Reachable Spaces and Controllability With Focus on Heat Equation Systems

Marius Tucsnak

Outline

- Well-posed linear time invariant control systems
- Reachable space and controllability
- Some remarks about HUM
- The reachable space for the constant coefficients heat equation.
- Robustness of the reachable space with respect to perturbations of the generator

- Concluding remarks

Well-posed linear time invariant control systems

Some notation

We consider control systems described by equations of the form

$$(SE) \qquad \dot{z}(t) = Az(t) + Bu(t), \qquad \text{with}$$

- X (the state space) and U (the input space) are complex Hilbert spaces. We have $X = \mathbb{C}^n$ and $U = \mathbb{C}^m$ for finite-dimensional control systems.

- $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a strongly continuous semigroup on X generated by A. We have $\mathbb{T}_t = \mathrm{e}^{tA}$ for finite-dimensional control systems. X_1 is $\mathcal{D}(A)$ endowed with the graph norm and X_{-1} is the dual of $\mathcal{D}(A^*)$ with respect to the pivot space X.

- $B \in \mathcal{L}(U; X_{-1})$ is the control operator.

Admissible control operators

The solution of (SE) writes:

$$z(t) = \mathbb{T}_t z(0) + \Phi_t u,$$

where $\mathbb{T}$ is the semigroup generated by A and

$$\Phi_t \in \mathcal{L}(L^2([0,\infty);U), X_{-1}), \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\,\mathrm{d}\sigma.$$

Definition. B *is called an admissble control operator for* $\mathbb{T}$ *if* $\operatorname{Ran}\Phi_t \subset X$ *for one (and hence all)* $t > 0$.

Example. *Take* $A = -A_0$ *with* $A_0 > 0$. *For* $\alpha > 0$, *denote* $X_\alpha = \mathcal{D}(A_0^\alpha)$ *and* $X_{-\alpha}$ *is the dual of* X_α *with respect to the pivot space* X. *Then every operator* $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$ *is admissible.*

Reachable space and controllability

Definition and first properties

- The *reachable space at time* τ is $\operatorname{Ran}\Phi_\tau$, it generally depends on τ and it is not, in general, a closed subspace of X.

- $\operatorname{Ran}\Phi_\tau$ is a Hilbert space when endowed with the norm

$$\|\eta\|_{\operatorname{Ran}\Phi_\tau} = \inf\left\{\|u\|_{L^2([0,\tau];U)},\ \text{s.t. } \Phi_\tau u = \eta.\right\}$$

- (Kalman, 1963) If X and U are finite dimensional then

$$\operatorname{Ran}\Phi_\tau = \operatorname{Ran}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

Controllability types

(A, B) is said *exactly controllable in time* τ if $\operatorname{Ran} \Phi_\tau = X$.

(A, B) is said *null controllable in time* τ if $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$. This is equivalent to the existence, for each $z_0 \in X$ of $u \in L^2([0, \tau]; U)$ such that the solution of

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0,$$

satisfies $z(\tau) = 0$.

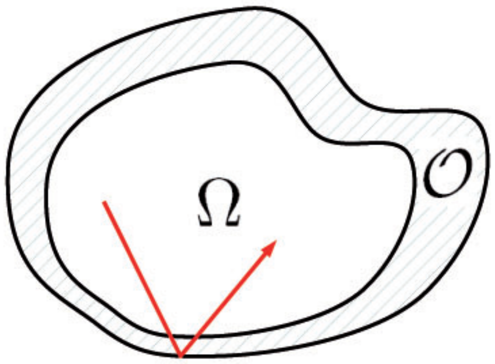
(A, B) is *approximatively controllable in time* τ if $\overline{\operatorname{Ran} \Phi_\tau} = X$.

The three above concepts coincide with the usual controllability concept in the case of finite dimensional LTIs.

A classical example

The system (Σ_{wave}) $\begin{cases} \ddot{v}(t,x) - \Delta v(t,x) &= u(t,x)\chi_{\mathcal{O}}(x) \qquad (t \geqslant 0, x \in \Omega), \\ v(t,x) &= 0 \qquad (t \geqslant 0, x \in \partial\Omega), \end{cases}$

is exactly controllable in time τ, see Bardos, Lebeau and Rauch (1992) iff any light ray traveling in Ω at unit speed and reflected according to geometric optics laws when it hits $\mathcal{O}$ will hit Ω in time $\leqslant \tau$.



There is almost no information about the reachable space when the Bardos-Lebeau-Rauch condition fails or when the time is small!

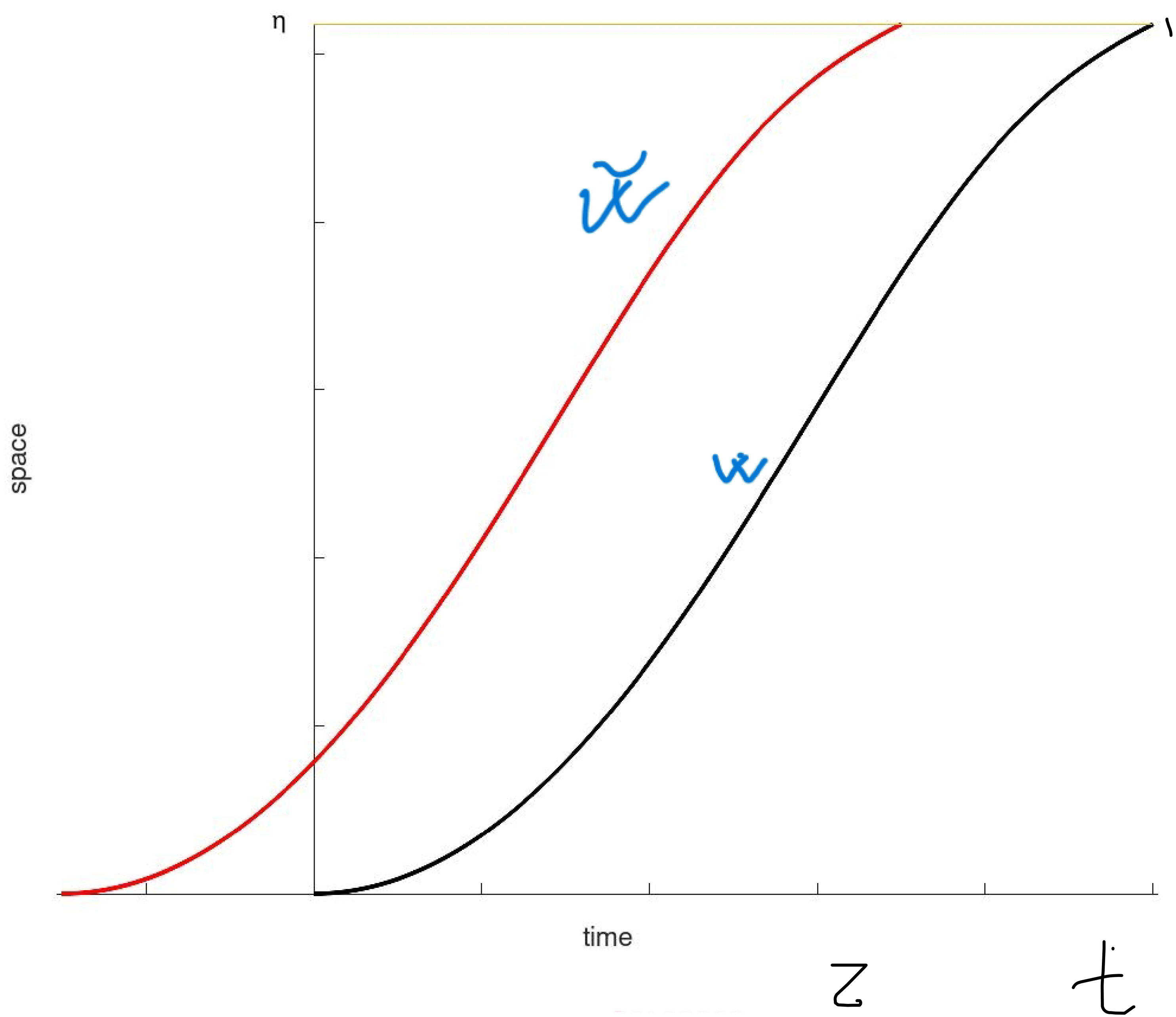
Null controllability and reachable space

Proposition. (Fattorini, Seidman) If (A, B) is null controllable in any time then $\operatorname{Ran}\Phi_\tau$ does not depend on $\tau > 0$. Given $\tau,\ \tau' > 0$, the norms $\|\cdot\|_{\operatorname{Ran}\Phi_\tau}$ and $\|\cdot\|_{\operatorname{Ran}\Phi_{\tau'}}$ are equivalent.
Proof. The fact that for $0 < t < \tau$, $\operatorname{Ran}\Phi_t \subset \operatorname{Ran}\Phi_\tau$ is an obvious one.

Let now $0 < \tau < t$, $\eta \in \operatorname{Ran}\Phi_t$ and
$\tilde{u}(\sigma) = u(\sigma + t - \tau), \qquad \tilde{w}(\sigma) = w(\sigma + t - \tau, \cdot)$.
Then $\eta = w(t, \cdot) = \tilde{w}(\tau, \cdot) = \mathbb{T}_\tau \tilde{w}(0, \cdot) + \Phi_\tau \tilde{u}$. Since $\operatorname{Ran}\Phi_\tau \supset \operatorname{Ran}\mathbb{T}_\tau$, we have $\eta \in \operatorname{Ran}\Phi_\tau$, thus $\operatorname{Ran}\Phi_t \subset \operatorname{Ran}\Phi_\tau$.

Remark. (Normand, 2019) under the above assumptions

$$\Phi_\tau(\sqrt{t}L^2([0, \tau]; U)) = \operatorname{Ran}\Phi_\tau.$$


η
space
time
w̃
w
1
z
t

Some remarks about HUM

(Hilbert Uniqueness Method)

A classical result (Douglas, 1966)

Proposition. *If Z, X are Hilbert spaces and $G \in \mathcal{L}(Z, X)$, then the following statements are equivalent:*
(a) *G is onto.*
(b) *G^* is bounded from below, i.e., there exists a constant $m > 0$ such that*

$$\|G^* x\|_Z \geqslant m\|x\|_X \qquad (x \in X).$$

(c) $GG^* > 0$.
Moreover, if these statements are true then $\|(GG^*)^{-1}\| \leqslant \frac{1}{m^2}$, where m is the constant appearing in statement (b).

Institut de Mathématiques de Bordeaux

The controllability Gramian

Definition. Let $(\mathbb{T}, \Phi)$ be a well-posed control LTI. The operator

$$R_\tau = \Phi_\tau \Phi_\tau^* \in \mathcal{L}(X)$$

is called the *controllability Gramian in time* τ.

Remark. If $B \in \mathcal{L}(U, X)$ then $R_\tau = \int_0^\tau \mathbb{T}_t B B^* \mathbb{T}_t^* \, \mathrm{d}t$.

Proposition. $(\mathbb{T}, \Phi)$ *is exactly controllable in time* $\tau > 0$ *if and only if* $R_\tau > 0$. *If*

$$u = \Phi_\tau^* R_\tau^{-1} z_0,$$

then $\Phi_\tau u = z_0$. *Moreover* u *is the unique minimal norm control.*

Basic HUM Method in Two Lines

1. *Computing* R_τ: For $\eta \in X$ we have $R_\tau \eta = \Phi_\tau v$ where

$$v(t) = B^* \mathbb{T}^*_{\tau - t} \eta \qquad (t \in [0, \tau]).$$

2. *Computing* u: With $\eta = R_\tau^{-1} z_0$ set

$$u(t) = B^* \mathbb{T}^*_{\tau - t} \eta \qquad (t \in [0, \tau]),$$

and you found the minimal norm control.

HUM and the reachable space

Proposition. *Assume that* (A, B) *is approximately controllable in some time* $\tau > 0$. *Then*

$$\|\eta\|_{(\mathrm{Ran}\,\Phi_\tau)'} = \|\Phi_\tau^* \eta\|_{L^2([0,\tau];U)} \qquad (\eta \in X).$$

The reachable space for the constant coefficients heat equation.

The heat equation on the half line (Dirichlet)

$$(LCH) \quad \begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geqslant 0, \ x \in (0,\infty), \\ w(t,0) = u(t), & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\infty)\,, \end{cases}$$

We have a well-posed system with $X = W^{-1,2}(0,\infty)$ and $U = \mathbb{C}$,

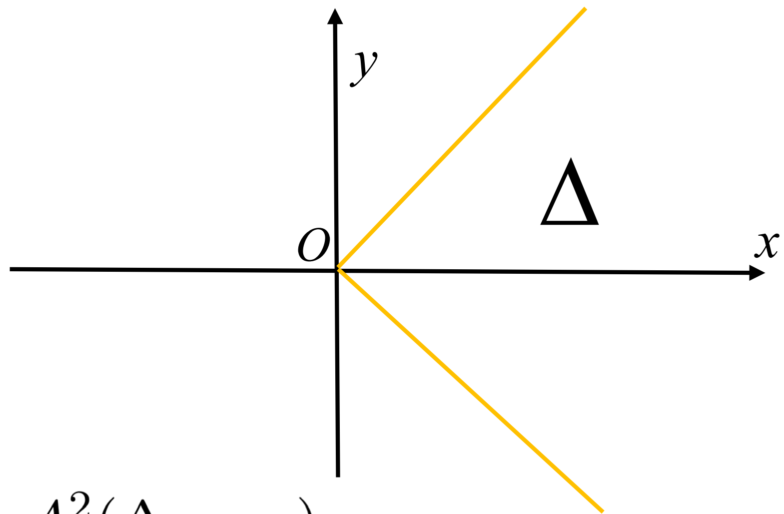
$$\left(\mathbb{T}_\tau^{\text{left}}\psi\right)(x) = \int_0^\infty \left[\frac{\mathrm{e}^{-\frac{(x-y)^2}{4\tau}}}{2\sqrt{\pi\tau}} - \frac{\mathrm{e}^{-\frac{(x+y)^2}{4\tau}}}{2\sqrt{\pi\tau}}\right]\psi(y)\,\mathrm{d}y$$

$$\left(\Phi_\tau^{\text{left}}u\right)(x) = \frac{1}{2\sqrt{\pi}}\int_0^\tau \frac{x\mathrm{e}^{-\frac{x^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{3/2}}u(\sigma)\,\mathrm{d}\sigma$$

A "typically infinite dimensional" example

- Approximately controllable in any positive time (duality)

- $\operatorname{Ran}\Phi_\tau^{\text{left}} \cap \operatorname{Ran}\mathbb{T}_\tau^{\text{left}} = \{0\}$ (an application of Hardy's uncertainty principle, see also Escauriaza, Seregin and Sverak (2003) or Dardé and Ervedoza (2020) for generalizations). No null controllability.



- $\Phi_\tau^{\text{left}}\left(\sqrt{t}L^2([0,\tau];U)\right) = A^2(\Delta, \omega_{0,\tau})$
$:= \{f \in \mathrm{HOL}(\Delta) \mid \int_\Delta |f(x+iy)|^2 \omega_{0,\tau}(x+iy)\,\mathrm{d}x\,\mathrm{d}y < \infty\},$
where $\omega_{0,\tau}(s) = \dfrac{\mathrm{e}^{\frac{\mathrm{Re}\,(s^2)}{2\tau}}}{\tau}$ for $s \in \Delta$ (Aikawa, Hayashi and Saitoh, 1990).

The heat equation on an interval

$$(BCH) \quad \begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geqslant 0, \ x \in (0,\pi), \\ w(t,0) = u_0(t), \quad w(t,\pi) = u_\pi(t) & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\pi), \end{cases}$$

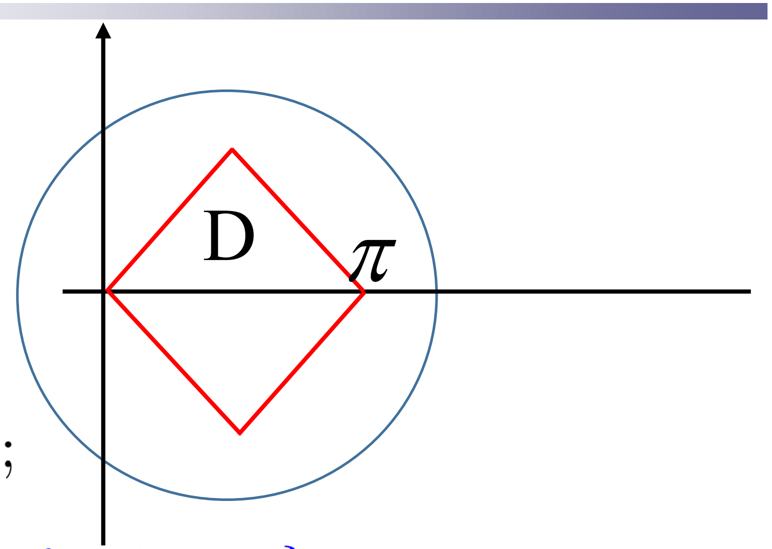
Given $\tau > 0$, define the *input to state map*

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = w(\tau,\cdot) \qquad (\tau > 0, \ u_0, \ u_\pi \in L^2[0,\tau]),$$

The above equations define a well-posed control LTIs with

$$X = H^{-1}(0,\pi) \text{ and } U = \mathbb{C}^2.$$

"Classical" results



Given $\tau > 0$ it is known that:

- $\operatorname{Ran}\Phi_\tau \subset \operatorname{Hol}(D)$, where $D = \{s = x + iy \in \mathbb{C} \ \mid \ |y| < x \ \text{ and } \ |y| < \pi - x\}$;
- $\operatorname{Ran}\Phi_\tau \supset \{\psi \in \operatorname{Hol}(S) \ \mid \ \psi^{(2k)}(0) = \psi^{(2k)}(\pi) = 0 \text{ for } k \in \mathbb{N}\}$, where $S = \{s = x + iy \in \mathbb{C} \ \mid \ |y| < \pi \ \}$ (Fattorini and Russell, 1971);
- $\operatorname{Ran}\Phi_\tau \supset \operatorname{Hol}(B)$, where $B = \left\{ s \in \mathbb{C} \ \mid \ \left|s - \frac{\pi}{2}\right| < \frac{\pi}{2} \mathrm{e}^{(2\mathrm{e})^{-1}} \ \right\}$ (Martin, Rosier and Rouchon, 2016);
- For every $\varepsilon > 0$ we have $\operatorname{Ran}\Phi_\tau \supset \operatorname{Hol}(D_\varepsilon)$, where D_ε is an ε-neighbourhood of the square D (Dardé and Ervedoza, 2016).
- $E^2(D) \subset \operatorname{Ran}\Phi_\tau \subset A^2(D)$ (Hartmann, Kellay and M.T., JEMS, 2021)

Hilbert spaces of analytic functions

Let $\Omega \subset \mathbb{C}$ be an open set with Lipschitz boundary.

The Hardy-Smirnov space $E^2(\Omega)$ is

$$E^2(\Omega) = \left\{ f \in Hol(\Omega) \;\middle|\; \int_{\partial\Omega} |f(\zeta)|^2 \, |\mathrm{d}\zeta| < \infty \right\},$$

The Bergman space with weight ω is

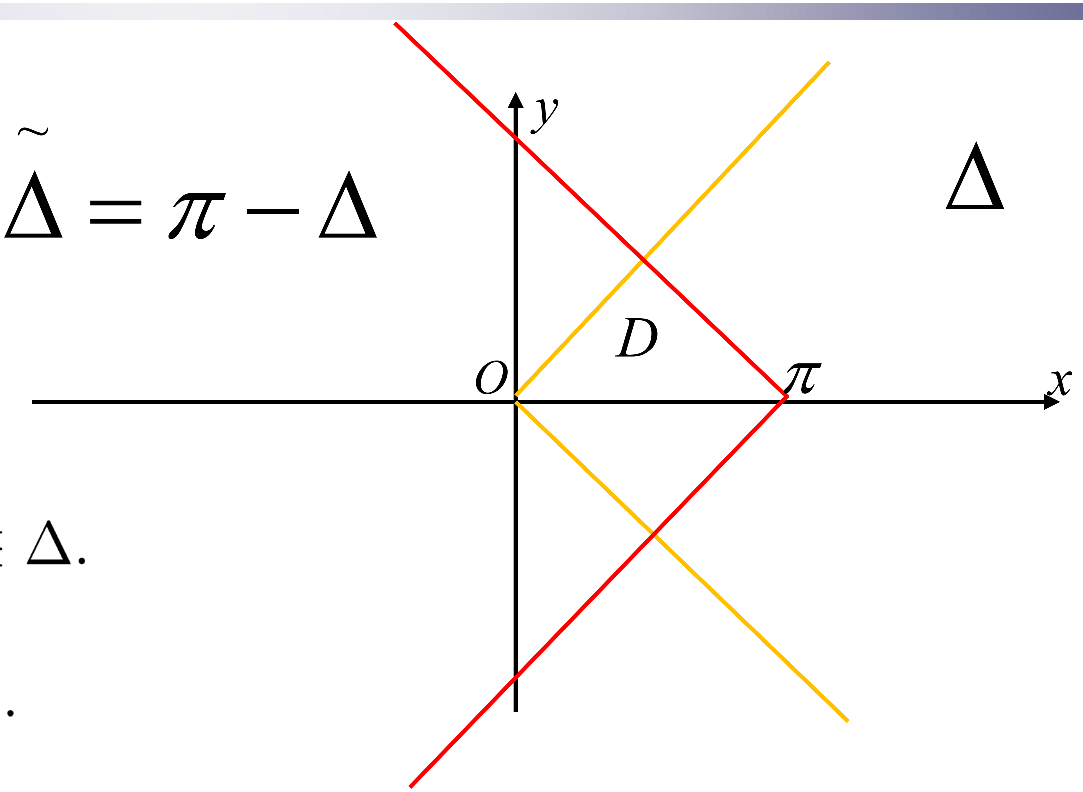
$$A^2(\Omega, \omega) = Hol(\Omega) \cap L^2(\Omega, \omega).$$

For $\omega = 1$ we simply write $A^2(\Omega)$.

Institut de Mathématiques de Bordeaux

Recent results (I): notation

$$\tilde{\Delta} = \pi - \Delta$$



Let $\delta > 0$. $\omega_{0,\delta}(s) = \frac{e^{\frac{\mathrm{Re}\,(s^2)}{2\delta}}}{\delta}$ for $s \in \Delta$.

Let $\omega_{\pi,\delta}(\tilde{s}) = \omega_{0,\delta}(\pi - \tilde{s})$ for $\tilde{s} \in \tilde{\Delta}$.

Let $X_\delta = A^2(\Delta, \omega_{0,\delta}) + A^2(\tilde{\Delta}, \omega_{\pi,\delta})$.

$$\|\varphi\|_\delta = \inf \left\{ \|\varphi_0\|_{A^2(\Delta,\omega_{0,\delta})} + \|\varphi_\pi\|_{A^2(\tilde{\Delta},\omega_{\pi,\delta})} \;\middle|\; \begin{array}{c} \varphi_0 + \varphi_\pi = \varphi \\ \varphi_0 \in A^2(\Delta, \omega_{0,\delta}) \\ \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \end{array} \right\}.$$

Recent results (II):

Theorem 1. (Kellay, Normand and M.T., Analysis & PDE, 2022)
For every $\tau > 0$ we have $\operatorname{Ran} \Phi_\tau = X_\tau$.

Proposition 1. (Kellay, Normand and M.T., 2022)
For every $\delta > 0$ we have $X_\delta = A^2(\Delta) + A^2(\tilde{\Delta})$.

Corollary 1. (Orsoni, 2019, Kellay, Normand and M.T., 2019)
We have $\operatorname{Ran} \Phi_\tau = A^2(\Delta) + A^2(\tilde{\Delta})$ for every $\tau, \ \delta > 0$.

Theorem 2. (Hartmann and Orsoni., 2020)
$A^2(\Delta) + A^2(\tilde{\Delta}) = A^2(D)$, thus $\operatorname{Ran} \Phi_\tau = A^2(D)$.

Institut de Mathématiques de Bordeaux

Proof of Proposition 1 (I)

$$(\Phi_\tau u)(x) = (\Phi_\tau^{\text{left}} u_0)(x) + \left(\Phi_\tau^{\text{right}} u_\pi\right)(x) + \int_0^\tau \frac{\partial \tilde{K}_0}{\partial x}(\tau-\sigma, x) u_0(\sigma)\,\mathrm{d}\sigma$$
$$+ \int_0^\tau \frac{\partial \tilde{K}_\pi}{\partial x}(\tau-\sigma, x) u_\pi(\sigma)\,\mathrm{d}\sigma \qquad (x \in (0,\pi)),$$

$$(\Phi_\tau^{\text{left}} u_0)(s) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{\mathrm{e}^{-\frac{s^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{3/2}} s u_0(\sigma)\,\mathrm{d}\sigma, \qquad (\Phi_\tau^{\text{right}} u_\pi)(s) = (\Phi_\tau^{\text{left}} u_\pi)(\pi - s)$$

$$\tilde{K}_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}^*} \mathrm{e}^{-\frac{(x+2m\pi)^2}{4\sigma}}, \qquad \tilde{K}_\pi(\sigma, x) = \sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}^*} \mathrm{e}^{-\frac{(x+(2m-1)\pi)^2}{4\sigma}}.$$

Since $\left(\Phi_\tau - \Phi_\tau^{\text{left}} - \Phi_\tau^{\text{right}}\right)(\sqrt{t}u)$ is “small” in X_τ, it suffices to consider only the first two terms in the right-hand side.

Several space dimensions

$$(BCHn)\qquad \begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \Delta w(t,x) & t \geqslant 0,\ x \in \Omega, \\[2ex] w(t,\cdot) = u, & t \in [0,\infty),\ x \in \partial\Omega \\[2ex] w(0,x) = 0 & x \in \Omega, \end{cases}$$

Given $\tau > 0$, define the *input to state map*

$$\Phi_\tau u = w(\tau,\cdot) \qquad\qquad (\tau > 0,\ u \in L^2([0,\tau]; L^2(\partial\Omega)).$$

Theorem (Strohmaier and Waters, 2020).
If Ω is a ball then for every $\tau > 0$ we have $\operatorname{Ran}\Phi_\tau \supset Hol\left(\overline{\mathcal{E}(\Omega)}\right)$, where

$$\mathcal{E}(\Omega) = \{x + iy \in \mathbb{C}^n \ \mid\ x \in \Omega,\ |y| < d(x,\partial\Omega)\}\,.$$

Robustness of the reachable space with respect to perturbations of the generator

Institut de Mathématiques de Bordeaux

Perturbations of the generator

- The reachable space of a finite dimensional LTI is not, in general, robust with respect to small perturbations of the generator (exercice using Kalman's matrix).
- For infinite dimensional LTIs the exact controllability property and thus the reachable space, are robust with respect to small perturbations of the generator.
- Can a similar robustness property be obtained for LTI's with a weaker controllability property?

Main results (Ervedoza, Le Balch', MT, JFA, 2022)

Theorem 3. Assume that (A, B) is null controllable in any time and that $P \in \mathcal{L}(X) \cap \mathcal{L}(\operatorname{Ran}\Phi_{\tau_0})$ for some $\tau_0 > 0$.
Then there exists $\delta_{\tau_0} > 0$ such that if $\|P\|_{\mathcal{L}(\operatorname{Ran}\Phi_{\tau_0})} \leq \delta_{\tau_0}$, then

$$\operatorname{Ran}\Phi_{\tau_0}^P = \operatorname{Ran}\Phi_{\tau_0}.$$

Theorem 4. Suppose that $A < 0$ has compact resolvents, $B \in \mathcal{L}(U, X_{-\alpha})$ for some $\alpha \in [0, 1/2]$, and that (A, B) is null-controllable in any time $\tau > 0$. Moreover, suppose that $P \in \mathcal{L}(X_{1-\alpha-\varepsilon}, \operatorname{Ran}\Phi_\tau)$, where $\alpha \in [0, 1/2]$ and $\varepsilon \in (0, 1-\alpha]$. Finally, suppose that the pair $(A + P, B)$ satisfies the Hautus type condition $\operatorname{Ker}(sI - A - P^*) \cap \operatorname{Ker} B^* = \{0\}$ for all $s \in \mathbb{C}$. Then for every $\tau > 0$ we have $\operatorname{Ran}\Phi_\tau^P = \operatorname{Ran}\Phi_\tau$, and $\operatorname{Ran}\mathbb{T}_\tau^P \subset \operatorname{Ran}\Phi_\tau^P$, that is the system $(A + P, B)$ is null-controllable in any time $\tau > 0$.

Main ingredient of the proof

Theorem 4.(Ervedoza, Le Balch and M.T., 2022)

Let $\Sigma = (\mathbb{T}, \Phi)$ be a well posed control system which is null controllable in any positive time. For $\tau > 0$ we denote by $\tilde{\mathbb{T}} = \left(\tilde{\mathbb{T}}_t\right)_{t \geqslant 0}$ the semigroup of operators defined by

$$\tilde{\mathbb{T}}_t = \mathbb{T}_t|_{\mathrm{Ran}\,\Phi_\tau}, \qquad (t \geqslant 0).$$

Then the family $\tilde{\mathbb{T}} = (\mathbb{T}_t|_{\mathrm{Ran}\,\Phi_\tau})_{t \geqslant 0}$ does not depend on the choice of $\tau > 0$, and forms a C^0 semigroup on $\mathrm{Ran}\,\Phi_\tau$. Moreover, the couple $\tilde{\Sigma} = (\tilde{\mathbb{T}}, \Phi)$ determines a well-posed control system with state space $\mathrm{Ran}\,\Phi_\tau$ and input space U. Finally, this system is exactly controllable in any positive time.

Idea of the proof (1)

It suffices to prove that for every $\tau > 0$ there exists a constant $c_\tau > 0$ such that

$$\|\mathbb{T}_t\|_{\mathcal{L}(\operatorname{Ran}\Phi_\tau)} \leqslant c_\tau \qquad (t \in (0,\tau]),$$

and then apply a classical result of Hille. First note that that

$$\|\mathbb{T}_t \eta\|_{\operatorname{Ran}\Phi_\tau} \leqslant c_\tau \|\mathbb{T}_t \eta\|_{\operatorname{Ran}\Phi_{2\tau}} \qquad (t \in (0,\tau],\ \eta \in \operatorname{Ran}\Phi_\tau).$$

On the other hand, we will see that

$$\|\mathbb{T}_t \eta\|_{\operatorname{Ran}\Phi_{2\tau}} \leqslant \|\eta\|_{\operatorname{Ran}\Phi_{2\tau-t}} \qquad (t \in (0,\tau],\ \eta \in \operatorname{Ran}\Phi_\tau).$$

We can thus combine the last two inequalities to obtain that

$$\|\mathbb{T}_t \eta\|_{\operatorname{Ran}\Phi_\tau} \leqslant c_\tau \|\eta\|_{\operatorname{Ran}\Phi_{2\tau-t}} \qquad (t \in (0,\tau],\ \eta \in \operatorname{Ran}\Phi_\tau).$$

Since $2\tau - t \geqslant \tau$, the last estimate implies the conclusion.

Idea of the proof (2)

Proof of the inequality:

$$\|\mathbb{T}_t\eta\|_{\mathrm{Ran}\,\Phi_{2\tau}} \leqslant \|\eta\|_{\mathrm{Ran}\,\Phi_{2\tau-t}} \qquad (t \in (0,\tau],\ \eta \in \mathrm{Ran}\,\Phi_\tau).$$

If $u \in L^2([0,2\tau-t];U)$ is such that $\Phi_{2\tau-t}u = \eta$ then $\tilde{u} \in L^2([0,2\tau];U)$ defined by

$$\tilde{u}(t) = \begin{cases} u(t) & (t \in [0,2\tau-t]), \\ 0 & (t \in (2\tau-t,2\tau]), \end{cases}$$

satisfies $\Phi_{2\tau}\tilde{u} = \mathbb{T}_t\eta$ and $\|\tilde{u}\|_{L^2([0,2\tau];U)} = \|u\|_{L^2([0,2\tau-t];U)}$. So we get

$$\|\mathbb{T}_t\eta\|_{\mathrm{Ran}\,\Phi_{2\tau}} = \|\Phi_{2\tau}\tilde{u}\|_{\mathrm{Ran}\,\Phi_{2\tau}} \leq \|\tilde{u}\|_{L^2([0,2\tau];U)} = \|u\|_{L^2([0,2\tau-t];U)},$$

for every $u \in L^2([0,2\tau-t];U)$ such that $\Phi_{2\tau-t}u = \eta$, then by taking the infimum of $u \in L^2([0,2\tau-t];U)$ such that $\Phi_{2\tau-t}u = \eta$ in both sides of the previous inequality we obtain the desired inequality.

Institut de Mathématiques de Bordeaux

Applications to the perturbed heat equation

1D heat equation with Neumann boundary control

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) = 0 & (t \geqslant 0,\ x \in (0,\pi)), \\ \dfrac{\partial z}{\partial x}(t,0) = u_0(t), \quad \dfrac{\partial z}{\partial x}(t,\pi) = u_\pi(t) & (t \geqslant 0), \\ z(0,x) = 0 & (x \in (0,\pi)), \end{cases}$$

$$A = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \text{ on } X = L^2[0,\pi],\ \mathcal{D}(A) = \left\{ z \in H^2(0,\pi),\ \frac{\mathrm{d}z}{\mathrm{d}x}(0) = \frac{\mathrm{d}z}{\mathrm{d}x}(\pi) = 0 \right\}.$$

$$B \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = -u_0 \delta_0 + u_\pi \delta_\pi .$$

Null-controllable in any time $\tau > 0$. [Fattorini Russell 1971]

Known result

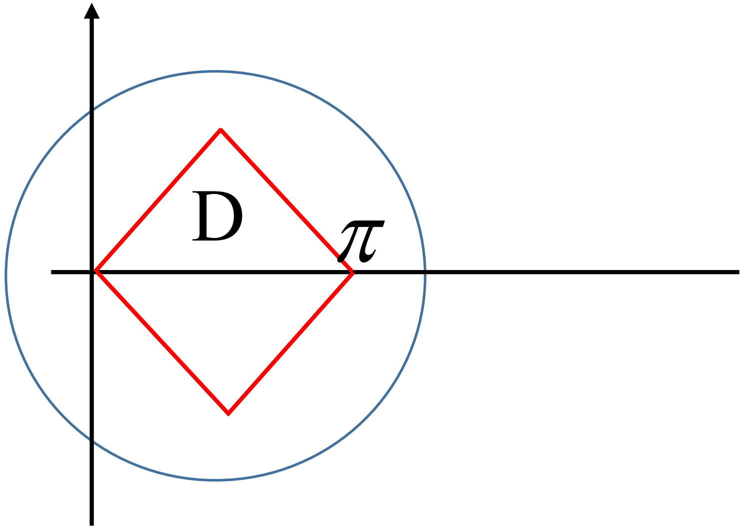
Theorem 4. ([Hartmann-Orsoni 2021], ...,[Hartman, Kellay, Tucsnak, 2021])
The reachable space of the above 1d heat equation is independent of the time horizon $\tau > 0$ and, for all $\tau > 0$,

$$\operatorname{Ran}\Phi_\tau = A^{1,2}(D),$$

where

$$D = \{s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x\}.$$

and $A^{1,2}(D) = \{f \in \operatorname{Hol}(D) \cap W^{1,2}(D)\}$



First application: small potentials

Proposition.
There exists $\varepsilon > 0$, such that if $p \in \mathrm{Hol}\,(D) \cap W^{1,\infty}(D)$ with $\|p\|_{W^{1,\infty}(D)} \leqslant \varepsilon$, the reachable set for the equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) + p(x) z(t,x) = 0 & (t \geqslant 0,\ x \in (0,\pi)), \\ \dfrac{\partial z}{\partial x}(t,0) = u_0(t), \quad \dfrac{\partial z}{\partial x}(t,\pi) = u_\pi(t) & (t \geqslant 0), \\ z(0,x) = 0 & (x \in (0,\pi)), \end{cases}$$

is independent of the time horizon and coincides with $A^{1,2}(D)$

Non local perturbations (inspired by [Cara and Zuazua, 2016])

Proposition. Let $K \in L^2([0,\pi]\times[0,\pi])$ be such that $x \mapsto K(x,y)$ is in $\mathrm{Hol}(D)\cap W^{1,2}(D)$ and that $K \in L^2_y([0,\pi]; W^{1,2}_x(D))$. Suppose that

$$\begin{cases} -\dfrac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(x) - s\psi(x) = \displaystyle\int_0^\pi \overline{K(y,x)}\psi(y)\,\mathrm{d}y, & (x\in[0,\pi]),\\ \psi(0) = \dfrac{\mathrm{d}\psi}{\mathrm{d}x}(0) = 0, \qquad \psi(\pi) = \dfrac{\mathrm{d}\psi}{\mathrm{d}x}(\pi) = 0. & \end{cases}$$

iff $\psi = 0$. Then the reachable space of

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) - \dfrac{\partial^2 z}{\partial x^2}(t,x) + \int_0^\pi K(x,y)z(t,y)\,\mathrm{d}y = 0 & (t\geqslant 0,\ x\in(0,\pi)),\\ \dfrac{\partial z}{\partial x}(t,0) = u_0(t), \quad \dfrac{\partial z}{\partial x}(t,\pi) = u_\pi(t) & (t\geqslant 0),\\ z(0,x) = 0 & (x\in(0,\pi)), \end{cases}$$

is independent of the time horizon and coincides with $A^{1,2}(D)$

Reachability with smooth inputs (nonlinear perturbations)

Proposition 3 . For $\tau > 0$ and $n \in \mathbb{N}$ we set

$$A_n^2(D) := \left\{ \psi \in A^2(D) \;\middle|\; \frac{\mathrm{d}^{2k}\psi}{\mathrm{d}s^{2k}} \in A^2(D) \text{ for } k = 1, \dots, n \right\} \qquad (n \geqslant 1).$$

$$W_{\mathrm{L}}^{n,2}(0,\tau) = \left\{ v \in W^{n,2}(0,\tau) \;\middle|\; v(0) = \cdots = \frac{\mathrm{d}^{n-1}v}{\mathrm{d}t^{n-1}}(0) = 0 \right\} \qquad (n \geqslant 1).$$

Then for every $\psi \in A_n^2(D)$ there exist $u_0,\ u_\pi \in W_{\mathrm{L}}^{n,2}(0,\tau)$ with $\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi$.

Remark. We conjecture that the following "analytic" version holds: for every $\psi \in Hol(\tilde{D})$, where $\tilde{D} \subset \mathbb{C}$ is an open set containing $\overline{D}$, there exist Gevrey type controls $u_0,\ u_\pi$, with all derivatives vanishing at $t = 0$, such that $\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi$.

Concluding remarks

Connections with the control cost

Assuming that the system $(\mathbb{T}, \Phi)$ is null controllable in some time $\tau > 0$ (this means that $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$.), *the cost of null controllability in time* τ is the number c_τ defined by $c_\tau = \sup_{\|\psi\|_X \leqslant 1} \|\mathbb{T}_\tau \psi\|_{\operatorname{Ran} \Phi_\tau}$.
For our boundary controlled heat equation we set

$$d_\tau = \sup_{\|\psi\|_{W^{-1,2}(0,\pi)} \leqslant 1} \|\mathbb{T}_\tau \psi\|_\tau.$$

Proposition 4.
With the above notation we have

$$\limsup_{\tau \to 0+} \frac{c_\tau}{d_\tau} \leqslant 1.$$

Questions to be studied

- Linear heat equations in several space dimensions
- Internal control
- Nonlinear heat equations (viscous Burgers,…)
- Other PDEs (Stokes, …)