* Proof of non-vanishing theorem

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*We give a sketch of the proof of the non-vanishing theorem. To study Zariski density in $\mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$, we consider a discrete valuation ring $W_\ell \subset \mathbb{C}_\ell$ finite over $W(\overline{\mathbb{F}}_\ell)$. Recall $\mathcal{X} = \mathcal{X}_v := \{\chi | \int_{Cl_n^-} \chi\psi d\varphi_f \neq 0$ for some n, $v(\chi) = v\}$ and the sequence $\underline{n} := \{\min(m) | \chi$ factors through Γ_m for $\chi \notin \mathcal{X}\}$ defines $\Xi = \{\mathbf{s}(\mathcal{A}) | [\mathcal{A}] \in \bigsqcup_{n \in \underline{n}} \mathrm{Ker}(\Gamma_n \to \Gamma_j)\}$ for some $j \geq r$. We show that $\underline{n}$ contains an arithmetic progression if $\dim \overline{\mathcal{X}} < d$ for the Zariski closure $\overline{\mathcal{X}}$ in $\mathbb{G}_m^d$.

§**0. Case** dim $\overline{\mathcal{X}} = 0$ **(and** $\Gamma \cong \mathbb{Z}_\ell$**).** We can take $j := r$ for r given by $\ell^r \,\|\, |\mathbb{F}_p[f, \lambda, \psi, \mu_\ell]^\times|$. If the Zariski closure $\overline{\mathcal{X}}$ has dimension 0, it is a finite set stable under $t \mapsto t^P$ ($\mathbb{F}_P = \mathbb{F}_p[f, \lambda, \psi, \mu_\ell]$); so, there exists an integer N such that if the order of χ is larger than ℓ^N, we have $\int_{Cl_n^-} \chi\psi d\varphi_f = 0$ for all n. Let $0 < m_0 \in \mathbb{Z}$ be the minimal integer such that $\mathfrak{l}^{m_0} = (\varphi\varphi^c)$ with $\varphi \in R$. Thus there exists a positive integer N' such that $\underline{n} \supset \{n \in m_0\mathbb{Z} | n \geq N'\}$, which is an arithmetic progression; so, the assertion follows from Density theorem if $\Gamma \cong \mathbb{Z}_\ell$.

Assume $0 <$ dim $\overline{\mathcal{X}} < d$; so, $O_\mathfrak{l} \neq \mathbb{Z}_\ell$ and dim $\overline{\mathcal{X}} < d$. We have $\chi^{\sigma^m} = \chi^{P^m} \in \mathcal{X}$ if $\chi \in \mathcal{X}$; i.e., $\mathcal{X}$ is σ-stable.

1. we may not have lower bound ℓ^N of the order of $\chi \in \mathcal{X}$,
2. the Galois action $\chi \mapsto \chi^{P^m}$ $m \in \mathbb{Z}$ only covers 1-dimensional segment starting with χ. In other words, $\mathrm{Tr}_{\mathbb{F}_P[\chi]/\mathbb{F}_P} \circ \chi$ only factor through $\mathrm{Ker}(\chi|_{\Gamma_n})\Gamma_n[\ell^r]$ whose order grows dependent on n (not a finite bounded sum of $f|\alpha(u/\varpi_\mathfrak{l}^r)([\mathcal{A}])$ over $u \in O/\mathfrak{l}^r$.

These are two points of difficulty which have to be addressed.

§1. Rigidity of torus.

Let W_ℓ be a discrete valuation ring finite flat over $W(\overline{\mathbb{F}}_\ell)$. We state the following theorem, which is a key to show that $\underline{n}$ contains arithmetic progression.

Rigidity Theorem. *Let $X = \mathrm{Spf}(\mathcal{T})$ be a closed formal subscheme of $\widehat{G} = \widehat{\mathbb{G}}^n_{m/W_\ell}$ flat geometrically irreducible over W_ℓ (i.e., $\mathcal{T} \cap \overline{\mathbb{Q}}_\ell = W_\ell$). Suppose there exists an open subgroup U of $\mathbb{Z}_\ell^\times$ such that X is stable under the action $\widehat{G} \ni t \mapsto t^u \in \widehat{G}$ for all $u \in U$. If X contains a Zariski dense subset $\Omega \subset X(\mathbb{C}_\ell) \cap \mu^n_{\ell^\infty}(\mathbb{C}_\ell)$, then there exist $\omega \in \Omega$ and a formal subtorus T such that $X = T\omega$.*

If time permits, we describe a sketch of the proof at the end (see my papers: JAMS **24** (2011) Rigidity lemma and Contemporary Math. **614** (2014) Lemma 4.1).

§**2. Use of rigidity.** Let, for a class $v \in O/\mathfrak{l}^j$ for $j \geq r$,

$\mathcal{X} = \mathcal{X}_v := \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) | \int_{Cl_n^-} \chi\psi d\varphi_f \neq 0 \ \exists n, v(\chi) = v\}$

$\mathcal{Z} = \mathcal{Z}_v := \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) | \int_{Cl_n^-} \chi\psi d\varphi_f = 0 \ \forall n, v(\chi) = v\}$

Write $\widehat{\mathcal{X}}$ for the formal Zariski closure of $\mathcal{X}$ in $\widehat{\mathbb{G}}^d_{m/W_\ell}$. Assume $\dim_{W_\ell} \widehat{\mathcal{X}} < d$ which leads to absurdity. Note $\dim_{W_\ell} \widehat{\mathcal{X}} = \dim \overline{\mathcal{X}}$.

Overcoming reducibility: Since $\chi \in \mathcal{X} \Rightarrow \chi^\sigma \in \mathcal{X}$ for $\sigma = \mathrm{Frob} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^r})$, σ permutes irreducible components of $\widehat{\mathcal{X}}$. Thus each irreducible component is fixed by some $\tau := \sigma^m$ for $m > 0$. Note $\sigma(x) = x^P$ for $P = p^r$, and put $\mathbf{P} = P^m$. Since $P \equiv 1 \mod \ell^r$ with $r > 0$, we have $\mathbf{P} \equiv 1 \mod \ell$. Since each irreducible component is formal, it is stable under $\tau^{\mathbb{Z}_\ell}$ which is an open subgroup of $\mathbb{Z}_\ell^\times$. By Rigidity Theorem, each irreducible component of $\widehat{\mathcal{X}}$ is of the form ωT for a subtorus T. Then $\widehat{\mathcal{X}} = \bigcup_{j \in J} \omega_j T_j$ for a finite index set J with $\omega_j \in \Omega$ and subtori T_j. The argument has been given if $\dim_{W_\ell} \widehat{\mathcal{X}} = \dim \overline{\mathcal{X}} = 0$; so, assume $0 < \dim_{W_\ell} \widehat{\mathcal{X}} < d$.

§**3. Tubular neighborhood.** Replacing $\widehat{\mathbb{G}}_m^d$ by $\widehat{\mathbb{G}}_m^d/\langle\omega_j\rangle_j$, we may assume that $\omega_j = 1$ for all j. Let V_j be the $\mathbb{Q}_\ell$-span of the Tate module of T_j as a subspace of $V := \mathbb{Q}_\ell(1)^d$. Since $0 < \dim V_j < d$, we claim to find a basis $B := \{e_1, \ldots, e_d\}$ of $\mathbb{Z}_\ell(1)^d$ such that $B' := B \cup \{e := \sum_j e_j\}$ is outside $\bigcup_j V_j$. Since $d > 1$, the set $\{B \in \mathrm{GL}_d(\mathbb{Z}_\ell) | B' \cap \bigcup_j V_j \neq \emptyset\}$ is a proper closed subset of $\mathrm{GL}_d(\mathbb{Z}_\ell)$ of dimension $d^2 - \max_j(\dim V_j) < d^2 = \dim \mathrm{GL}(d)$. This shows a plenty of the choice B.

Let $\Gamma_{\mathbf{P}} = \mathbf{P}^{\mathbb{Z}_\ell}$. Then $U := \Gamma_{\mathbf{P}} e_1 + \cdots + \Gamma_{\mathbf{P}} e_d$ is an open tubular neighborhood of the line $\mathbb{Z}_\ell \cdot e$. By replacing $\mathbf{P}$ by its power (i.e., shrinking U), the image $Cone(U)$ of $\bigcup_{u \in U} \mathbb{Q}_\ell \cdot u$ in $(\mathbb{Q}_\ell(1)/\mathbb{Z}_\ell(1))^d$ is disjoint from $\mathcal{X}[\ell^{N'}]^\times = \mathcal{X}[\ell^{N'}] - \mathcal{X}[\ell^{N'-1}]$ for all sufficiently large $N' > 0$.

§4. Proof of non-vanishing theorem.

Let $Cone(U)[\ell^M]^\times$ be the set of order ℓ^M elements in $Cone(U)$ and χ_i be the order ℓ^M element corresponding to $\frac{1}{\ell^M}e_i$. Write $\mathbf{P} = p^j$ $(j \geq r)$ and define $\mathcal{Z} = \mathcal{Z}_v$ for $v \in O/\mathfrak{l}^j$. Then for $M \geq N'$, writing $m := \dim_{\mathbb{F}_\mathbf{P}} \mathbb{F}_\mathbf{P}[\mu_{\ell^M}]$

$$Cone(U)[\ell^M]^\times = \{\prod_{i=1}^{d} \chi_i^{u_i} | u_i \in \Gamma_\mathbf{P}\} = \{\prod_{i=1}^{d} \chi_i^{\mathbf{P}^{m_i}} | 0 \leq m_i \leq m-1\} \subset \mathcal{Z}.$$

Thus if $\chi|_{\Gamma_n[\ell^j]} = \chi_v$ for $v \in O/\mathfrak{l}^j$,

$$\sum_{\chi \in Cone(U)[\ell^M]^\times} \chi = \prod_{i=1}^{d} \mathrm{Tr}_{\mathbb{F}_\mathbf{P}[\mu_{\ell^M}]/\mathbb{F}_\mathbf{P}}(\chi_i) \overset{\text{Trace rel.}}{=} [\mathbb{F}_\mathbf{P}[\mu_{\ell^M}] : \mathbb{F}_\mathbf{P}]^d \chi_v,$$

where $\chi_v = 0$ outside $\Gamma_n[\ell^j]$. This j depends on $\mathfrak{l}$ and Ξ contains $\underline{n} = \{n \in \mathbb{Z} | n \geq N'\}$. Thus if $a(\xi, f) \neq 0$ for $(\xi \mod \mathfrak{l}^j) = -v$, we get the contradiction. □

§**5. Preliminary to the proof of Rigidity Theorem.** The regular locus of X° is open dense in the generic fiber $\mathrm{Spec}(\mathcal{T})_{/K}$ (for the field $K = \mathrm{Frac}(W)$ for $W = W_\ell$). Then $\Omega^\circ := X^\circ \cap \Omega$ is Zariski dense in $\mathrm{Spec}(\mathcal{T})_{/K}$. Write $X^s := \mathrm{Spec}(\mathcal{T})_{/K} - X^\circ$ (the singular locus). The stabilizer U_ζ of $\zeta \in \Omega$ in U is an open subgroup of U. By $t \mapsto t\zeta^{-1}$, we assume that the identity $\mathbf{1} \in \Omega^\circ$.

By adding subscript an, X_{an} denotes the rigid analytic spaces associated to X. Then $X^\circ_{an} = X_{an} - X^s_{an}$ is an open rigid analytic subspace of X_{an}. Apply the logarithm $\log : \widehat{G}^{an}(\mathbb{C}_\ell) \to \mathbb{C}_\ell^n = Lie(\widehat{G}^{an}_{/\mathbb{C}_\ell})$ sending $(t_j)_j \in \widehat{G}^{an}(\mathbb{C}_\ell)$ to $(\log_\ell(t_j))_j) \in \mathbb{C}_\ell^n$ for the ℓ-adic logarithm map $\log_\ell : \mathbb{C}_\ell^\times \to \mathbb{C}_\ell$. Then for each smooth point $x \in X^\circ(W)$, taking a small analytic open ball G_x centered at x in $\widehat{G}_{an}$ so that $V_x = G_x \cap X^\circ(W)$ for a d-dimensional open ball in $X^\circ(W)$ centered at $x \in X^\circ(W)$. Then $\log(X^\circ(W))$ contains the origin $0 \in \mathbb{C}_\ell^n$. Take $\zeta \in \Omega^\circ$. Write T_ζ for the Tangent space at ζ of X. Then $X_\zeta \cong W^d$ for $d = \dim_W X$. The space $T_\zeta \otimes_W \mathbb{C}_\ell$ is canonically isomorphic to the tangent space T_0 of $\log(V_\zeta)$ at 0.

§6. Proof in case: $\dim_W X = 1$. If $\dim_W X = 1$, there exists an infinite order element $t_1 \in X(W)$. We write $U = (1 + \ell^m \mathbb{Z}_\ell)$ for $0 < m \in \mathbb{Z}$. Then X is the (formal) Zariski closure $\overline{t_1^U}$ of

$$t_1^U = \{t_1^{1+\ell^m z} | z \in \mathbb{Z}_\ell\} = t_1 \{t_1^{\ell^m z} | z \in \mathbb{Z}_\ell\},$$

which is a coset of a formal subgroup Z. Since t_1^U is an infinite set, we have $\dim_W Z > 0$. From irreducibility and $\dim_W X = 1$, we conclude $X = t_1 Z$ and $Z \cong \widehat{\mathbb{G}}_m$. Since X contains roots of unity $\zeta \in \Omega \subset \mu^n_{\ell^\infty}(W)$, we confirm that $X = \zeta Z$ for $\zeta \in \Omega \cap \mu^n_{\ell^{m'}}$ for $m' \gg 0$. Replacing t_1 by $t_1^{\ell^m}$ for m as above if necessary, we have the translation $\mathbb{Z}_\ell \ni s \mapsto \zeta t_1^s \in Z$ of one parameter subgroup $\mathbb{Z}_\ell \ni s \mapsto t_1^s$. Thus we have $\log(t_1) = \frac{dt_1^s}{ds}|_{s=0} \in T_\zeta$, which is sent by "$\log : \widehat{G} \to \mathbb{C}_\ell^n$" to $\log(t_1) \in T_0$. This implies that $\log(t_1) \in T_0$ and hence $\log(t_1) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ (under the identification of the tangent space at any $x \in \widehat{G}$ with $Lie(\widehat{G})$). Therefore T_ζ's over $\zeta \in \Omega^\circ$ can be identified canonically.

§7. Proof in case $\dim_W X > 1$**.** Consider the Zariski closure Y of t^U for an infinite order element $t \in V_\zeta$ (for $\zeta \in \Omega^\circ$). Since U permutes finitely many geometrically irreducible components, each component of Y is stable under an open subgroup of U. Therefore $Y = \bigcup_j \zeta_j \mathbb{T}_j$ is a union of formal subtori $\mathbb{T}_j$ of dimension ≤ 1, where ζ_j runs over a finite set inside $\mu^n_{\ell^\infty}(\mathbb{C}_\ell) \cap X(\mathbb{C}_\ell)$. Since $\dim_W Y = 1$, we can pick $\mathbb{T}_j$ of dimension 1 which we denote simply by $\mathbb{T}$. Then $\mathbb{T}$ contains t^u for some $u \in U$. Applying the argument in the case of $\dim_W X = 1$ to $\mathbb{T}$, we find $u \log(t) = \log(t^u) \in T_\zeta$; so, $\log(t) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ and $t \in V_\zeta$. Summarizing our argument, we have found

(T) The Zariski closure of t^U in X for an element $t \in V_\zeta$ of infinite order contains a coset $\xi\mathbb{T}$ of one dimensional subtorus $\mathbb{T}$, $\xi^{\ell^{m'}} = 1$ and $t^{\ell^{m'}} \in \mathbb{T}$ for some $m' > 0$;

(D) Under the notation as above, we have $\log(t) \in T_\zeta$.

Moreover, the image $\overline{V}_\zeta$ of V_ζ in $\widehat{G}/\mathbb{T}$ is isomorphic to $(d-1)$-dimensional open ball.

§**8. Induction on** d**.** If $d > 1$, therefore, we can find $\bar{t}' \in \overline{V}_\zeta$ of infinite order. Pulling back $\bar{t}'$ to $t' \in V_\zeta$, we find $\log(t), \log(t') \in T_\zeta$, and $\log(t)$ and $\log(t')$ are linearly independent in T_ζ. Inductively arguing this way, we find infinite order elements $t_1, \dots, t_d$ in V_ζ such that $\log(t_i)$ span over the quotient field K of W the tangent space $T_{\zeta/K} = T_\zeta \otimes_W K \hookrightarrow T_0$ (for any $\zeta \in \Omega^\circ$). We identify $T_{1/K} \subset T_0$ with $T_{\zeta/K} \subset T_0$. Thus the tangent bundle over $X^\circ_{/K}$ is constant as it is constant over the Zariski dense subset Ω°. Therefore X° is close to an open dense subscheme of a coset of a formal subgroup. See Contemporary Math. **614** (2014) Lemma 4.1 for more details to conclude that X° is indeed an open dense subscheme of a coset of a formal subgroup.