Infinite dimensional linear systems (1)
General Introduction and Admissible control operators (or well posed control LTI's)

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Aim and contents

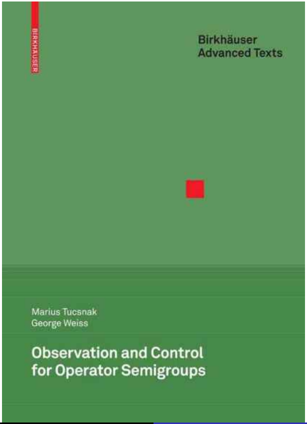
The general aim of these lectures is to coherently cover subjects going from introductory material to new challenging topics and open questions.

Contents:

1. Admissible control operators (or well posed control LTI's)
2. Reachable spaces and controllability, with focus on heat equation systems
3. Hautus conditions and perturbations for time reversible systems
4. Time optimal control problems.

Today Lecture: Admissible control operators (or well posed control LTI's)

Main reference : https://link.springer.com/book/10.1007/978-3-7643-8994-9



First definitions

Definition 1

A family $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ in $\mathcal{L}(X)$, with X a Hilbert space, is a *strongly continuous (operator) semigroup* on X if

(1) $\mathbb{T}_0 = I$,

(2) $\mathbb{T}_{t+\tau} = \mathbb{T}_t \mathbb{T}_\tau$ for every $t, \tau \geqslant 0$ (the semigroup property),

(3) $\lim\limits_{t \to 0,\, t>0} \mathbb{T}_t z_0 = z_0$, for all $z_0 \in X$ (strong continuity).

Definition 2

$A : \mathcal{D}(A) \to X, \ \ \mathcal{D}(A) = \left\{ z \in X \;\middle|\; \lim_{t\to 0,\ t>0} \frac{\mathbb{T}_t z - z}{t} \text{ exists} \right\},$

$$Az = \lim_{t \to 0,\ t>0} \frac{\mathbb{T}_t z - z}{t} \qquad\qquad (z \in \mathcal{D}(A)),$$

is called the *generator* of $\mathbb{T}$.

First examples associated to differential equations

Example 3

For $A \in \mathcal{L}(X)$ we put $\mathbb{T}_t = \mathrm{e}^{tA} := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$.

Example 4

Let $\tau > 0$, take $X = L^2[0, a]$ and for every $t \in \mathbb{R}$ and $z \in X$ define

$$(\mathbb{T}_t z)(x) = \begin{cases} z(x+t) & \text{if } \ x + t \leqslant a, \\ 0 & \text{else.} \end{cases}$$

Moreover,

$$A = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad \mathcal{D}(A) = \{z \in W^{1,2}(0, a) \mid z(a) = 0\}.$$

Sufficient conditions for semigroup generation

Theorem 5

Let $\mathcal{D}(A)$ be a dense subspace of X. If $A : \mathcal{D}(A) \to X$ satisfies one of the conditions:

- $A = A^* \leqslant m\mathbb{I}_X$, *for some* $m \in \mathbb{R}$.
- $A = -A^*$,
- $A = \tilde{A} + P$, *where* $\tilde{A} : \mathcal{D}(A) \to X$ *generates a* C^0*-semigroup on* X *and* $P \in \mathcal{L}(X)$,

then A *generates a* C^0*-semigroup on* X*. In the second case* $\mathbb{T}$ *can be extended to a unitary group.*

Further examples

Example 6

The Dirichlet Laplacian on an open set Ω with “smooth“ $\partial\Omega$ is selfadjoint and non negative on $L^2(\Omega)$,

Example 7

The *wave operator*

$$A = \begin{bmatrix} 0 & \mathbb{I} \\ \Delta & 0, \end{bmatrix}$$

is skew-adjoint on $X = W_0^{1,2}(\Omega) \times L^2(\Omega)$.

The space X_{-1}

Definition 8

Let $V \subset X$ be a dense subspace. Then the dual V' of V with respect to the *pivot space* X is the completion of X with respect to the norm on X defined by $\|f\|_{V'} = \sup_{\|\varphi\|_V \leqslant 1} |\langle \varphi, f \rangle_X|$. For $V = \mathcal{D}(A^*)$, with A a semigroup generator, V' is denoted by X_{-1}.

Theorem 9

Suppose that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. Let X_1 be $\mathcal{D}(A)$ endowed with the graph norm. Then for every $t \geqslant 0$, $\mathbb{T}_t$ has a restriction which is in $\mathcal{L}(X_1)$ and a unique extension $\widetilde{\mathbb{T}}_t$ which is in $\mathcal{L}(X_{-1})$. Moreover, these new families of operators are similar to the original semigroup.

Extrapolation spaces or "towers of Hilbert spaces"

Remark 1

The construction of X_1 and X_{-1} can be iterated, in both directions, so that we obtain the infinite sequence of spaces

$$\dots X_2 \subset X_1 \subset X \subset X_{-1} \subset X_{-2} \dots$$

each inclusion being dense and with continuous embedding. For each $k \in \mathbb{Z}$, the original semigroup $\mathbb{T}$ has a restriction (or an extension) to X_k which is the image of $\mathbb{T}$ through the unitary operator $(\beta I - A)^{-k} \in \mathcal{L}(X, X_k)$. The space X_{-2} occasionally arises in the proof of theorems in infinite-dimensional systems theory.

Linear differential equations in Hilbert spaces (I)

Definition 10

Consider the differential equation

$$\dot{z}(t) = Az(t) + f(t), \tag{1}$$

where $f \in L^1_{\mathrm{loc}}([0,\infty); X_{-1})$. A *solution of* (1) *in* X_{-1} is a function

$$z \in L^1_{\mathrm{loc}}([0,\infty); X) \cap C([0,\infty); X_{-1})$$

which satisfies the following equations in X_{-1}:

$$z(t) - z(0) = \int_0^t \left[Az(\sigma) + f(\sigma)\right] \mathrm{d}\sigma \qquad (t \in [0,\infty)). \tag{2}$$

Linear differential equations in Hilbert spaces (II)

Remark 2

We could also define the concept of a "weak solution of (1) in X_{-1}", by requiring instead of (2) that for every $\varphi \in X_1^d$ (which designs $\mathcal{D}(A^*)$ endowed with the graph norm) and every $t \geqslant 0$,

$$\langle z(t)-z(0), \varphi\rangle_{X_{-1},X_1^d} = \int_0^t \left[\langle z(\sigma), A^*\varphi\rangle_X + \langle f(\sigma), \varphi\rangle_{X_{-1},X_1^d}\right] \mathrm{d}\sigma.$$

Remark 3

If $f \in L^1_{\text{loc}}([0,\infty); X)$ then the concept of a solution of (1) in X can be defined similarly, by replacing everywhere in Definition 10 the space X_{-1} by X and X by X_1. This concept of a solution appears often in the literature, being designed as *weak solution*.

Mild=Weak and a regularity result

Proposition 1

With the notation of Definition 10*, suppose that* z *is a solution of* (1) *in* X_{-1} *and denote* $z_0 = z(0)$*. Then* z *is given by*

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\sigma} f(\sigma) \mathrm{d}\sigma. \tag{3}$$

Theorem 11

If $z_0 \in X$ *and* $f \in W^{1,\infty}_{loc}((0,\infty); X_{-1})$*, then the equation* (1) *has a unique solution in* X_{-1}*, denoted* z*, that satisfies* $z(0) = z_0$*. Moreover, this solution is such that*

$$z \in C([0,\infty); X) \cap C^1([0,\infty); X_{-1}),$$

Basic ingredients

- The Hilbert spaces U, X and Y
- U is the *input space*, X is the *state space* and Y is the *output space*
- A family of operators

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \end{bmatrix} : X \times L^2([0,\infty);U) \to X$$

- Some notation: for $u, v \in L^2_{\text{loc}}([0,\infty);W)$ and $\tau \geqslant 0$, the *τ-concatenation* of u and v, denoted $u \underset{\tau}{\diamondsuit} v$, is the function defined by

$$(u \underset{\tau}{\diamondsuit} v)(t) = \begin{cases} u(t) & \text{for } t \in [0,\tau], \\ v(t-\tau) & \text{for } t > \tau. \end{cases}$$

First basic definition

Definition. $(\Sigma_t)_{t\geqslant 0} = (\mathbb{T}_t, \Phi_t)_{t\geqslant 0}$ define a *well-posed linear system* withe state space X and input space U if

1. $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is an operator semigroup on X,
2. The *input maps* $(\Phi_t)_{t\geqslant 0}$ are in $\mathcal{L}\left(L^2([0,\infty);U), X\right)$ and

$$\Phi_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v \qquad (u,\ v \in L^2([0,\infty);U)).$$

If U, X are finite dimensional then $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$ and

$$\mathbb{T}_t = \mathrm{e}^{tA},\ \Phi_t u = \int_0^t \mathrm{e}^{(t-\sigma)A} B u(\sigma)\,\mathrm{d}\sigma \qquad (t \geqslant 0).$$

We thus retrieve the standard description of finite dimensional LTIs

$$\dot{z}(t) = Az(t) + Bu(t).$$

Representation of infinite dimensional systems

Theorem 12 (Weiss [2] (1989), M.T. and Weiss, [1] (2014))

Let $(\mathbb{T}, \Phi)$ *be a well posed system. Then there is a unique* $B \in \mathcal{L}(U, X_{-1})$ *such that*

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\,\mathrm{d}\sigma \qquad \left(t \geqslant 0,\ u \in L^2([0,\infty); U)\right). \tag{4}$$

Moreover,for any $z_0 \in X$ *the function*

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \qquad (t \geqslant 0)$$

is the (unique) mild solution of $\dot{z}(t) = Az(t) + Bu(t)$ *with* $z(0) = z_0$.

Admissible control operators

Let $B \in \mathcal{L}(U, X_{-1})$. For $\tau \geq 0$, we define $\Phi_\tau \in \mathcal{L}(U, X_{-1})$ by (4).

Definition 13

Let $(\mathbb{T}, \Phi)$ be a well posed system. Then there is a unique $B \in \mathcal{L}(U, X_{-1})$ such that

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\, \mathrm{d}\sigma \qquad \left(t \geqslant 0,\ u \in L^2([0,\infty); U)\right).$$

Moreover,for any $z_0 \in X$ the function

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \qquad (t \geqslant 0)$$

is the (unique) mild solution of $\dot{z}(t) = Az(t) + Bu(t)$ with $z(0) = z_0$.

A duality result

Theorem 14

Suppose that $B \in \mathcal{L}(U, X_{-1})$. Then B is an admissible control operator for $\mathbb{T}$ if and only if B^* is an admissible observation operator for $\mathbb{T}^*$, i.e., *for evert $\tau > 0$ there exists $k_\tau > 0$ with*

$$\int_O^\tau \|B^*\mathbb{T}_t^* f\|_U^2 \,\mathrm{d}t \leqslant k_\tau^2 \|f\|_X^2 \qquad (f \in \mathcal{D}(A^*)).$$

Remark 4

In the above theorem, and in most of the results to follow, X and U are identified with their duals. This means in particular, that the dual of any subspace of X will be taken with respect to the pivot space X.

Heat equation with Dirichlet b.c. on a half-line (I)

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(t,x) = \dfrac{\partial^2 \varphi}{\partial x^2}(t,x) & (t \geqslant 0,\ x \in (0,\infty)), \\ \varphi(t,0) = 0, & (t \in [0,\infty)), \end{cases} \tag{5}$$

Proposition 2

Let X be one of the spaces $W_0^{1,2}(0,\infty)$, $L^2(0,\infty)$ or $W^{-1,2}(0,\infty)$. Then equations (5) determine a C^0-semigroup $\mathbb{T}^{\mathrm{left}}$ on X with

$$\left(\mathbb{T}^{\mathrm{left}}_\tau \psi\right)(x) = \int_0^\infty \left[\frac{\mathrm{e}^{-\frac{(x-y)^2}{4\tau}}}{2\sqrt{\pi\tau}} - \frac{\mathrm{e}^{-\frac{(x+y)^2}{4\tau}}}{2\sqrt{\pi\tau}} \right] \psi(y)\,\mathrm{d}y.$$

Heat equation with Dirichlet b.c. on a half-line (II)

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \dfrac{\partial^2 z}{\partial x^2}(t,x) & (t \geqslant 0,\ x \in (0,\infty)), \\[2ex] z(t,0) = u(t), & (t \in [0,\infty)), \end{cases} \tag{6}$$

Proposition 3

(6) *determines a well-posed control LTI, denoted* $(\mathbb{T}^{\text{left}}, \Phi^{\text{left}})$ *with* $X = W^{-1,2}(0,\pi)$, $U = \mathbb{C}$ *where* $\mathbb{T}^{\text{left}}$ *is given in Proposition 2 and*

$$(\Phi_\tau^{\text{left}} u)(x) = \frac{1}{2\sqrt{\pi}} \int_0^\tau \frac{\mathrm{e}^{-\frac{x^2}{4(\tau-\sigma)}}}{(\tau-\sigma)^{3/2}} x u(\sigma)\,\mathrm{d}\sigma.$$

Heat equation with Dirichlet b.c. on an interval (I)

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(t,x) = \dfrac{\partial^2 \varphi}{\partial x^2}(t,x) & (t \geqslant 0,\ x \in (0,\pi)), \\ \varphi(t,0) = 0,\ \varphi(t,\pi) = 0, & (t \in [0,\infty)), \end{cases} \tag{7}$$

Proposition 4

Let X be one of the spaces $W_0^{1,2}(0,\pi)$, $L^2(0,\pi)$ or $W^{-1,2}(0,\pi)$. Then equations (7) determine a C^0-semigroup $\mathbb{T}$ on X with

$$\begin{aligned} (\mathbb{T}_\tau \psi)(x) = \frac{1}{2\sqrt{\tau\pi}} \int_0^\pi \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\frac{(x-\xi+2m\pi)^2}{4\tau}} \psi(\xi)\,\mathrm{d}\xi \\ - \frac{1}{2\sqrt{\tau\pi}} \int_0^\pi \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\frac{(x+\xi+2m\pi)^2}{4\tau}} \psi(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{8}$$

Heat equation with Dirichlet b.c. on an interval (II)

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \dfrac{\partial^2 z}{\partial x^2}(t,x) & (t \geqslant 0,\ x \in (0,\pi)), \\ z(t,0) = u_0(t),\ z(t,\pi) = u_\pi(t) & (t \in [0,\infty)), \end{cases} \tag{9}$$

Proposition 5

(9) *determines a well-posed control LTI, denoted* $(\mathbb{T}, \Phi)$ *with* $X = W^{-1,2}(0,\pi)$, $U = \mathbb{C}$ *where* $\mathbb{T}$ *is given in Proposition 4 and*

$$(\Phi_\tau u)(x) = (\Phi_\tau^{\text{left}} u_0)(x) + \left(\Phi_\tau^{\text{right}} u_\pi\right)(x) + \int_0^\tau \frac{\partial \tilde{K}_0}{\partial x}(\tau - \sigma, x) u_0(\sigma)\, \mathrm{d}\sigma + \int_0^\tau \frac{\partial \tilde{K}_\pi}{\partial x}(\tau - \sigma, x) u_\pi(\sigma)\, \mathrm{d}\sigma,$$

Continued

$$\tilde{K}_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}^*} \mathrm{e}^{-\frac{(x+2m\pi)^2}{4\sigma}},$$

$$\tilde{K}_\pi(\sigma, x) = \sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}^*} \mathrm{e}^{-\frac{(x+(2m-1)\pi)^2}{4\sigma}}.$$

Proof.

It suffices to solve the Cauchy problem with the initial data

$$\tilde{\psi}(\eta) = \begin{cases} \psi(\eta + 2m\pi) & \eta \in [-2m\pi, -(2m-1)\pi] \\ -\psi(\eta + 2m\pi) & \eta \in [-(2m+1)\pi, -2m\pi], \end{cases}$$

with $m \in \mathbb{Z}$. □

M. Tucsnak and G. Weiss, *Well-posed systems-the LTI case and beyond*, Automatica, 50 (2014), pp. 1757–1779.

G. Weiss, *Admissibility of unbounded control operators*, SIAM J. Control Optim., 27 (1989), pp. 527–545.