Control and stabilization of partial differential equations systems with essential quadratic components

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Recent advances on control theory of PDE systems
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Outline

A KdV control system

Following L. Rosier (1997), we consider the KdV control system

(1) $$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \; t \in [0,T], \; x \in [0,L], \\ y(t,0) = y(t,L) = 0, \; y_x(t,L) = u(t), \; t \in [0,T], \end{cases}$$

where, at time $t \in [0,T]$, the state is $y(t,\cdot) \in L^2(0,L)$ and the control is $u(t) \in \mathbb{R}$.

Controllability of the linearized control system

The linearized control system (around 0) is

$$(2) \qquad \begin{cases} y_t + y_x + y_{xxx} = 0,\ t \in [0,T],\ x \in [0,L], \\ y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T], \end{cases}$$

where, at time $t \in [0,T]$, the state is $y(t,\cdot) \in L^2(0,L)$ and the control is $u(t) \in \mathbb{R}$.

Theorem (L. Rosier (1997))

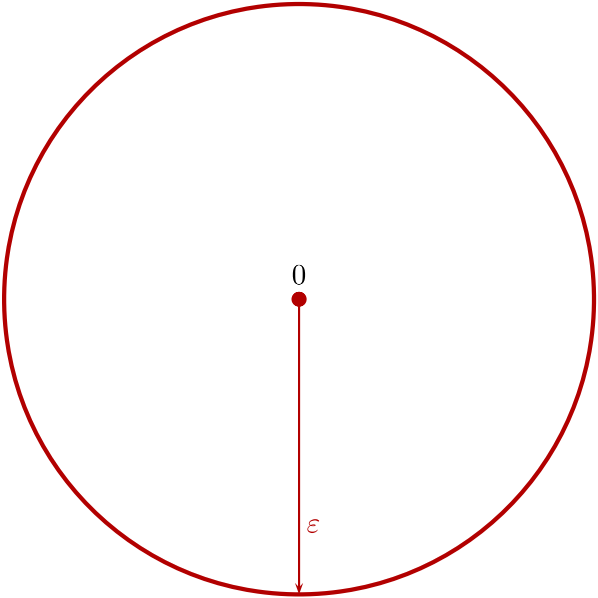
For every $T > 0$, the linearized control system is controllable in time T if and only

$$(3) \qquad L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}},\ k \in \mathbb{N}^*,\ l \in \mathbb{N}^* \right\}.$$

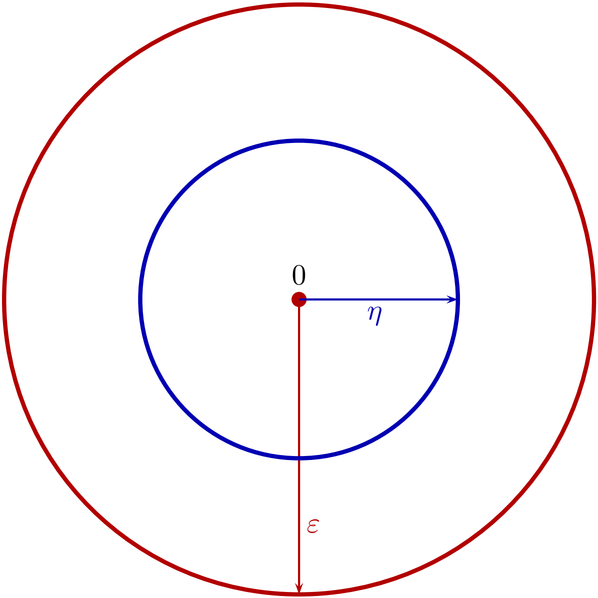
Moreover, if $L \in \mathcal{N}$, the uncontrollable part is a linear space (later denoted $\mathcal{M}$) of finite dimension.

Local controllability in time T

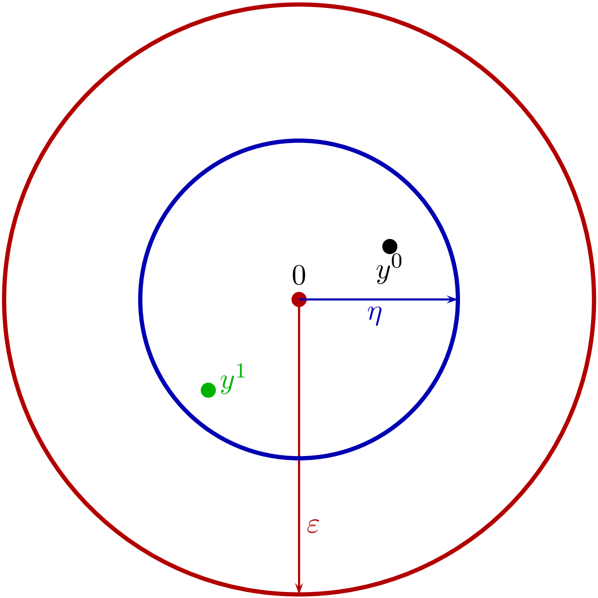
Local controllability in time T



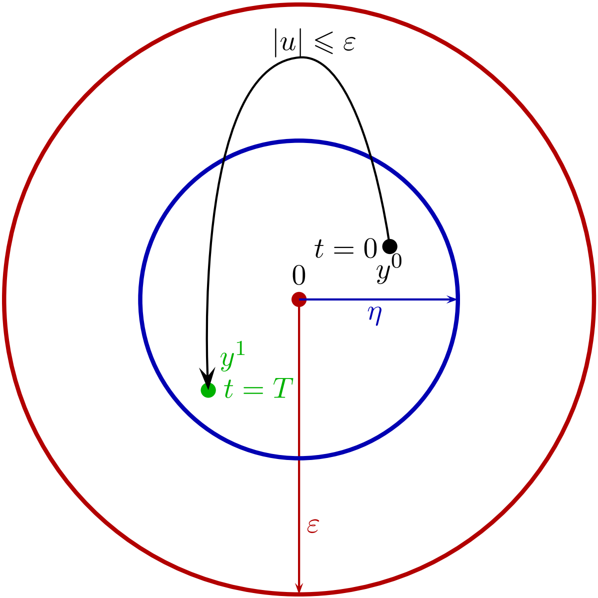
Local controllability in time T



Local controllability in time T



Local controllability in time T



Application to the nonlinear system

Theorem (L. Rosier (1997))

For every $T > 0$, the KdV control system is locally controllable (around 0) in time T if $L \notin \mathcal{N}$ for the L^2-norm for the state and the L^2-norm for the control.

Remark

The above controllability property is called Small-Time Local Controllability, STLC in short: the time, the state, and the controls are small (for suitable norms).

Controllability when $L \in \mathcal{N}$

Theorem (STLC if $\dim(\mathcal{M}) = 1$, JMC and E. Crépeau (2004))

If the uncontrollable part $\mathcal{M}$ of the linearized system is of dimension 1, for every $T > 0$ the KdV control system is locally controllable (around 0) in time T.

Remark

If $L = 2\pi$, $\mathcal{M}$ is of dimension 1 and there are infinitely many L such that $\mathcal{M}$ is of dimension 1.

Theorem (Local controllability in large time, E. Cerpa (2007), E. Cerpa and E. Crépeau (2008))

For every $L \in \mathcal{N}$, there exists $T > 0$ such that the KdV control system is locally controllable (around 0) in time T.

The proofs of these theorems rely on the power series expansion method. In the first theorem an expansion to the order 3 is required, while in the secund theorem an expansion to the order 2 is used. For the order 3 the computations are more complicate but the fact that this order is odd helps to get the local controllability in small time.

STLC for every $L \in \mathcal{N}$?

Question (Small-time local controllability)

Assume that dim$(\mathcal{M}) > 1$. *Is is true that for every* $T > 0$ *the control system*

(4) $$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0,\ t \in [0,T],\ x \in [0,L], \\ y(t,0) = y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in [0,T], \end{cases}$$

is locally controllable in time T*?*

Theorem (JMC, A. Koenig and H.-M. Nguyen (2022))

Let $k, l \in \mathbb{N} \setminus \{0\}$ *be such that* $2k + l \notin 3\mathbb{N}$. *Assume that*

$$L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}. \tag{5}$$

Then our KdV control system is not small-time locally null-controllable with controls in H^1 *and initial datum in* $H^3(0, L) \cap H_0^1(0, L)$, *i.e., there exist* $T_0 > 0$ *and* $\varepsilon_0 > 0$ *such that, for every* $\delta > 0$, *there is* $y_0 \in H^3(0, L) \cap H_0^1(0, L)$ *with* $\|y_0\|_{H^3(0,L)} < \delta$ *such that for every* $u \in H^1(0, T_0)$ *with* $\|u\|_{H^1(0,T_0)} < \varepsilon_0$ *and* $u(0) = y_0'(L)$, *we have*

$$y(T_0, \cdot) \neq 0, \tag{6}$$

where $y \in C\big([0, T_0]; H^3(0, L)\big) \cap L^2\big([0, T_0]; H^4(0, L)\big)$ *is the unique solution of our control system for the control* u *and starting from* y_0.

Open problem (Regularity and small-time local controllability)

Is the KdV control system small-time locally null controllable with initial state in $L^2(0,L)$ and control in $L^2(0,T)$ for a critical length as in the previous theorem?

Related to this problem let us recall that M. Bournissou (2022) gave a PDE example (a Schrödinger equation) such that the functional spaces do matter for the small-time controllability: She gives the first example for a pde control system where the cubic term (which gives the positive result: it is STLC) wins against the quadratic term (which gives the negative result: not STLC).
Note that with the boundary conditions $y(t,0) = y_x(t,L) = 0$ and $y(t,0) = u(t)$, Hoai-Minh Nguyen (2023) got the optimal result: obstruction to the small-time local controllability in the good spaces for all critical lengths!

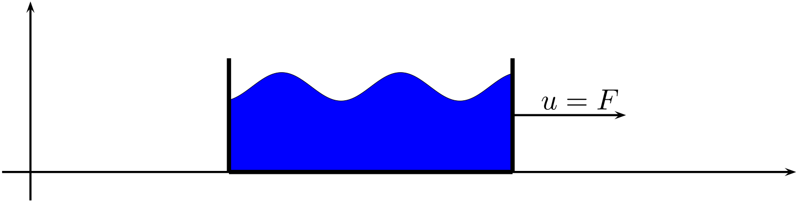
Open problem (Just $\dim \mathcal{M} > 1$)

Can the assumption $2k + l \notin 3\mathbb{N}$ be replaced by the weaker assumption $\dim \mathcal{M} > 1$?

Open problem (Optimal time)

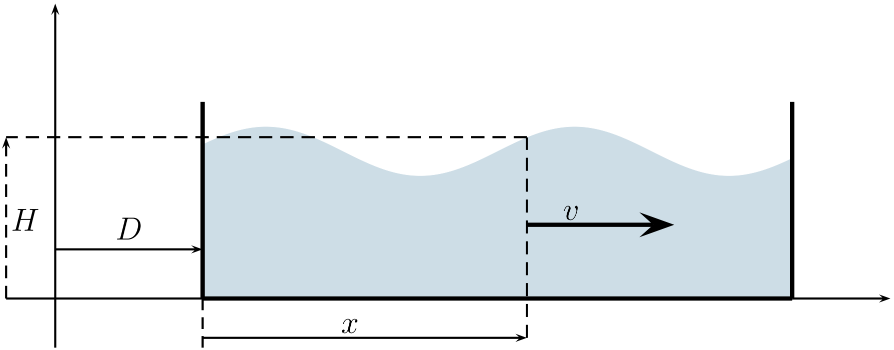
What is the minimal time for local controllability?

The water tank control system



The modelling is done with the Saint-Venant equations. See F. Dubois, N. Petit and P. Rouchon (1999).

Saint-Venant equations: Notations



The horizontal velocity v is taken with respect to the one of the tank.

The model: Saint-Venant equations

(1) $$H_t + (Hv)_x = 0,\ t \in [0, T],\ x \in [0, L],$$

(2) $$v_t + \left(gH + \frac{v^2}{2}\right)_x = -u(t),\ t \in [0, T],\ x \in [0, L],$$

(3) $$v(t, 0) = v(t, L) = 0,\ t \in [0, T],$$

(4) $$\dot{s}(t) = u(t),\ t \in [0, T],$$

(5) $$\dot{D}(t) = s(t),\ t \in [0, T].$$

- $u(t)$ is the horizontal acceleration of the tank in the absolute referential,
- g is the gravity constant,
- s is the horizontal velocity of the tank,
- D is the horizontal displacement of the tank.

State space

(1) $$\frac{d}{dt}\int_0^L H\,(t,x)\,dx = 0,$$

(2) $$H_x(t,0) = H_x(t,L) \quad (= -u(t)/g).$$

Definition (State space)

The state space (denoted $\mathcal{Y}$) is the set of

$$Y = (H,v,s,D) \in C^1([0,L]) \times C^1([0,L]) \times \mathbb{R} \times \mathbb{R}$$

satisfying

(3) $$v(0) = v(L) = 0,\ H_x(0) = H_x(L),\ \int_0^L H(x)dx = LH_e.$$

Local controllability in large time

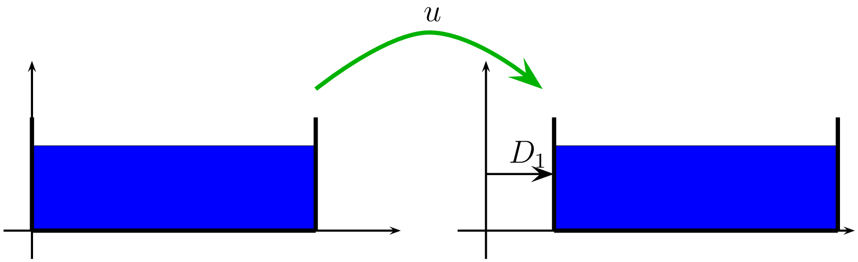
Theorem (JMC, (2002))

For $T > 0$ large enough the water-tank control system is locally controllable in time T around $(Y_e, u_e) := ((H_e, 0, 0, 0), 0)$.

Prior work: F. Dubois, N. Petit and P. Rouchon (1999):

1. The linearized control system around (Y_e, u_e) is not controllable,
2. Steady state controllability of the linearized control system around (Y_e, u_e).

Steady-state controllability



F. Dubois, N. Petit and P. Rouchon proved in 1999 that for the linearized system around the equilibrium $H = H_e$, speed=0, position of the tank at the origin, the steady-state controllability is valid for every time T such that

$$T > \sqrt{\frac{LH_e}{g}}. \tag{1}$$

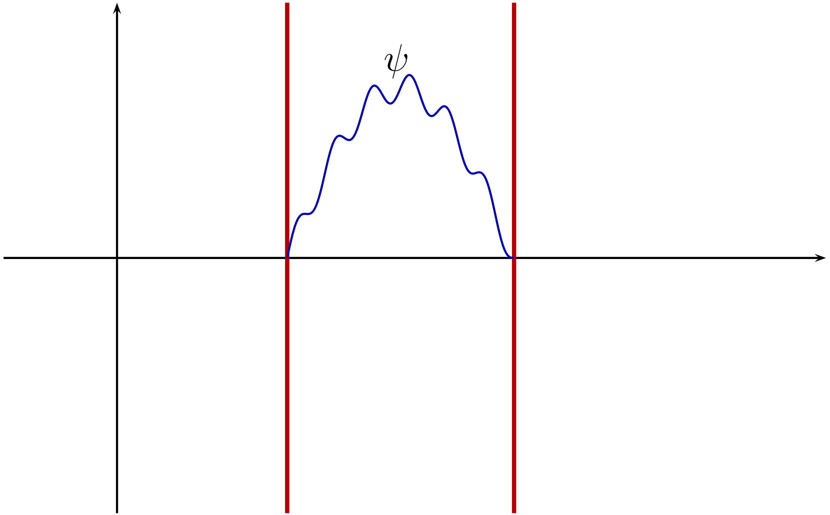
However we have the following theorem

Theorem (JMC, A. Koenig and H.-M. Nguyen (2023))

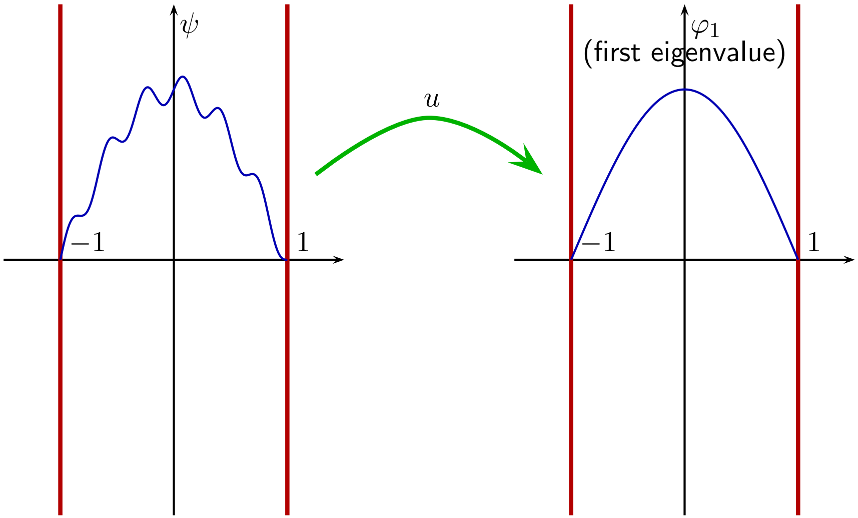
For every $T < 2\sqrt{\frac{LH_e}{g}}$ the steady-state controllability with small (for the C^0-norm) control does not hold in time T even if the two steady states are arbitrary close but different.

A quantum particle in a moving box

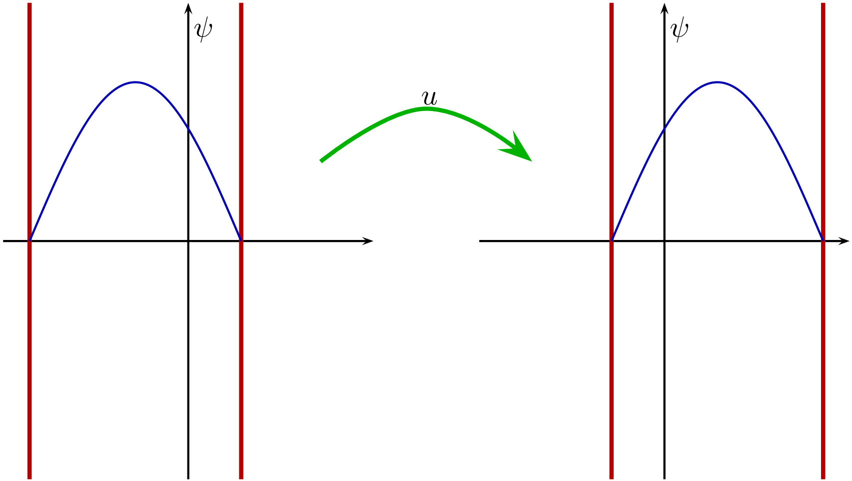
(Suggested by P. Rouchon)



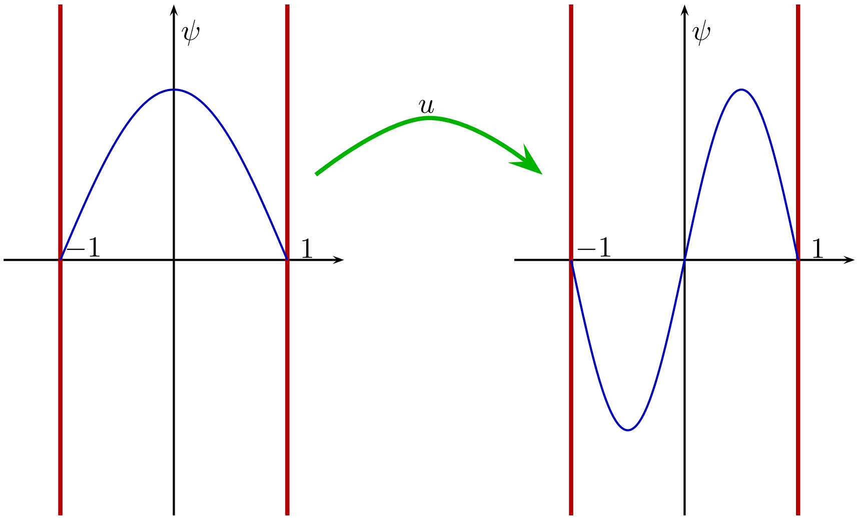
Local ("null") controllability



Motion of the box



Global controllability: From the first eigenfunction to the second one

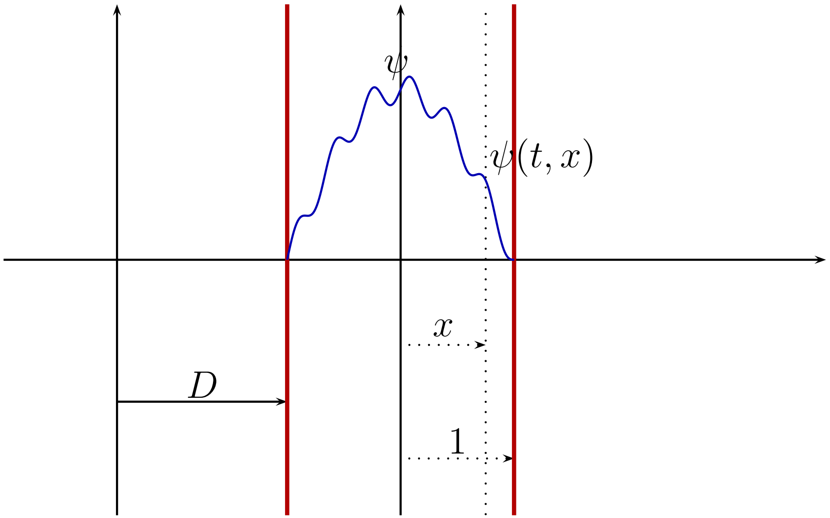


Controllability results

Theorem

- *The steady-state motion of the box for the linearized control system around the first eigenfunction holds in small time: P. Rouchon (2003). However this result does not hold for the (nonlinear) system JMC (2006) for small controls and arbitrary small but not 0 displacement.*
- *Large time local controllability: Without* (S, D)*: K. Beauchard (2005); with* (S, D)*: K. Beauchard and JMC (2006),*
- *Large time controllability between eigenfunctions: K. Beauchard and JMC (2006),*
- *Large time global controllability: V. Nersesyan (2008).*

Not STLC: Notations



Not STLC: Equations and definitions

Let $\varepsilon \in (0,1]$. Let $u : (0,T) \to \mathbb{R}$ be such that

$$|u(t)| < \varepsilon,\ t \in (0,T). \tag{2}$$

Let (ψ, S, D) be the solution of the Cauchy problem (the control system (P. Rouchon))

$$i\psi_t = -\psi_{xx} - u(t)x\psi,\ (t,x) \in (0,T) \times (-1,1), \tag{3}$$

$$\psi(t,-1) = \psi(t,1) = 0,\ t \in (0,T), \tag{4}$$

$$\dot{S}(t) = u(t),\ \dot{D}(t) = S(t),\ t \in (0,T), \tag{5}$$

$$\psi(0,x) = \varphi_1(x),\ x \in (-1,1), \tag{6}$$

$$S(0) = 0,\ D(0) = 0. \tag{7}$$

We assume that $S(T) = 0$. Let $\theta : [0,T] \times (-1,1) \to \mathbb{C}$ be defined by

$$\theta(t,x) := e^{i\lambda_1 t}\psi(t,x),\ (t,x) \in (0,T) \times (-1,1). \tag{8}$$

Not STLC: A quantity with a sign whatever is the control

One defines $V(t) := -i + i\int_{-1}^{1}\theta(t,x)\varphi_1(x)$. Simple computations show that

$$V(t) = \int_0^t S(\tau)V_0(\tau)d\tau + \int_0^t S(\tau)^2 V_1(\tau)d\tau \tag{9}$$

with

$$V_0(\tau) := 2i\int_{-1}^{1}\theta(\tau,x)\varphi_{1x}(x)dx,\ V_1(\tau) := -\frac{i}{2}\int_{-1}^{1}\theta_t(\tau,x)x^2\varphi_1(x)dx.$$

Standard estimates lead to

$$V_0(t) = S(t) + O(\|S\|_{L^1(0,t)} + \varepsilon\|S\|_{L^2(0,t)} + \varepsilon|S(t)|), \tag{10}$$

$$V_1(t) = O(\varepsilon). \tag{11}$$

... Hence the real part of $V(t)$ is positive for t small enough and $S \not\equiv 0$ on $[0,t]$. Typical obstruction: a quantity which should be unsigned has a sign whatever is the control. It is very classical for finite dimensional control systems.

A Burgers control system

Let us consider the following Burgers control system (introduced by S. Guerrero)

$$(12) \qquad \begin{cases} y_t - y_{xx} + yy_x = u(t),\ t \in [0,T],\ x \in [0,L], \\ y(t,0) = 0,\ y(t,L) = 0,\ t \in [0,T], \end{cases}$$

where, at time t, the state is $y(t,\cdot) \in L^2(0,L)$ and the control is $u(t) \in \mathbb{R}$.

Theorem (F. Marbach (2018))

The control system (12) *is not small-time locally controllable.*

Marbach's proof relies on scaling, power series expansions and new quadratic estimates (for kernel operators with singular kernels) leading to a quantity which should be unsigned but has a sign whatever is the control.

Open problem

Is the control system (12) *locally null controllable in large enough time?*

Main novelties of our obstruction to STLC for KdV

- This is the first case dealing with boundary controls. In our case one does not know what are the iterated Lie brackets even heuristically. Let us take this opportunity to point out that, even if they are not expected to be in the state space (see JMC (2007)), that would be very interesting to understand what are these iterated Lie brackets.
- It sounds difficult to perform the change of time-scale introduced by F. Marbach (2018) for a Burgers control system in our situation. Indeed this change will also lead to a boundary layer. However one can no longer use the maximum principle to study this boundary layer. Moreover if the change of time-scale, if justified, allows simpler computations, the advantage for not using it might be to get better or more explicit time for the obstruction to small-time local controllability.
- The linear drift term of the linearized control system is neither self-adjoint nor skew-adjoint.

A toy example

Let us consider the finite dimensional control system

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_3, \quad \dot{y}_3 = -y_2 + 2y_1 u, \tag{13}$$

where the state is $(y_1, y_2, y_3)^{\mathrm{tr}} \in \mathbb{R}^3$ and the control is $u \in \mathbb{R}$. The linearized control system of our toy control system around $(0,0) \in \mathbb{R}^3 \times \mathbb{R}$ is

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_3, \quad \dot{y}_3 = -y_2, \tag{14}$$

which is clearly not controllable. An obstruction to small-time local controllability of our toy control system (13) can be obtained by pointing out that if $(y, u) : [0, T] \rightarrow \mathbb{R}^3 \times \mathbb{R}$ is a trajectory of the toy control system (13) such that $y(0) = 0$, then

$$y_2(T) = \int_0^T \cos(T - t) y_1^2(t) dt, \tag{15}$$

$$y_3(T) = y_1(T)^2 - \int_0^T \sin(T - t) y_1^2(t) dt. \tag{16}$$

Hence,

$$y_2(T) \geq 0 \text{ if } T \in [0, \pi/2] \tag{17}$$

$$y_3(T) \leqslant 0 \text{ if } T \in [0, \pi] \text{ and } y_1(T) = 0, \tag{18}$$

which both show that our toy control system is not small-time locally controllable. More precisely, using (18), is not locally controllable in time $T \in [0, \pi]$ ((17) gives only an obstruction for $T \in [0, \pi/2]$). For the toy control system one knows that it is locally controllable in a large enough time and the optimal time for local controllability is also known: this control system is locally controllable in time T if and only if $T > \pi$. Moreover, if there are higher order perturbations (with respect to the weight $(r_1, r_2, r_3) = (1, 2, 2)$ for the state and 1 for the control) one can still get an obstruction to small-time local controllability by pointing out that the two previous obstructions respectively imply the following coercivity properties

$$(19) \qquad \forall T \in (0,\pi/2),\ \exists \delta > 0 \text{ s. t. } y_2(T) \geq \delta |u|^2_{H^{-1}(0,T)},$$

$$(20) \qquad \forall T \in (0,\pi],\ \exists \delta > 0 \text{ s. t. } (y_1(T) = 0 \Rightarrow y_3(T) \leqslant -\delta |u|^2_{H^{-2}(0,T)}).$$

Note that inequality (19) does not require any condition on the control, while (20) requires that the control is such that $y_1(T) = 0$. On the other hand it is (20) which gives the largest time for the obstruction to local controllability in time T: (19) gives an obstruction for $T \in [0, \pi/2)$, while (20) gives an obstruction for $T \in [0, \pi]$, which in fact optimal as mentioned above.

Remark

The fact that our toy system is not STLC follows from a necessary condition due to H. Sussmann (1983) relying on iterated Lie brackets. See also more general obstructions to STLC due to K. Beauchard and F. Marbach (2018), K. Beauchard J. Le Borgne and F. Marbach (2022, 2023). See also K. Beauchard's lectures in this conference. Unfortunately, as already mentioned, iterated Lie brackets are not so well understood for PDE controls, especially for boundary controls.

Our approach is inspired by the power series expansion method introduced by JMC and E. Crépeau (2004). The idea of this method is to search/understand a control u of the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots . \tag{21}$$

The corresponding solution then formally has the form

$$y = \varepsilon y_1 + \varepsilon^2 y_2 + \cdots , \tag{22}$$

and the non-linear term yy_x can be written as

$$yy_x = \varepsilon^2 y_1 y_{1,x} + \cdots . \tag{23}$$

One then obtains the following systems ($x \in (0,L)$ and $t \in (0,T)$)

(24)
$$\begin{cases} y_{1,t}(t,x) + y_{1,x}(t,x) + y_{1,xxx}(t,x) = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ y_{1,x}(t,L) = u_1(t), \end{cases}$$

(25)
$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) + y_1(t,x)y_{1,x}(t,x) = 0, \\ y_2(t,0) = y_2(t,L) = 0, \\ y_{2,x}(t,L) = u_2(t). \end{cases}$$

Let us recall that for the local controllability in large time the idea (JMC and E. Crépeau (2004), E. Cerpa (2007) and E. Cerpa and E. Crépeau (2009) is then to find u_1 and u_2 such that, if $y_1(0, \cdot) = y_2(0, \cdot) = 0$, then $y_1(T, \cdot) = 0$ and the $L^2(0, L)$-orthogonal projection of $y_2(T)$ on $\mathcal{M}$ is a given (non-zero) element in $\mathcal{M}$. In JMC and E. Crépeau an expansion up to the order 3 is necessary since y_2 belongs to the orthogonal space of $\mathcal{M}$ in this case. The three papers rely on contradiction arguments using the structure of the KdV systems.

Here instead of using a contradiction argument, the strategy is to characterize all possible u_1 which steers the linearized control system from 0 at time 0 to 0 at time T. This is done by taking the Fourier transform with respect to time of the solution y_1 and applying Paley-Wiener's theorem. We then prove, in the case $2k + l \neq 3\mathbb{N} \setminus \{0\}$, if the time T is sufficiently small, $y_2(T, \cdot)$ has to be in some explicit open half-space if $u_1 \neq 0$.

Notations

For $z \in \mathbb{C}$, let $(\lambda_j)_{1 \leqslant j \leqslant 3} = \big(\lambda_j(z)\big)_{1 \leqslant j \leqslant 3}$ be the three solutions repeated with the multiplicity of

$$\lambda^3 + \lambda + iz = 0. \tag{26}$$

Set

$$Q(z) := \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{pmatrix}, \tag{27}$$

$$P(z) := \sum_{j=1}^{3} \lambda_j (e^{\lambda_{j+2} L} - e^{\lambda_{j+1} L}) = \det \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}, \tag{28}$$

with the convention $\lambda_4 = \lambda_1$.

$$\Xi = \Xi(z) := -(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3), \tag{29}$$

$$G(z) = P(z)/\Xi(z) \quad \text{and} \quad H(z) = \det Q(z)/\Xi(z). \tag{30}$$

Characterization of the controls steering the linearized control system from 0 to 0

Proposition

Let $L > 0$, $T > 0$, and $u \in L^2(-\infty, +\infty)$. Assume that u has a compact support included in $[0, T]$, and u steers the linearized control system from 0 at the time 0 to 0 at the time T. Then $\hat{u}$ and $\hat{u}G/H$ satisfy the Paley-Wiener conditions

(31) $$\hat{u} \text{ and } \hat{u}G/H \text{ are entire functions,}$$

and

(32) $$|\hat{u}(z)| + \left|\frac{\hat{u}G(z)}{H(z)}\right| \leq Ce^{T|\Im(z)|},$$

for some positive constant C.

Some definitions

Let $k,\, l \in \mathbb{N} \setminus \{0\}$ be such that be such that

(33) $$L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}.$$

Let us define

(34) $$\eta_1 = -\frac{2\pi i}{3L}(2k+l), \quad \eta_2 = \eta_1 + \frac{2\pi i}{L}k, \quad \eta_3 = \eta_2 + \frac{2\pi i}{L}l,$$

(35) $$p = \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}},$$

(36) $$E := \frac{40\pi^3}{3L^3}(e^{\eta_1 L}-1)ikl(k+l),$$

(37) $$\varphi(x) := \sum_{j=1}^{3}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x} \text{ for } x \in [0,L], \qquad \text{with } \eta_4 := \eta_1,$$

(38) $$\Psi(t,x) := \Re(E\varphi(x)e^{-ipt}).$$

Some properties

$$(E \neq 0) \Leftrightarrow (2k + l \notin 3\mathbb{N}), \tag{39}$$

$$(\Psi(0,\cdot) \neq 0) \Leftrightarrow (E \neq 0), \tag{40}$$

$$\Psi_t + \Psi_x + \Psi_{xxx} = 0, \tag{41}$$

$$\Psi(t,0) = \Psi(t,L) = \Psi_x(t,0) = \Psi_x(t,L) = 0. \tag{42}$$

Coercivity

Proposition

If $E \neq 0$, there exists $T_* > 0$ and $C > 0$ such that, for any (real) $u \in L^2(-\infty, +\infty)$ with $u(t) = 0$ for $t \notin (0, T_*)$ and $y(T_*, \cdot) = 0$ where y is the unique solution of the linearized KdV control system starting from 0 at time 0 and using the control u, we have

$$\int_0^{T_*} \int_0^L y^2(t,x)\Psi_x(t,x)dxdt \geq C\|u\|^2_{H^{-2/3}(\mathbb{R})}. \tag{43}$$

This is one of the main steps of our proof of the obstruction to small time local controllability.

Obstruction to the small-time local controllability of the power expansion up to the order 2

(44) $y = \varepsilon y_1 + \varepsilon^2 y_2 + \ldots,\ y_1$ being the order $1, y_2$ being the order 2,

(45) $u = \varepsilon u_1 + \varepsilon^2 u_2,\ u_1$ being the order $1, u_2$ being the order 2.

We have

(46) $y_{1t} + y_{1x} + y_{1xxx} = 0,\ y_1(t,0) = y_1(t,L) = 0,\ y_{1x}(t,L) = u_1(t),$

(47)
$$y_{2t} + y_{2x} + y_{2xxx} = -y_1 y_{1x},\ y_2(t,0) = y_2(t,L) = 0,\ y_{2x}(t,L) = u_2(t),$$

We require that $y_1(0,x) = y_1(T^*,x) = 0$. So u_1 steers the linearized control system from 0 to 0, as u in the previous proposition. Applying this proposition one gets that

(48)
$$\int_0^{T_*}\int_0^L y_1^2(t,x)\Psi_x(t,x)dxdt \geq C\|u\|^2_{H^{-2/3}(\mathbb{R})}.$$

However multiplying (47) by Ψ, using the equation and the boundary conditions satisfies by Ψ (see above) and integration by parts on gets that the left hand side of the previous inequality is

$$\int_0^L y_2(T^*, x)\Psi(T^*, x)dx - \int_0^L y_2(0, x)\Psi(0, x)dx. \tag{49}$$

Hence

$$\int_0^L y_2(T^*, x)\Psi(T^*, x)dx - \int_0^L y_2(0, x)\Psi(0, x)dx \geq C\|u\|^2_{H^{-2/3}(\mathbb{R})} \tag{50}$$

which gives an obstruction to the null-controllability of the order 2 if $\Psi(0, \cdot) \neq 0$, i.e. if $2k + l \notin 3\mathbb{N}$. Moreover it gives an inequality which is crucial to deal with the remaining terms.
It remains to indeed take care of the remaining terms. Not an easy task in fact. However when we are confident that something should work it is often a question of time (may be large time...) to prove it. We finally succeed to perform it.

The case of the water tank control system

Without loss of generality $g = H_e = L = 1$. The linearized control system around the equilibrium $H = 1$, $V = 0$ is

$$\text{(51)} \qquad \begin{cases} \partial_t h_1 + \partial_x v_1 = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ \partial_t v_1 + \partial_x h_1 = -u(t) & \text{for } (t,x) \in (0,T) \times (0,1), \\ v_1(t,0) = v_1(t,1) = 0 & \text{for } t \in (0,T). \end{cases}$$

while the second order term is the control system (we forget the control u_2, the control in some sense is (h_1, v_1) which is a solution of (51))

$$\text{(52)} \qquad \begin{cases} \partial_t h_2 + \partial_x v_2 = -\partial_x (h_1 v_1) & \text{for } (t,x) \in (0,T) \times (0,1), \\ \partial_t v_2 + \partial_x h_2 = -\partial_x \left(\frac{v_1^2}{2} \right) & \text{for } (t,x) \in (0,T) \times (0,1), \\ v_2(t,0) = v_2(t,L) = 0 & \text{for } t \in (0,T). \end{cases}$$

The main idea is to prove that if a control steers the linearized system from 0 to 0, this second order always lies in some half-space, at least when $T < 2$. More precisely, for $T < 2$, we prove that for well-chosen functions ϕ, ψ (they are explicit), there exists $c > 0$ such that for every control u that steers the linearized system from 0 to 0 and such that $\int_0^T u(s)ds = 0$, we have the coercivity estimate

(53) $$(h_2(T, \cdot), \phi) + (v_2(T, \cdot), \psi) \geq c\|U\|_{L^2}^2,$$

with

(54) $$U(t) := \int_0^t u(s)ds.$$

...

Return to our KdV control system

(1) $$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0,\ t \in (0,T),\ x \in (0,L), \\ y(t,0) = 0,\ y(t,L) = 0,\ y_x(t,L) = u(t),\ t \in (0,T). \end{cases}$$

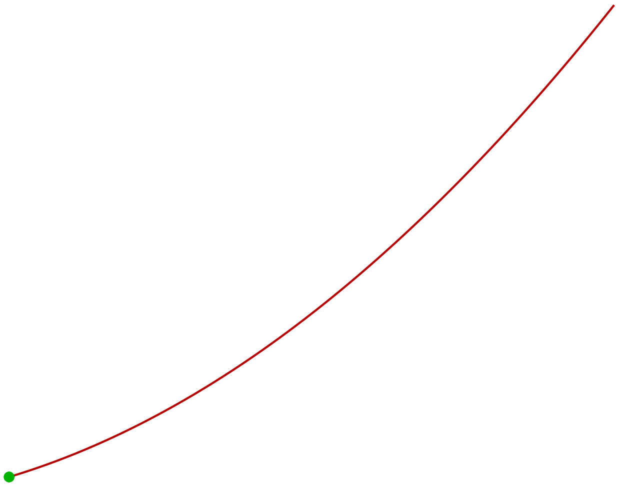
We assume that

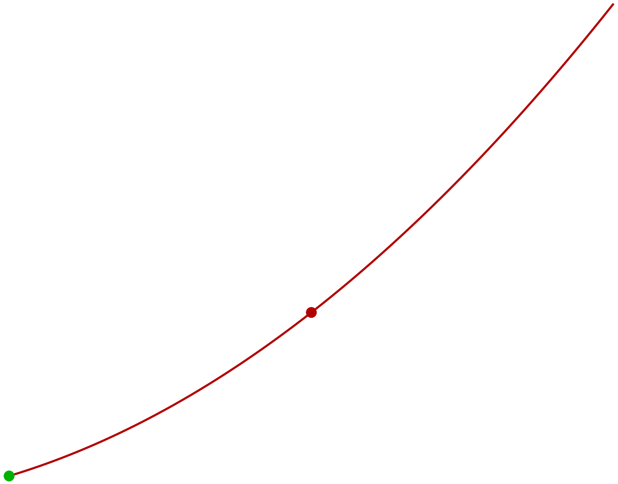
(2) $$L \in \mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}},\ k \in \mathbb{N}^*,\ l \in \mathbb{N}^* \right\}.$$

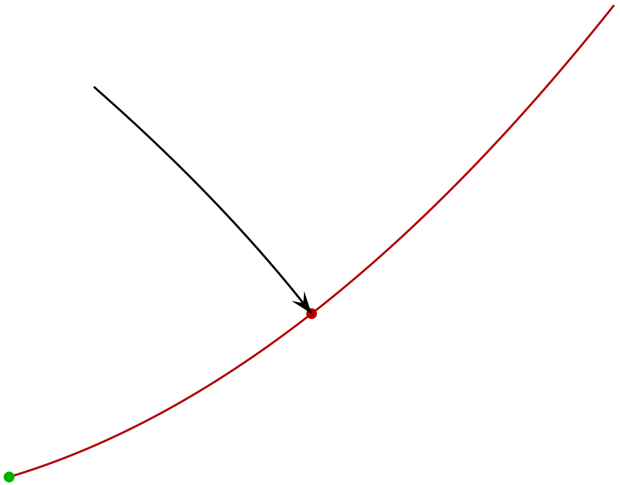
Then (see above) the nonlinear control system is locally controllable in large time (but not necessarily in small time). We are interested in the stabilization of the nonlinear system. Same problem for the water tank control system.

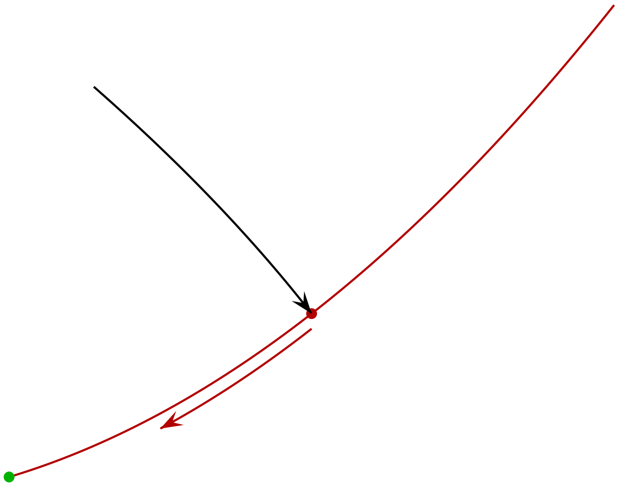
Phantom tracking

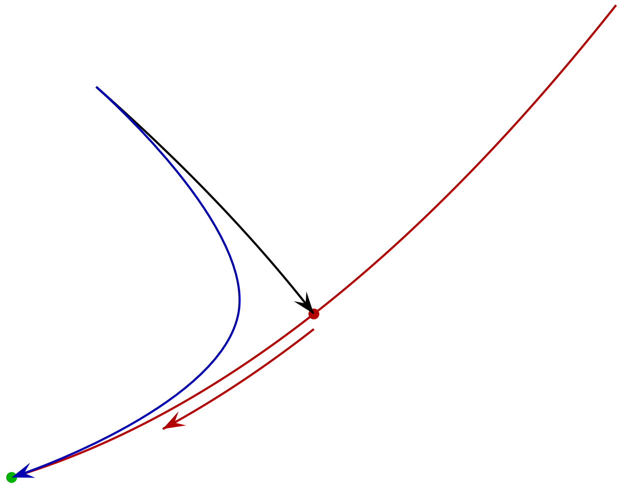
Roughly speaking this method can be described as follows. Let us assume that there exists a curve of equilibria $\gamma \in [0, \bar{\gamma}] \mapsto (x^\gamma, u_\gamma)$ of the control system $\dot{x} = f(x, u)$ such that $(x_0, u_0) = (0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$. We assume that for every $\gamma \in (0, \bar{\gamma}]$ there exist a feedback law u^γ which asymptotically stabilizes x^γ (with a large enough basin of attraction: it must contains 0). The idea is then to use for the control system $\dot{x} = f(x, u)$ the feedback law $u^{\tilde{\gamma}}$ where $\tilde{\gamma} : \mathbb{R}^n \to [0, \bar{\gamma}]$ is a well chosen function. Roughly speaking one steers the control system to the "phantom" x^γ with a γ which is moving to 0.











Phantom tracking method: An example

Let us go back to our toy control system.

$$\dot{y}_1 = u,\ \dot{y}_2 = y_3,\ \dot{y}_3 = -y_2 + 2y_1 u, \tag{1}$$

where the state is $(y_1, y_2, y_3)^{\text{tr}} \in \mathbb{R}^3$ and the control is $u \in \mathbb{R}$. The point $(y^\gamma, u_\gamma) := ((\gamma, 0, 0,)^{\text{tr}}, 0)$ is an equilibrium of the control system. The linearized control system at this equilibrium is the linear control system

$$\dot{y}_1 = u,\ \dot{y}_2 = y_3,\ \dot{y}_3 = -y_2 + 2\gamma u, \tag{2}$$

where the state is $(y_1, y_2, y_3)^{\text{tr}} \in \mathbb{R}^3$ and the control is $u \in \mathbb{R}$. This linear control system is controllable if (and only if) $\gamma \neq 0$. Therefore, if $\gamma \neq 0$ the equilibrium can be asymptotically stabilized for the nonlinear control system.

Stabilization of y^γ

One considers the following control Lyapunov function $V^\gamma : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$(1) \qquad V^\gamma(y) := (y_1 - \gamma)^2 + y_2^2 + y_3^2,\ \forall y = (y_1, y_2, y_3)^{\text{tr}} \in \mathbb{R}^3.$$

The time derivative of V^γ along the trajectory of our control system is

$$(2) \qquad \dot{V}^\gamma = 2u(y_1 - \gamma + 2y_1y_3).$$

Hence, in order to asymptotically stabilize x^γ for our control system, it is natural to consider the feedback law $u^\gamma : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$(3) \qquad u^\gamma := -y_1 + \gamma - 2y_1y_3.$$

One gets $\dot{V}^\gamma = -2(y_1 - \gamma + 2y_1y_3)^2$. Using the LaSalle invariance principle, one gets that this feedback law globally asymptotically stabilizes x^γ.

Let us now follow the phantom tracking strategy. In fact, instead of using $u^{\tilde{\gamma}}$ with a suitable $\tilde{\gamma} : \mathbb{R}^3 \to \mathbb{R}$ it is better to use directly a control Lyapunov of the type $V^{\tilde{\gamma}}$. Theoretically, the best way to choose $\tilde{\gamma}$ is to define it implicitly by proceeding in the following way. There exits an open neighborhood Ω of $0 \in \mathbb{R}^3$ and $V \in C^\infty(\Omega; \mathbb{R})$ such that

$$V(0) = 0, \forall x \in \Omega \setminus \{0\}, \tag{1}$$

$$V(y) = (y_1 - V(y))^2 + y_2^2 + y_3^2, \forall y = (y_1, y_2, y_3)^{\text{tr}} \in \Omega. \tag{2}$$

Therefore our choice of $\tilde{\gamma} = V(y)$, i.e. is such that $\tilde{\gamma}(x) = V^{\tilde{\gamma}(y)}$, $\tilde{\gamma}(0) = 0$. For the existence of V: use the implicit function theorem. In this simple case, V can be computed explicitly. One has
$\dot{V} = 2(y_1 - V)(u - \dot{V}) + 2y_2y_3 + 2y_3(-y_2 + 2y_1u)$,
i.e.,$(1 + 2y_1 - 2V)\dot{V} = 2u(y_1 - V + 2y_1y_3)$. We define a feedback law $u : \Omega \to \mathbb{R}$ by $u := -y_1 + V - 2y_1y_3$, which leads to
$(1 + 2x_1 - 2V)\dot{V} = -2(V - y_1 - 2y_1y_3)^2 \leqslant 0$. One can conclude that the feedback law u locally asymptotically our control system by using once more the LaSalle invariance principle.

Two possible improvements:

(i) One can get global asymptotic stability. It suffices to modify V by requiring $V = V(y) = (y_1 - \theta(V(y)))^2 + y_2^2 + y_3^2$, with a well chosen function $\theta : [0, +\infty) \to [0, +\infty)$.

(ii) One can get explicit feedback laws by using a dynamic extension: Replace the initial control system by the following one

$$\dot{y}_1 = u,\ \dot{y}_2 = y_3,\ \dot{y}_3 = -y_2 + 2y_1 u,\ \dot{\gamma} = v \tag{1}$$

where the state is $(y_1, y_2, y_3, \gamma)^{\text{tr}} \in \mathbb{R}^3$ and the control is $(u, v)^{\text{tr}} \in \mathbb{R}^2$. For $z = (y_1, y_2, y_3, \gamma)^{\text{tr}} \in \mathbb{R}^4$, one defines

$$\varphi(z) = (y_1 - \gamma)^2 + y_2^2 + y_3^2 + \gamma^2, \tag{2}$$

$$W(z) := \varphi(z) + (\gamma - \varphi(z))^2. \tag{3}$$

Compute $\dot{W}$ etc.

Applications of the phantom tracking method

1. Asymptotic stabilization of the Euler equations of incompressible fluids: JMC (1996) -introduction of the method-, O. Glass (2005),
2. Quantum control systems: K. Beauchard, JMC, M. Mirrahimi and P. Rouchon (2007).
3. Quantum box: K. Beauchard and M. Mirrahimi (2009).
4. Camassa-Holm equation: O. Glass (2008), V. Perrollaz (2013).

Open problem

Is it possible to apply this method to the water-tank control system?

Asymptotic stabilization of the linearized control system around "y^γ" for γ small but not 0 : JMC, A. Hayat, S. Xiang and Ch. Zhang (2022).

Remark

For the KdV control system even with critical lengths, $u = 0$ already gives the asymptotic stability (with a decay rate of $1/\sqrt{t}$ for $(k, l) = (1, 1)$ (J. Chu, JMC, Shang (2015)) and $(k, l) = (1, 2)$ (S. Tang, J. Chu, P. Shang and JMC (2018)).

Quadratic systems

Let us consider finite dimensional control systems of the following form

$$(1) \qquad \dot{x} = Ax + Bu \text{ and } \dot{y} = Ly + Q(x,x),$$

where n, m and k are three positive integers, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $L \in \mathbb{R}^{k\times k}$ and Q is a quadratic map from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^k$. For the control system (1), the state is $(x^{\text{tr}}, y^{\text{tr}})^{\text{tr}} \in \mathbb{R}^{n+k}$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, and the control is $u \in \mathbb{R}^m$.
Let us point out that the linearized control system around the trajectory $(\bar{x}, \bar{y}, \bar{u}) := (0,0,0)$ is

$$(2) \qquad \dot{x} = Ax + Bu \text{ and } \dot{y} = Ly,$$

a linear control system which is never controllable.

The assumptions

We assume the existence of $T > 0$ such that the following three properties hold

$(\mathcal{Q}_1)$ There exists $\rho_1 \in (0,1)$ such that

$$(\dot{x} = Ax) \Rightarrow \left(|x(T)|^2 \leqslant \rho_1 |x(0)|^2\right), \tag{1}$$

$(\mathcal{Q}_2)$ $|e^{TL}y| \leq |y|$ for every $y \in \mathbb{R}^k$,

$(\mathcal{Q}_3)$ Let $\mathbb{S}^{k-1} := \{b \in \mathbb{R}^k;\ |b| = 1\}$. There exist $\delta > 0$, $C_0 > 0$ and $v : [0,T] \times \mathbb{S}^{k-1} \to \mathbb{R}^m$ such that

$$v \in L^\infty([0,T] \times \mathbb{S}^{k-1}; \mathbb{R}^m), \tag{2}$$

$$|v(t,b) - v(t,b')| \leqslant C_0 |b - b'|,\ \forall t \in (0,T),\ \forall b, b' \in \mathbb{S}^{k-1}, \tag{3}$$

$$\left(\dot{\tilde{x}} = A\tilde{x} + Bv(t,b),\ \dot{\tilde{y}} = L\tilde{y} + Q(\tilde{x},\tilde{x}),\ \tilde{x}(0) = 0,\ \tilde{y}(0) = 0\right) \Rightarrow \left(\tilde{x}(T) = 0,\ \tilde{y}(T) \cdot e^{TL} b \leqslant -\delta\right),\ \forall b \in \mathbb{S}^{k-1}. \tag{4}$$

Definition of feedback laws

For $\varepsilon > 0$, let us consider the following periodic time-varying feedback law $u_\varepsilon : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^m$

(1)

$$u_\varepsilon(t,y) := \begin{cases} \varepsilon\sqrt{|e^{-tL}y|}\,v\left(t, \dfrac{e^{-tL}y}{|e^{-tL}y|}\right), & \forall t \in [0,T),\ \forall y \in \mathbb{R}^k \setminus \{0\}, \\ 0, & \forall t \in [0,T),\ y = 0 \in \mathbb{R}^k. \end{cases}$$

(2)

$$u_\varepsilon(t+T,y) = u_\varepsilon(t,y),\ \forall t \in \mathbb{R},\ \forall y \in \mathbb{R}^k.$$

We are interested in the asymptotic behavior of the solutions to the closed loop system

(closed-loop) $\dot{x} = Ax + Bu_\varepsilon(t,y)$ and $\dot{y} = Ly + Q(x,x)$,

Weighted exponential stability of the closed loop system

Theorem (JMC and I. Rivas (2015))

Let us assume that ($\mathcal{Q}_1$), ($\mathcal{Q}_2$) and ($\mathcal{Q}_3$) hold. Then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, there exist $C > 0$ and $\lambda > 0$ such that, for every $t_0 \in \mathbb{R}$, for every solution (x, y) of the closed loop system defined at time t_0, one has

(1) $$|x(t)|^2 + |y(t)| \leqslant Ce^{-\lambda(t-t_0)} \left(|x(t_0)|^2 + |y(t_0)|\right), \forall t \in [t_0, +\infty).$$

More general systems

Our next result allows to stabilize nonlinear control systems for which a quadratic "approximation" is given by our previous quadratic system and satisfies the assumptions of the previous theorem. The control system takes now the following more general form

(1) $$\dot{x} = Ax + R_x(x,y) + Bu,\ \dot{y} = Ly + Q(x,x) + R_y(x,y),$$

where the state is $(x^{\mathrm{tr}}, y^{\mathrm{tr}})^{\mathrm{tr}} \in \mathbb{R}^{n+k}$, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, and the control is $u \in \mathbb{R}^m$. We assume that $R_x : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $R_y : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ are both continuous. Our next result deals with the asymptotic stability of 0 for the closed loop system

(2) $$\dot{x} = Ax + Bu_\varepsilon(t,y) + R_x(x,y) \text{ and } \dot{y} = Ly + Q(x,x) + R_y(x,y).$$

Theorem (JMC and I. Rivas (2015))

Let us assume that $(\mathcal{Q}_1)$, $(\mathcal{Q}_2)$ *and* $(\mathcal{Q}_3)$ *hold. Let us also assume the existence of* $\eta > 0$ *and* $M > 0$ *such that,* $\forall \varepsilon \in (0,1)$, $\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^k$ *such that* $|x| + |y| \leqslant 1$,

$$|R_x(\varepsilon x, \varepsilon^2 y)| \leq M\varepsilon^{1+\eta}, \tag{1}$$

$$|R_y(\varepsilon x, \varepsilon^2 y)| \leq M\varepsilon^{2+\eta}. \tag{2}$$

Then, there exists $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0]$, *there exist* $C > 0$, $r > 0$ *and* $\lambda > 0$ *such that, for every* $t_0 \in \mathbb{R}$, *for every solution* (x, y) *of*

$$\dot{x} = Ax + Bu_\varepsilon(t, y) + R_x(x, y) \text{ and } \dot{y} = Ly + Q(x, x) + R_y(x, y). \tag{3}$$

with $|x(t_0)|^2 + |y(t_0)| \leqslant r$, *one has*

$$|x(t)|^2 + |y(t)| \leqslant Ce^{-\lambda(t-t_0)} \left(|x(t_0)|^2 + |y(t_0)|\right), \forall t \in [t_0, +\infty). \tag{4}$$

Applications to classical systems

- An example satisfying the necessary Brockett condition but not my necessary condition (JMC (1992)) for asymptotic stabilizability by means of **stationary** feedback laws

(1) $$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{y}_1 = x_1^2 - x_2^2,\ \dot{y}_2 = 2x_1x_2,$$

- The baby stroller control system

(2) $$\dot{x}_1 = x_3 \cos x_2,\ \ \dot{x}_2 = x_4,\ \ \dot{x}_3 = u_1,\ \ \dot{x}_4 = u_2,\ \dot{y} = x_3 \sin x_2.$$

- The underactuated surface vessel system.

“Application” to the KdV control system

Even if the proof of the two previous theorems require that the control system is finite dimensional, one can consider the same type of feedback laws for systems modeled by means of partial differential equations. This was done for the following KdV equation (already seen above)

$$
(1) \quad \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, & t \in [0,T],\, x \in [0,L], \\ y(t,0) = y(t,L) = 0,\, y_x(t,L) = u(t), & t \in [0,T], \end{cases}
$$

where, at time $t \in [0,T]$, the control is $u \in \mathbb{R}$ and the state is $y(t,\cdot) \in L^2(0,L)$.

Theorem (JMC, I Rivas and S. Xiang (2016))

Assume that L is a critical length such that $L \notin 2\pi\mathbb{N}$. Then the KdV control system has a quadratic approximation satisfying $(\mathcal{Q}_1)$, $(\mathcal{Q}_2)$ and $(\mathcal{Q}_3)$. Moreover there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, there exist $C > 0$, $r > 0$ and $\lambda > 0$ such that, for every $t_0 \in \mathbb{R}$ and for every solution y of the closed loop system

$$(1) \qquad \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0,\ t \in [t_0, +\infty),\ x \in [0, L], \\ y(t,0) = y(t,L) = 0,\ y_x(t,L) = u_\varepsilon(t,y),\ t \in [t_0, +\infty), \end{cases}$$

such that $|y(t_0)|_{L^2} \leqslant r$ one has, for every $t \in [t_0, +\infty)$,

$$(2) \qquad |y_c(t)|^2_{L^2} + |y_u(t)|_{L^2} \leqslant Ce^{-\lambda(t-t_0)} \left(|y_c(t_0)|^2_{L^2} + |y_u(t_0)|_{L^2}\right),$$

where, for $z \in L^2(0, L)$, z_c and z_u are the orthogonal projection of z on the controllable and uncontrollable linear subspace of the linearized control systems of (1) at 0 (these two linear spaces are orthogonal complements, so that $z = z_c + z_u$).

Open problem (Rapid stabilization)

For the same L, is it true that, for every $\lambda > 0$ there exists $r > 0$ and a periodic time-varying feedback law $u(t, y)$ such that for every $t_0 \in \mathbb{R}$ for every solution y of the closed loop system

$$(1) \qquad \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \ t \in [t_0, +\infty), \ x \in [0, L], \\ y(t, 0) = y(t, L) = 0, \ y_x(t, L) = u(t, y), \ t \in [t_0, +\infty) \end{cases}$$

such that $|y(t_0)|_{L^2} \leqslant r$ one has, for every $t \in [t_0, +\infty)$,

$$(2) \qquad |y_{\boldsymbol{c}}(t)|^2_{L^2} + |y_{\boldsymbol{u}}(t)|_{L^2} \leqslant Ce^{-\lambda(t-t_0)} \left(|y_{\boldsymbol{c}}(t_0)|^2_{L^2} + |y_{\boldsymbol{u}}(t_0)|_{L^2}\right)?$$

Open problem (The water tank control system)

Is it possible to adapt this strategy for the water tank control system?