$.

- The pair $(A, C)$ is *final state observable in time* $\tau$ if there exists $K_\tau > 0$ such that

$$K_\tau^2 \int_0^\tau \|C\mathbb{T}_t z_0\|_Y^2 \mathrm{d}t \geqslant \|\mathbb{T}_\tau z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A).$$

**Remark.** If $X = \mathbb{C}^n$ the 3 concepts coincide and are time independent.

# Perturbations of exactly observable systems

**Proposition (Haad and Duprez-Olive).** Let $\tau > 0$ and let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$. Assume that the pair $(A, C)$ is *exactly observable in time* $\tau_0$ and let $P \in \mathcal{L}(X)$. Then $(A + P, C)$ is exactly observable in any time $\tau > \tau_0$ if

- $\|P\|_{\mathcal{L}(X)} << 1$.
- $P$ compact and $\mathrm{Ker}\,(s\mathbb{I} - A - C) = \{0\}$ for every $s \in \mathbb{C}$.

**Remark.** Our perturbation results do not require any of the above assumptions .

# The duality observability-controllability

**Proposition 2 (Dolecki and Russell, 1973) .** *The pair* $(A, C)$ *is exactly observable in time* $\tau$ *iff the pair* $(A^*, C^*)$ *is exactly controllable in time* $\tau$ *(this means that for every* $z_0 \in X$ *there exists* $u \in L^2([0, \tau]; U)$ *s.t.*

$$\dot{z}(t) = A^* z(t) + C^* u(t), \;\; z(0) = 0, \; z(\tau) = z_0).$$

*Moreover, the control cost for* $(A^*, C^*)$ *coincides with the observation cost of* $(A, C)$*.*

# The classical Hautus test

Let $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$, $A : X \in \mathcal{L}(X)$ and $C \in \mathcal{L}(X, Y)$.

$$\dot{w}(t) = Aw(t), \;\; y(t) = Cw(t).$$

**Theorem (Hautus).** The following conditions are equivalent:

- $(A, C)$ observable;
- rang $[s\mathbb{I} - A \;\; C] = n$ for every $s \in \mathbb{C}$.
- There exists $m > 0$ such that $\|s\varphi - A\varphi\|_X^2 + \|C\varphi\|_Y^2 \geqslant m^2\|\varphi\|_X^2$ for every $\varphi \in X$.
- $C\varphi \neq 0$ for every eigenvector $\varphi$ of $A$.

# Hautus test for skew-adjoint generators (I)

Let $\mathbb{T}$ be a group of unitary operators on $X$, with generator $A$.
Let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$.

Theorem. (Miller (2005))
The pair $(A, C)$ is exactly observable iff there exists $M$, $m > 0$ s.t.

$$M^2\|(i\omega I - A)z_0\|^2 + m^2\|Cz_0\|^2 \geqslant \|z_0\|^2 \qquad \forall\, \omega \in \mathbb{R},\ z_0 \in \mathcal{D}(A).$$

If the above estimate holds then
$(A, C)$ is exactly observable in any time $\tau > M\pi$.

# Hautus test for skew-adjoint generators (II)

Assume that there exists an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ formed of eigenvectors of $A$ and the corresponding eigenvalues $\lambda_k$ satisfy $\lim|\lambda_k| = \infty$. Let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$. For some $\alpha > 0$ denote $E_\alpha = \operatorname{span}\{\phi_k \mid |\mu_k| \leqslant \alpha\}^\perp$

Proposition 3 (Tucsnak and Weiss (2009)).
Asssume that

1. There exist $M,\ m,\ \alpha > 0$ s.t. for every $\omega \in \mathbb{R}$ with $|\omega| > \alpha$, we have

$$M^2\|(i\omega I - A)z_0\|^2 + m^2\|Cz_0\|^2 \geqslant \|z_0\|^2 \qquad \forall\, z_0 \in E_\alpha \cap \mathcal{D}(A),$$

2. $C\phi \neq 0$ for every eigenvector $\phi$ of $A$.

Then $(A, C)$ is exactly observable in any time $\tau > M\pi$

# A spectral test for skew-adjoint generators

Assume that there exists an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ formed of eigenvectors of $A$ and the corresponding eigenvalues $\lambda_k$ satisfy $\lim|\lambda_k| = \infty$. Let $C \in \mathcal{L}(\mathcal{D}(A), Y)$ be an admissible observation operator for $\mathbb{T}$. For $\omega \in \mathbb{R}$ and $r > 0$, set $J(\omega, r) = \{k \in \Lambda \text{ such that } |\mu_k - \omega| < r\}$.

Proposition. The following statements are equivalent:

**(S1)** There exist $r$, $\delta > 0$ such that for all $\omega \in \mathbb{R}$ and for every wave packet of $A$ of parameters $\omega$ and $r$, denoted by $z$, we have $\|Cz\|_Y \geqslant \delta\|z\|_X$.

**(S2)** $(A, C)$ is exactly observable.

Moreover, if **(S1)** holds for some $r$, $\delta > 0$, then $(A, C)$ is exactly observable in any time $\tau > \pi\sqrt{\dfrac{1}{r^2} + \dfrac{4K^2(r)}{r\delta^2}}$, where

$$K(r) = \sup_{s\in\mathbb{C}_r} \sqrt{\operatorname{Re} s}\,\|C(sI - A)^{-1}\|_{\mathcal{L}(X,Y)}.$$

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# Exact controllability for a Schrödinger system

**Theorem.** (Jaffard (1990), Bourgain,Burq and Zworski (2013))
*Let* $\Omega = [0,1]^2$ *and let* $\mathcal{O} \subset \Omega$ *be a set of positive measure. Then the system*

$$\begin{cases} \dot{z} + i\Delta z & = u\chi_{\mathcal{O}} \qquad\qquad (t \geqslant 0, x \in \Omega), \\ z(t,x) & = 0 \qquad\qquad (t \geqslant 0, x \in \partial\Omega), \end{cases}$$

*is exactly controllable in any time* $\tau > 0$.

*Proof.* It almost suffices to combine the spectral test with the following results of Zygmund (1972):

**Theorem.** With the above notation, there exists a constant $K_{\mathcal{O}} > 0$ such that for every $R > 0$ and $(c_{mn}) \in l^2$, we have

$$K_{\mathcal{O}}^2 \int_{\mathcal{O}} \left| \sum_{m^2+n^2=R^2} c_{mn} \mathrm{e}^{2\pi i(mx+ny)} \right| \mathrm{d}x\,\mathrm{d}y \geqslant \sum_{m^2+n^2=R^2} |c_{mn}|^2.$$

# Zygmund's proof (I)

**Lemma 1.** *Let* $\widehat{\chi}(\nu) = \int_{\mathcal{O}} \mathrm{e}^{-2\pi i \nu\cdot\xi}$. *Then there exists* $\varepsilon > 0$ *such that*

$$|\widehat{\chi}(\nu)| < \mu(\mathcal{O}) - \varepsilon \qquad\qquad (\nu \in \mathbb{R}^2 \setminus \{(0,0)\}).$$

**Lemma 2.** For any three lattice points $\lambda,\ \mu,\ \nu$ situated on a circumference of radius $R$ we have

$$|\lambda - \mu|\,|\mu - \nu|\,|\nu - \lambda| \geqslant 2R.$$

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# Zygmund's proof (II)

$$\int_{\mathcal{O}} \left| \sum_{m^2+n^2=R^2} c_{mn} \mathrm{e}^{2\pi i(mx+ny)} \right|^2 \mathrm{d}x\,\mathrm{d}y = \underbrace{|\mathcal{O}| \sum_{m^2+n^2=R^2} |c_{mn}|^2}_{P} + \underbrace{\sum_{\lambda \neq \mu} c_\mu \overline{c_\nu} \widehat{\chi}(\nu - \mu)}_{Q}.$$

Let $\Delta = \{|\lambda - \nu| \;\mid\; |\lambda| = |\nu| = R\}$ and let $R_0$ be such that

$$\left( 2 \sum_{|\lambda| \geqslant R_0} |\widehat{\chi}(\lambda)|^2 \right)^{\frac{1}{2}} \leqslant \frac{\varepsilon}{2}.$$

Let $\Delta' = \{\alpha \in \Delta \;\mid\; |\alpha| \leqslant R_0\}$ and $\Delta'' = \Delta \setminus \Delta'$. Set

$$Q = Q' + Q''.$$

# Zygmund's proof (III)

$$|Q''| \leqslant \left(\sum |c_\mu \bar{c}_\nu|^2\right)^{\frac{1}{2}} \left(|\sum \widehat{\chi}(\nu-\mu)|^2\right)^{\frac{1}{2}} \leqslant \sum_{|\lambda|=R} |c_\lambda|^2 \left(2 \sum_{|\lambda \geqslant R_0} |\widehat{\chi}(\lambda)|^2\right)^{\frac{1}{2}} \leqslant \frac{\varepsilon}{2} \sum_{|\nu|=R} |c_\nu|^2,$$

since a circle has at most two chords of prescribed lenghth and direction.

With $R_0$ fixed we can choose, by Lemma 2, $R$ large enough to split the lattice points of $C(0,R)$ into "distant" pairs $(\mu,\nu)$ such that $|\mu-\nu| \leqslant R_0$. For each of these pairs we use Lemma 1 to obtain that

$$|c_\mu \bar{c}_\nu \widehat{\chi}(\nu-\mu) + c_\nu \bar{c}_\mu \widehat{\chi}(\mu-\nu)| \leqslant (|c_\mu|^2 + |c_\nu|^2)(|\mathcal{O}| - \varepsilon).$$

Thus $|Q'| \leqslant (|\mathcal{O}| - \varepsilon) \sum_{|\nu|=R} |c_\nu|^2$.

# From the wave to the Schrödinger equation

# The wave equation with Neumann boundary observation

Theorem.(Bardos, Lebeau, Rauch, 1992).
Assume that $\Gamma \subset \partial\Omega$ and that $\tau > 0$ and consider the system

$$\begin{cases} \ddot{w} - \Delta w = 0 & (x \in \Omega,\ t \geq 0), \\ w = 0 & (x \in \partial\Omega,\ t \geq 0). \end{cases}$$

Then the following conditions are equivalent:

1. There exists $K_{\tau,\Gamma} > 0$ s.t.

$$K_{\tau,\Gamma}^2 \int_0^\tau \int_\Gamma \left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}\Gamma \mathrm{d}t \geqslant \int_\Omega \left(|\nabla w(0)|^2 + \dot{w}(0)|^2\right) \mathrm{d}x,$$

   for every solution $w$.

2. $\Gamma$ satisfies the geometric optics condition (also called Bardos, Lebeau, Rauch condition).

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# Notation

- $A_0 : \mathcal{D}(A_0) \to H$ with compact resolvents and $A_0 = A_0^* > 0$.
- $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ with the norm $\|w\|_{\frac{1}{2}} = \|A_0^{\frac{1}{2}} w\|$
- $X = H_{\frac{1}{2}} \times H$, $A : \mathcal{D}(A) \to X$, $A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}$
- $C_1 \in \mathcal{L}(H_1, Y)$, $C \in \mathcal{L}(X_1, Y)$, $C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$.

**Proposition (Miller (2005), Tucsnak and Weiss(2009)).**
If $(A, C)$ is exactly observable then $(iA_0, C_1)$, with the state space $H_{\frac{1}{2}}$, is exactly observable in any time $\tau > 0$.

**Proof.** By Theorem 2 there exist $M$, $m > 0$ s.t.

$$M^2\|(i\sqrt{\omega}I - A)\widetilde{z}\|^2 + m^2\|C\widetilde{z}\|^2 \geq \|\widetilde{z}\|^2 \qquad \forall\, \omega > 0,\ \widetilde{z} \in \mathcal{D}(A)$$

If we choose $\widetilde{z} = \begin{bmatrix} z \\ iA_0^{\frac{1}{2}} z \end{bmatrix}$, with $z \in \mathcal{D}(A_0^{\frac{3}{2}})$, we obtain

$$\frac{M^2}{\omega}\|(\omega I - A_0)z\|^2_{\frac{1}{2}} + \frac{m^2}{2}\|C_1 z\|^2 \geq \|z\|^2_{\frac{1}{2}} \qquad \forall\, \omega > 0,\ z \in \mathcal{D}(A_0^{\frac{3}{2}}).$$

We conclude using Hautus.

# The Schrödinger equation with Neumann boundary observation

Theorem. Assume that $\Gamma \subset \partial\Omega$ satisfies the Bardos-Lebeau-Rauch condition. Then for every $\tau > 0$, there exists $K_{\tau,\Gamma} > 0$ s.t.

$$K_{\tau,\Gamma}^2 \int_0^\tau \int_\Gamma \left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}\Gamma \geq \int_\Omega |\nabla w(0)|^2 \mathrm{d}x\mathrm{d}t,$$

for every solution $w$.

$$\begin{cases} \dot{w} + i\Delta w = 0 & (x \in \Omega,\ t \geq 0), \\ w = 0 & (x \in \partial\Omega,\ t \geq 0). \end{cases}$$

Remark. The above result has been obtained by micro-local analysis by Lebeau in 1992. A "softer" approach has been obtained by Burq and Zworski (2005) and precised in Miller (2006) and Tucsnak and Weiss (2007).

# A frequency dependent Hautus test

# The abstract context (I)
# Notation

- $\mathcal{H}$ and $Y$ are two Hilbert spaces;
- $A_0 : \mathcal{D}(A_0) \to \mathcal{H}$ is a strictly positive operator with compact resolvents;
- For $\alpha > 0$, we denote by $\mathcal{H}_\alpha$ the space $\mathcal{D}(A_0^\alpha)$ endowed with the graph norm of $A_0^\alpha$. Note that for every $\alpha \in \mathbb{R}$ the operator $A_0$ can be restricted (or extended) to a unitary operator $\mathcal{L}(\mathcal{H}_\alpha, \mathcal{H}_{\alpha-1})$.
- $C_0 \in \mathcal{L}(\mathcal{H}, Y)$ is an observation operator.

# The abstract context (II)
# Main result

**Theorem 1.** Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H} \times \mathcal{H}_{-1}$ and $C \in \mathcal{H} \times \mathcal{H}_{-1}$ be defined by $\mathcal{D}(\mathcal{A}) = \mathcal{H}_1 \times \mathcal{H}$ and $\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0^2 & 0 \end{bmatrix}$, $\quad C = \begin{bmatrix} C_0 & 0 \end{bmatrix}$. Suppose that the system $(\tilde{A}, C)$, defined by

$$\mathcal{D}(\tilde{A}) = \mathcal{H}_{\frac{1}{2}} \times \mathcal{H}_{-\frac{1}{2}}, \qquad \tilde{A} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix},$$

with state space $\mathcal{H} \times \mathcal{H}_{-\frac{1}{2}}$ is exactly observable (in some time).

Then there exists a continuous function $M_1 : \mathbb{R} \to [0, +\infty)$, which tends to zero when $|\omega| \to \infty$, and a constant $m_1 > 0$ such that

$$M_1^2(\omega) \|(i\omega I - \mathcal{A}) z_0\|^2_{\mathcal{H} \times \mathcal{H}_{-1}} + m_1^2 \|C z_0\|^2_Y \geqslant \|z_0\|^2_{\mathcal{H} \times \mathcal{H}_{-1}} \qquad (\omega \in \mathbb{R},\ z_0 \in \mathcal{D}(\mathcal{A})).$$

# Linear perturbations

# Another abstract result

**Theorem 2.** With the notation and assumptions in Theorem 1, assume that $P_0 \in \mathcal{L}(\mathcal{H}, \mathcal{H}_{-1}) \cap \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be such that $P_0$, with domain $\mathcal{H}$, is a symmetric operator on $\mathcal{H}_{-1}$. Let $P := \begin{bmatrix} 0 & 0 \\ P_0 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{H} \times \mathcal{H}_{-1})$ and let $\mathcal{A}_P : \mathcal{D}(\mathcal{A}_P) \to \mathcal{H} \times \mathcal{H}_{-1}$ be the operator defined by

$$\mathcal{D}(\mathcal{A}_P) = \mathcal{D}(\mathcal{A}), \qquad \mathcal{A}_P = \mathcal{A} - P.$$

Moreover, let $C \in \mathcal{L}(\mathcal{H} \times \mathcal{H}_{-1})$ be defined by $C = \begin{bmatrix} C_0 & 0 \end{bmatrix}$ and suppose that

$$\operatorname{Ker}(s^2 I + A_0^2 + P_0) \cap \operatorname{Ker} C_0 = \{0\} \qquad (s \in \mathbb{C}).$$

Then the pair $(\mathcal{A}_P, C)$ is exactly observable in any time $\tau > 0$.

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# Application to perturbed plates

Let $P_0 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ be the operator defined by

$$P_0\varphi = \sum_{k,l=1}^{n} a_{kl} \frac{\partial^2 \varphi}{\partial x_k \partial x_l} \qquad (\varphi \in \mathcal{H}_1).$$

$\varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $s \in \mathbb{C}$ are such that

$$s^2\varphi + \Delta^2\varphi + P_0\varphi = 0 \qquad (\text{in} \quad \Omega),$$

$$\varphi = 0, \qquad \Delta\varphi = 0 \qquad (\text{on} \quad \partial\Omega),$$

$$\varphi = 0, \qquad (\text{in} \quad \mathcal{O}),$$

then $\varphi = 0$. This follows from a slight variation of the global Carlemman estimates for the Laplacian.

# The Von Karman system

# Linearized von Karman system

Setting $w(t,x) = \eta(x) + \varepsilon\delta(t,x)$, we obtain the first order approximation:

$$\ddot{\delta}(t,x) + \Delta^2\delta(t,x) + [\delta, \Phi(\eta,\eta)] + 2[\eta, \Phi(\eta,\delta)] = u(t,x)\chi_{\mathcal{O}}(x),$$

$$\delta(t,x) = \Delta\delta(t,x) = 0 \qquad (x \in \partial\Omega,\ t \in (0,\infty))$$

$$\delta(0,x) = \delta_0(x), \quad \dot{\delta}(0,x) = \delta_1(x) \qquad (x \in \Omega).$$

There are <u>two</u> perturbation operators:

$$P_0\psi = \sum_{k,l=1}^{2} a_{kl} \frac{\partial^2\psi}{\partial x_k \partial x_l} \qquad (\psi \in H^2(\Omega) \cap H_0^1(\Omega)),$$

$$Q_0\psi = \frac{\partial^2\eta}{\partial x_1^2}\frac{\partial^2}{\partial x_2^2}\left(\Phi\left[\eta,\psi\right]\right) + \frac{\partial^2\eta}{\partial x_2^2}\frac{\partial^2}{\partial x_1^2}\left(\Phi\left[\eta,\psi\right]\right) - 2\frac{\partial^2\eta}{\partial x_1\partial x_2}\frac{\partial^2}{\partial x_1\partial x_2}\left(\Phi\left[\eta,\psi\right]\right).$$

# Concluding remarks

# Some open questions

- Arbitrary zero order perturbations for Schrödinger;
- Symmetric second order perturbations for plates when $\Omega$ is rectangular and $\mathcal{O}$ arbitrary;
- Less restrictive assumptions on the stationary state $\eta$ for the von Kármán system.
- Arbitrary second order pertbation for plates;
- Controllability around buckled states for von Karman (bilinear control).