Cyclic characters of alternating group

joint work with Amritanshu Prasad and Velmurugan S

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Cyclic Characters

Let G be a group and $g \in G$ of order m.

Consider the cyclic subgroup $C_m = \langle g \rangle$ with irreducible characters

$$\begin{aligned} \sigma_i : C_m &\to \mathbb{C} \\ g &\mapsto \zeta_m^i, \end{aligned}$$

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The induced characters,

$$\psi := \operatorname{Ind}_{C_m}^{G} \zeta_m^i$$

are called cyclic characters of G.

Litreature review

- In 1987, Kraśkiewicz and Weyman [KW01] described the decomposition of cyclic characters of Coxeter groups obtained by inducing characters of the cyclic group generated by a Coxeter element.
- In 1989 Stembridge [Ste89] did it for all cyclic characters of symmetric groups, wreath product groups, and a few cyclic characters of complex reflection groups.
- Garsia [Gar90] has given a different proof for the case of the cyclic subgroup generated by the largest n cycle of the symmetric group.
- Jöllenbeck and Schocker [JS00] gave a new approach to Stembridge's result on the symmetric group.

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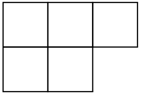
partition $\rightarrow$ Young diagram

$\lambda = (3,2) \vdash 5$



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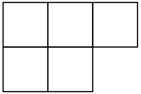
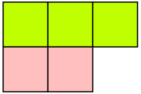
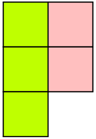


partition $\lambda \quad \rightarrow \quad$ conjugate λ'





Descents and Major index

Standard Young Tableau (SYT): $T = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2} \end{array} \in SYT(3,2)$

Entries increase along rows and columns

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$$T = \begin{array}{|c|c|c|}\hline \colorbox{yellow}{1} & 3 & \colorbox{yellow}{4} \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \rightarrow \quad \mathrm{Des}(T) = \{1, 4\}.$$

Major index of T:

$$\mathrm{Maj}(T) = \sum_{x \in \mathrm{Des}(T)} x$$

$$\mathrm{Maj}(T) = 5.$$

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Example

$T \in SYT(3, 2)$

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We have,

$$a_{\lambda,0} = a_{\lambda,1} = a_{\lambda,2} = a_{\lambda,3} = a_{\lambda,4} = 1.$$

Irreducible characters of alternating group

If $\lambda \neq \lambda'$ then the restriction

$$\operatorname{Res}_{A_n}^{S_n} \chi_\lambda = \operatorname{Res}_{A_n}^{S_n} \chi_{\lambda'} = \chi_\lambda$$

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We denote

$$a_{\lambda,i}^+ := \langle \mathrm{Ind}_{C_n}^{A_n} \zeta_n^i, \chi_\lambda^+ \rangle_{A_n} \text{ and } a_{\lambda,i}^- := \langle \mathrm{Ind}_{C_n}^{A_n} \zeta_n^i, \chi_\lambda^- \rangle_{A_n}$$

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There exists a pair of irreducible characters χ_λ^+ and χ_λ^-.

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- If $\lambda = (\frac{n+1}{2}, 1, \ldots, 1)$ then

$$\chi_\lambda^\pm(w_\sigma^-) = \chi_\lambda^\mp(w_\sigma^+).$$

Observe,

$$a_{\lambda,i} = a^{+}_{\lambda,i} + a^{-}_{\lambda,i}.$$

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Let

$$\Delta_{\lambda,i} = a^+_{\lambda,i} - a^-_{\lambda,i}.$$

$$
\begin{aligned}
\Delta_{\lambda,i} &= a^{+}_{\lambda,i} - a^{-}_{\lambda,i} \\
&= \langle \zeta_n^i, \operatorname{Res}^{A_n}_{C_n} \chi_\lambda^{+} \rangle_{C_n} - \langle \zeta_n^i, \operatorname{Res}^{A_n}_{C_n} \chi_\lambda^{-} \rangle_{C_n} \\
&= \frac{1}{n} \left(\sum_{\substack{k=1 \\ w \sim w^k}}^{n} \zeta_n^{ik} (\chi_\lambda^{+}(w^k) - \chi_\lambda^{-}(w^k)) + \sum_{\substack{k=1 \\ w \nsim w^k}}^{n} \zeta_n^{ik} (\chi_\lambda^{+}(w^k) - \chi_\lambda^{-}(w^k)) \right) \\
&= \frac{1}{n} (\chi_\lambda^{+}(w) - \chi_\lambda^{-}(w)) \left(\sum_{\substack{k=1 \\ w \sim w^k}}^{n} \zeta_n^{ik} - \sum_{\substack{k=1 \\ w \nsim w^k}}^{n} \zeta_n^{ik} \right).
\end{aligned}
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Conjugacy Classes

$w_\sigma \sim w_\sigma^k$ if and only if $\gcd(k, n) = 1$ and there exists $\tau_k \in A_n$ such that $\tau_k w_\sigma \tau_k^{-1} = w_\sigma^k$.

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Suppose $n = p^a$ and k is a generator of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$, then

$$\tau_k = (1\ k\ k^2 \ldots k^{(p^a - p^{a-1})})(p\ kp\ k^2p \ldots k^{p^{(a-1)} - p^{(a-2)}} p) \ldots (p^{(a-1)}\ kp^{(a-1)} \ldots k^{(p-2)} p^{(a-1)})(p^a)$$

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Jacobi symbol: $\left(\frac{k}{p^a}\right) = \left(\frac{k}{p}\right)^a$, where

$$\left(\frac{k}{p}\right) = \begin{cases} 0 & \text{if } k \equiv 0 \mod p, \\ 1 & \text{if } k \not\equiv 0 \mod p \text{ and } k \equiv x^2 \mod p \text{ for some integer } x, \\ -1 & \text{otherwise.} \end{cases}$$

Lemma

Let $n = \prod_{j=1}^{s} p_j^{a_j}$, *for odd primes* p_j, $1 \leq j \leq s$. *Then* w *and* w^k *are conjugate to each other in* A_n *if and only if* $\left(\frac{k}{n}\right) = 1$.

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So we have,

$$\Delta_{\lambda,i} = \frac{1}{n}(\chi_\lambda^+(w^k) - \chi_\lambda^-(w^k)) \left(\sum_{\substack{k=1 \\ w \sim w^k}}^{n} \zeta_n^{ik} - \sum_{\substack{k=1 \\ w \nsim w^k}}^{n} \zeta_n^{ik} \right)$$

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$$= \sqrt{\frac{\epsilon_n}{n}} \sum_{k=1}^{n} \left(\frac{k}{n}\right) \zeta_n^{ik}$$

where $\epsilon_n = \pm 1$, for $n \equiv \pm 1 \mod 4$

Lemma

Let $n = p^a$*, where* p *is an odd prime and* n *is a natural number and let* $d = \gcd(i, n)$*, then*

$$\sum_{k=1}^{n} \left(\frac{k}{n}\right) \zeta_n^{ik} = \begin{cases} 0 & \text{if } n/d \text{ is not square free,} \\ -p^{a-1} & \text{if } n/d = p \quad a \text{ is even,} \\ p^a - p^{a-1} & \text{if } n/d = 1 \quad a \text{ is even,} \\ 0 & \text{if } n/d \neq p \quad \text{and } a \text{ is odd,} \\ p^{a-1} \left(\frac{i/d}{p}\right) \sqrt{\epsilon_p p} & \text{if } n/d = p \quad \text{and } a \text{ is odd,} \end{cases}$$

where $\epsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4, \\ -1 & \text{if } p \equiv 3 \mod 4. \end{cases}$

Proposition

For any odd prime p and $a > 0$, let $n = p^a$, $d = \gcd(n, i)$ and $\lambda = (\frac{n+1}{2}, 1, \ldots, 1)$ be the hook partition. Then we have the following:

1. *If a is even*

$$\Delta_{\lambda,i} = \begin{cases} -p^{\frac{a-2}{2}} & \textit{if } d = p^{a-1} \\ p^{\frac{a-2}{2}}(p-1) & \textit{if } d = p^a \\ 0 & \textit{otherwise.} \end{cases}$$

2. *If a is odd,*

$$\Delta_{\lambda,i} = \begin{cases} \epsilon_p p^{a-1} \left(\frac{i/d}{p}\right) & \textit{if } d = p^{a-1} \\ 0 & \textit{otherwise,} \end{cases}$$

where $\epsilon_p = \pm 1$ for $p \equiv \pm 1 \mod 4$.

Theorem ([PPS])

Let $\lambda = (\frac{n+1}{2}, 1, \ldots, 1)$ *be a partition of* $n = \prod_{j=1}^{s} p_j$ *and* $d = \gcd(n, i)$*. If* $\frac{n}{d}$ *is not square free then we have* $\Delta_{\lambda,i} = 0$*. Otherwise, we have the following cases:*

- *if* n *is a square then*

$$\Delta_{\lambda,i} = \frac{(-1)^s}{\sqrt{n}} \prod_{j\in[s]} p_j^{a_j-1} \prod_{j\notin J} (p_j - 1)$$

- *if* n *is not a square then*

$$\Delta_{\lambda,i} = \begin{cases} \frac{\epsilon_n \sqrt{n}}{n_0} (-1)^{|J_e|} \prod_{j=1}^{s} \left(\frac{f_j}{p^{a_j}}\right) \prod_{j\notin J} (p_j - 1) \prod_{j\in J_o} \left(\frac{i/p_j^{a_j-1}}{p_j}\right) \sqrt{p_j} & \text{if } a_j \text{ is even } \forall j \notin J \\ 0 & \text{otherwise.} \end{cases}$$

where $J := \{j \in [s] : p_j \mid \frac{n}{d}\}$, $J_e = \{j \in J : a_j \text{ is even}\}$, $J_o = \{j \in J : a_j \text{ is odd}\}$, *and* n_0 *is the square free part of* n*.*

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