2-categorical (abelian) 3d Mirror Symmetry

joint w/ Ben Gammage / Aaron Mazel-Gee

arose out of discussions with Tudor Dimofte

$$
\begin{tikzcd}
 & \text{3d } \mathcal{N}=4 \text{ gauge theory} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{3d A-model} & \text{3d holomorphic topological twist} \arrow[l] \arrow[r] & \text{3d B-model}
\end{tikzcd}
$$

de Rham A/B-model defined on 3-manifolds with a transverse holomorphic foliation. Give rise to algebraic field theory (see Raskin).

Betti A/B-model defined on all 3-manifolds. Gives rise to a fully extended functorial field theory.

Q1: What 2-categories do these field theories assign to a point?

Q2: Can we verify 3d mirror symmetry?

3d B-model

Kapustin, Rozansky, Saulina (2008, 2009)

really should collapse $\mathbb{Z}$ cohomological grading to $\mathbb{Z}/2\mathbb{Z}$

X derived symplectic stack ← even shift

Roughly, there should be a 2-category $KRS(X)$ with

objects: $(L, \mathcal{L})$ where $L \to X$ derived Lagrangian and $\mathcal{L}$ is a quasi-coherent sheaf of categories on L.

morphisms: $\mathrm{Hom}((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) = \Gamma(L_1 \cap L_2, \mathcal{H}om(\mathcal{L}_1, \mathcal{L}_2))$

<u>Note</u> To my knowledge even when $X = T^*Y$ there is no fully satisfactory mathematical construction of $KRS(X)$

- Rozansky–Oblomkov have model in terms of matrix factorizations concrete but difficult to verify formal properties
- Arinkin has proposed a $\mathbb{Z}$-graded lift known as the 2-category of (ind) coherent sheaves of categories on Y.

Coherent Sheaves of Categories (after Arinkin)

First, a reminder on quasi-coherent sheaves of categories as defined by Gaitsgory

1) If $Y = \operatorname{Spec} A$, then $2QCoh(Y) = A$-linear categories

2) In general $2QCoh(Y) = \lim_{\operatorname{Spec} A \to Y} 2QCoh(\operatorname{Spec} A)$

3) $2QCoh(Y)$ satisfies smooth descent

$\Longrightarrow$

If G is a smooth algebraic group $2QCoh(BG) \cong QCoh(G)$-comod

Just as for quasi-coherent sheaves there is an adjunction

$$\mathrm{Loc} : QCoh(Y)\text{-mod} \rightleftarrows 2QCoh(Y) : \Gamma$$

Def We say that Y is 1-affine if the above adjunction is an equivalence.

Thm (Gaitsgory) V/G is 1-affine for G reductive.

Ex $\quad EG = pt \to pt/G = BG$

mild lie should be $QCoh(G)$-mod

$$\underset{\Gamma = \mathrm{Hom}(N,-)}{QCoh(BG)\text{-mod} \;\cong\;} 2QCoh(BG) \overset{\mathrm{Hom}(D,-)}{\cong} QCoh(G)\text{-comod}$$

$$\begin{array}{ccccc} QCoh(BG) & \longleftarrow & N = 2G_{BG} & \longmapsto & Vect \\ Vect & \longleftarrow & D = 2G_{EG} & \longmapsto & QCoh(G) \end{array}$$

Can consider $2G_{V/G} \in 2QCoh(BG)$ via $p: V/G \to BG$

$$QCoh(V/G) \longleftarrow 2G_{V/G} \longmapsto QCoh(V)$$

Thm (ungauging)

$$QCoh(V) \cong \mathrm{Hom}(D, 2G_{V/G}) \cong \mathrm{Hom}_{QCoh(BG)\text{-mod}}(Vect, QCoh(V/G))$$

Thm (gauging)

$$QCoh(V/G) \cong \mathrm{Hom}(N, 2G_{V/G}) \cong \mathrm{Hom}_{QCoh(G)\text{-comod}}(Vect, QCoh(V))$$

This example is related to B-twisted 3d pure gauge theory

Key observation that leads to Arinkin's approach is

Thm (Gaitsgory) Let $p: X \to Y$ be a closed embedding of smooth stacks. Then

$$2QCoh(Y^{\wedge}_X) \cong QCoh(X \times_Y X)\text{-mod}$$

monoidal under convolution

formal neighborhood of X in Y

Ex $0 \times_{\mathbb{A}^1} 0 = \mathbb{A}^1[-1] \Rightarrow 2QCoh(\mathbb{A}^{1\wedge}_0) \cong \underbrace{QCoh(\mathbb{A}^1[-1])}_{\mathbb{C}[\varepsilon]\text{-mod}}\text{-mod}$

deg -1

Ex If first infinitesimal neighborhood of X in Y is split we have

$$X \times_Y X = N_X[-1]Y \Rightarrow 2QCoh(Y^{\wedge}_X) \cong QCoh(N_X[-1]Y)\text{-mod}$$

Even though X, Y are smooth $X \times_Y X$ doesn't have to be. Thus we often have

$$QCoh(X \times_Y X) \neq IndCoh(X \times_Y X)$$

Both monoidal under convolution!

deg 2

Ex $IndCoh(\mathbb{A}^1[-1]) \cong \mathbb{C}[y]\text{-mod}$

Ex $IndCoh(N_X[-1]Y) \cong$ "$QCoh(N_X^*[2]Y)$"

not quite true since $N^*_X[2]Y$ is a coaffine stack but same idea

Def Let $\{X_i \to Y\}_{i \in I}$ be a collection of closed embeddings of smooth stacks such that $X = \amalg X_i \to Y$ is surjective. Then

$$2\mathrm{IndCoh}(Y, \Lambda) := \mathrm{IndCoh}(X \times_Y X)\text{-mod}$$

also called singular support

where $\Lambda = \bigcup_{i \in I} N^*_{X_i}[2]Y \subseteq T^*[2]Y$ is the Lagrangian skeleton.

There is also a small version $2\mathrm{Coh}(Y, \Lambda)$ using $\mathrm{Coh}(X \times_Y X)$-mod instead.

Claim $2\mathrm{Coh}(Y, \Lambda)$ should be thought of as a $\mathbb{Z}$-graded lift of $\mathrm{KRS}(T^*Y, \Lambda)$

Q: What is Λ physically?

When mass/FI params are turned on I usually choose Λ to be the attracting sets.

In abelian theory with no mass/FI will choose Λ to be union of attracting sets for all possible phases.

Ex The free hypermultiplet $T^*\mathbb{A}^1$ has attracting sets $\mathbb{A}^1 = N^+_{\mathbb{A}^1}\mathbb{A}^1$ when $m_{\mathbb{R}} \geq 0$ and $T^+_0\mathbb{A}^1 = N^+_0\mathbb{A}^1$ when $m_{\mathbb{R}} < 0$

$$\Lambda = N^+_{\mathbb{A}^1}\mathbb{A}^1 \cup N^+_0\mathbb{A}^1$$

$$X = \mathbb{A}^1 \sqcup 0 \longrightarrow \mathbb{A}^1 = Y$$

$$X \times_Y X = \begin{pmatrix} \mathbb{A}^1 & 0 \\ 0 & \mathbb{A}^1[-1] \end{pmatrix} \qquad \mathrm{IndCoh}(X \times_Y X) \simeq \begin{pmatrix} \mathrm{QCoh}(\mathbb{A}^1) & \mathrm{Vect} \\ \mathrm{Vect} & \text{“}\mathrm{QCoh}(\mathbb{A}^1[-1])\text{”} \end{pmatrix}$$

$$\text{“}\mathrm{QCoh}(\mathbb{A}^1[-1])\text{”} = \mathbb{C}[y]\text{-mod}, \quad |y| = 2$$

Ex For gauged hypermultiplet $T^*(\mathbb{A}^1/\mathbb{G}_m)$ get

$$X = \mathbb{A}^1/\mathbb{G}_m \sqcup B\mathbb{G}_m \longrightarrow \mathbb{A}^1/\mathbb{G}_m = Y$$

$$X \times_Y X = \begin{pmatrix} \mathbb{A}^1/\mathbb{G}_m & B\mathbb{G}_m \\ B\mathbb{G}_m & \mathbb{A}^1[-1]/\mathbb{G}_m \end{pmatrix} \qquad \mathrm{IndCoh}(X \times_Y X) \simeq \begin{pmatrix} \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) & \mathrm{QCoh}(B\mathbb{G}_m) \\ \mathrm{QCoh}(B\mathbb{G}_m) & \text{“}\mathrm{QCoh}(\mathbb{A}^1[-1]/\mathbb{G}_m)\text{”} \end{pmatrix}$$

Note In this case Λ gives whole space.

Note Can upgrade to object of $3\mathrm{QCoh}(B\mathbb{G}_m) \simeq (\mathrm{QCoh}(B\mathbb{G}_m)\text{-mod})\text{-mod}$ by remembering "diagonal" $\mathrm{QCoh}(B\mathbb{G}_m)$ in $\mathrm{IndCoh}(X \times_Y X)$. There is an analogue of ungauging that gives back the free hyper.

3d A-model

Kapustin, Vyas, Setter (2016) Defined the field theory in general. Only examined boundary conditions for pure $U(1)$-gauge theory and $\mathbb{R}^3 \times S^1$ valued hypermultiplet.

Teleman (2014)

studied A-twist of pure G gauge theory. The 2-category of boundary conditions is

$$\underbrace{G\text{-cat}}_{2\mathrm{Fuk}(T^*BG)} := \text{2-category of categories with Betti } G\text{-actions}$$

He also proved the following version of 3d mirror symmetry

$$2\mathrm{Fuk}(T^*BG) \cong 2\mathrm{QCoh}(L_{G^\vee})$$

$$\begin{tikzcd} L_{G^\vee} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_C(0, G^\vee) \arrow[d] \\ 0 \arrow[r] & \mathfrak{t}^\vee / W^\vee \end{tikzcd}$$

And gave some hints about how to generalize to an equivalence involving $KRS(\mathcal{M}_C(0, G^\vee))$.

In abelian case he proved the other direction as well

$$2\mathrm{Fuk}(T^*\mathbb{G}_m) \cong 2\mathrm{QCoh}(B\mathbb{G}_m)$$

Betti G-actions on (∞,n)-categories

Def: Let BG be the $(\infty,1)$-category with a single object $*$ and $\mathrm{End}(*) = G$

$$G\text{-}n\mathrm{Cat} := \mathrm{Hom}_{(n+1)\text{-}\mathrm{cat}}(BG, n\mathrm{Cat})$$

Basically objects are $\mathcal{C} \in n\mathrm{Cat}$ together with $S^1 \to \mathrm{End}(\mathcal{C})$.

Note
- All n-categories will be assumed to be stable. Might have to linearize BG at various points.
- I will ignore issues of size / presentability for the whole talk

n=0 $\quad G \to C_\bullet(G) \to \mathrm{End}(\mathcal{C})$ $\qquad G\text{-}0\mathrm{Cat} \cong C_\bullet(G)\text{-mod}$

E_1-map

n=1 $\quad G \to \mathrm{End}(\mathcal{C})$

$1 \mapsto 1_{\mathcal{C}}$

$\Omega_1 G \to C(\Omega_1 G) \to \mathrm{End}(1_{\mathcal{C}})$

E_2-map

Teleman's theorem follows from

$C_\bullet(\Omega_1 G) \cong \mathbb{C}[L_{G^L}]$

Digression 3-categorical Betti Langlands

$$S^1\text{-}2\mathrm{Cat} \cong 3\mathrm{Coh}(B\mathbb{G}_m)$$

Exchanging gauging and ungauging

Pf Compare the 3-stabilization of BS^1 with the 3-category with one object and one morphism associated to $(\mathrm{QCoh}(B\mathbb{G}_m), \otimes)$.

$$\pi_1 \mathbb{G}_m = \mathbb{Z} = X^*(\mathbb{G}_m)$$

A-twisted free hypermultiplet

Hard to get a version of $2\mathrm{Fuk}(T^*\mathbb{A}^1)$ that is non-trivial without choosing a Lagrangian skeleton.

Choose $\Lambda = \mathbb{A}^1 \cup T^*_0\mathbb{A}^1$ as on B-side.

Q: what is $2\mathrm{Fuk}(T^*\mathbb{A}^1, \Lambda)$?

2 obvious candidates

1) constructible sheaves of categories on $(\mathbb{A}^1, \overbrace{\mathbb{A}^1 - \{0\} \cup \{0\}}^{S})$

$$2\mathrm{Shv}_c(\mathbb{A}^1, S) = \mathrm{Hom}_{2\text{-cat}}(\mathrm{Exit}(\mathbb{A}^1, S), \mathrm{St})$$

defined by Treumann, Lurie.

makes sense for any conically stratified space ✗

2) Perverse schobers on $(\mathbb{A}^1, S)$

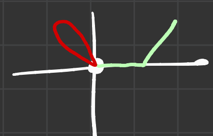
$2\mathrm{Perv}(\mathbb{A}^1, S)$ defined by Kapranov, Schechtman. ✓

makes sense in limited situations

Sheaves via exit paths

(X,S) stratified space

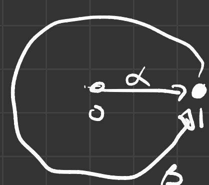
Def A path $\gamma: I \to X$ has the exit property with respect to S if for all $0 \leq t_0 < t_1 \leq 1$ the dimension of the stratum containing $\gamma(t_0)$ is less than or equal to the dimension of the stratum containing $\gamma(t_1)$.



Can generalize this to simplices $\sigma: \Delta^n \to X$ and define an $(\infty,1)$-category
$$\mathrm{Exit}(X,S) \subseteq \mathrm{Sing}(X)$$

Thm (Treumann) $\mathrm{Hom}(\mathrm{Exit}(X,S), \mathrm{Vect}) \cong \mathrm{Shv}_c(X,S)$

Ex $(X,S) = (\mathbb{A}^1, \mathbb{A}^1 - \{0\} \cup \{0\})$

$\pi_1(\mathrm{Exit}(X,S)) =$ 

$\alpha = \beta \circ \alpha$

Perverse Sheaves

$$\mathrm{Shv}_c(\mathbb{A}^1, 0)$$

$$\mathrm{Perv}(\mathbb{A}^1, S) \quad\cup\!\!\!/ \qquad \cup\!\!\!\backslash \quad \mathrm{Shv}_c(\mathbb{A}^1, S)^{\heartsuit}$$

Nearby / Vanishing cycles give exact functors $\Psi, \Phi : \mathrm{Perv}(\mathbb{A}^1, 0) \to \mathrm{Vect}$

<u>Thm</u> (MacPherson, Vilonen)

$$\mathrm{Perv}(\mathbb{A}^1, 0) \simeq \left\{ \Psi \underset{\mathrm{can}}{\overset{\mathrm{var}}{\rightleftarrows}} \Phi \;\middle|\; \underset{(1-\mathrm{var}\circ\mathrm{can})}{\overset{T_\Psi}{\|}} \text{ and } \underset{(1-\mathrm{can}\circ\mathrm{var})}{\overset{T_\Phi}{\|}} \text{ are invertible} \right\}$$

<u>Note</u> By passing to a larger category can find projective objects co-representing Φ, Ψ.

<u>Note</u> $S^1 \circlearrowright \mathbb{A}^1$ preserving S. Therefore one gets an S^1-action on $\mathrm{Perv}(\mathbb{A}^1, S)$ which is generated by the natural endomorphism (T_Ψ, T_Φ) of $1_{\mathrm{Perv}(\mathbb{A}^1, S)}$.

Perverse Schober = spherical functor

Def A functor $\Phi \xrightarrow{S} \Psi$ is spherical if it has a right adjoint R such that the twist functors

$$T_\Phi = \mathrm{fib}\,(1_\Phi \to R \circ S) \qquad T_\Psi = \mathrm{cofib}\,(S \circ R \to 1_\Psi)$$

are invertible.

Note 1) In fact S has an infinite sequence of adjoints

$$\cdots\; S_{-2} \dashv S_{-1} \dashv \underset{\substack{\parallel \\ S}}{S_0} \dashv \underset{\substack{\parallel \\ R}}{S_1} \dashv \;\cdots$$

2) $S \circ T_\Phi[2] = T_\Psi \circ S$

Riehl-Verity

Can define a stable $(\infty, 2)$-category $2\mathrm{Fun}(\mathrm{Adj}, \mathrm{St})$ of adjunctions such that $2\mathrm{Perv}(\mathbb{A}^1, 0)$ is a full subcategory.

(see Dyckerhoff, Kapranov, Schechtman, Soibelman)

By 2) the S^1-action on $\mathrm{Perv}(\mathbb{A}^1, S)$ categorifies to an S^1-action on $2\mathrm{Perv}(\mathbb{A}^1, S)$

3d Mirror Symmetry (GH, BG, AMG)

- $$\mathrm{KRS}(T^*\mathbb{A}^1/\mathbb{G}_m, \Lambda) \cong 2\mathrm{Fuk}(T^*\mathbb{A}^1, \Lambda)$$

- As boundary conditions for 4d $N=4$

$$\begin{array}{ccc} \mathrm{KRS}(T^*\mathbb{A}^1/\mathbb{G}_m) & \in & 3\mathrm{QCoh}(B\mathbb{G}_m) \\ \| & & \wr\| \\ 2\mathrm{Fuk}(T^*\mathbb{A}^1, \Lambda) & \in & S^1\text{-}2\mathrm{Cat} \end{array}$$

Relate to be $2\mathrm{Fuk}(T^*\mathbb{A}^1, S)^{S^1}$

$$\Longrightarrow \quad \mathrm{KRS}(T^*\mathbb{A}^1, \Lambda) \cong 2\mathrm{Fuk}(T^*(\mathbb{A}^1/\mathbb{G}_m), \Lambda)$$

- Deleting components from Λ $\Longleftrightarrow$ passing to open set

mass FI

Rough idea of proof

consider

$$QCoh(\mathbb{A}^1/\mathbb{G}_m) \xrightarrow{Res} QCoh(B\mathbb{G}_m)$$

free category equipped with invertible endofunctor

T with $1 \to T$ multiplication by $x \in \mathcal{O}_{\mathbb{A}^1}$

grading shift

Thus

$$\mathrm{Hom}(Res, S) \xrightarrow{\sim} \Phi$$

$$\begin{tikzcd}
QC(\mathbb{A}^1/\mathbb{G}_m) \arrow[r, "F"] \arrow[d, "Res"'] & \Xi \arrow[d] \\
QC(B\mathbb{G}_m) \arrow[r, "G"] & \Xi
\end{tikzcd} \longleftrightarrow F(\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m})$$

Thus Res categorifies one of the projections in $Rep(\mathbb{A}^1, S)$.

Other projection is similar and endomorphisms give desired matrix algebra.