The odd integrable system

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INTEGRABLE SYSTEMS

- principal G-bundle P, Higgs field $\Phi \in H^0(C, \mathrm{ad}(P) \otimes K)$

- invariant polynomials $p_1, \ldots, p_k$ degrees $d_1 = 2, d_3, \ldots, d_k$

$$p_i(\Phi) \in H^0(C, K^{d_i})$$

- $f \in H^0(C, K^{d_i})^* \cong H^1(C, K^{1-d_i})$ = function on

  $\mathcal{M}$ = moduli space of Higgs bundles

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  $\mathcal{M}$ = moduli space of Higgs bundles

- $\mathcal{M}$ symplectic, f_1, f_2 Poisson commute

- $\mathcal{N}$ = moduli space of stable bundles

  $T^*\mathcal{N} \subset \mathcal{M}$, $\mathbf{C}^*$-action on fibres

- p_i homogeneous of degree d_i, $f \in H^1(C, K^{1-d_i})$

  $\Rightarrow$ defines a symmetric tensor $S \in H^0(\mathcal{N}, S^{d_i}T)$

- Poisson bracket $\equiv$ *Schouten bracket*

  S_1, S_2 sections of $S^m T, S^n T$, $[S_1, S_2]$ section of $S^{m+n-1}T$

  $m = n = 1$ = Lie bracket

- invariant forms $\varphi_1, \ldots, \varphi_k \in \Lambda^{2d_i-1}\mathfrak{g}^*$

  $\varphi_i(\Phi_1, \ldots, \Phi_{2d_i-1}) \in H^0(C, K^{2d_i-1})$

- $f \in H^0(C, K^{2d_i-1})^* \cong H^1(C, K^{2-2d_i})$

  $\Rightarrow \rho \in H^0(\mathcal{N}, \Lambda^{2d_i-1}T)$ = polyvector field

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- *Schouten-Nijenhuis bracket*

  ρ_1, ρ_2 sections of $\Lambda^m T, \Lambda^n T$, $[\rho_1, \rho_2]$ section of $\Lambda^{m+n-1}T$

  $m = n = 1$ = Lie bracket

- **Fact:** These sections commute

- basic example: symmetric $p_1(X,Y) =$ Killing form $B(X,Y)$

alternating $\rho(X,Y,Z) = B([X,Y],Z)$

- basis $\Phi_i \in H^0(C, \mathrm{ad}(P) \otimes K)$ $\rho_{123} = \mathrm{tr}([\Phi_1, \Phi_2]\Phi_3)$

NJH, *Stable bundles and polyvector fields*, in "Complex and Differential Geometry", W. Ebeling et al (eds.) Springer Proceedings in Mathematics **8**, 135–156, Springer Verlag, Heidelberg (2011).

Symmetric 2-tensor: $H^0(\mathcal{N}, S^2T)$

- $$\begin{array}{ccc} H^1(C,K^{-1}) \otimes H^0(C,\mathrm{ad}(P)\otimes K) & \to & H^1(C,\mathrm{ad}(P)) \\ \quad\quad\quad\quad\wr\| & & \wr\| \\ \quad\quad\quad\quad T^* & & T \end{array}$$

$$\alpha \otimes \Phi \mapsto \alpha\Phi$$

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Alternating 3-tensor: $H^0(\mathcal{N}, \Lambda^3T)$

- $H^1(C,K^{-2}) \otimes H^0(C,\mathrm{ad}(P)\otimes K^2) \to H^1(C,\mathrm{ad}(P))$

$$\beta \otimes [\Phi_1,\Phi_2] \mapsto \beta[\Phi_1,\Phi_2]$$

Hyperkähler metrics and supersymmetry NJ Hitchin, A Karlhede, U Lindström, M Roček Communications in Mathematical Physics 108 (4), 535-589	1525	1987
Generalized calabi–yau manifolds N Hitchin Quarterly Journal of Mathematics 54 (3), 281-308	1489	2003
Stable bundles and integrable systems N Hitchin	1229	1987
Harmonic spinors N HITCHIN	1215	1974

SL (2) over the octonions N Hitchin Mathematical Proceedings of the Royal Irish Academy 118 (1), 21-38	6	2018
Stable bundles and polyvector fields N Hitchin Complex and Differential Geometry: Conference held at Leibniz Universität …	6	2011
Vector bundles and the icosahedron N Hitchin Vector bundles and complex geometry 522, 71-87	6	2010

EXAMPLES

- E rank n, $\Phi \in H^0(C, \mathrm{End}_0\, E \otimes K)$

  unchanged by $E \to E \otimes L$, $L^n \cong \mathcal{O}$

- $\Rightarrow$ polyvector field $\rho \in H^0(\mathcal{N}, \Lambda^k T)$ is

  invariant by action of $H^1(C, \mathbf{Z}_n) \cong \mathbf{Z}_n^{2g}$

- C genus 2, E rank 2, $\Lambda^2 E$ trivial

  moduli space $\mathcal{N} \cong \mathsf{P}^3$, $\Lambda^3 T \cong \mathcal{O}(4)$ anticanonical divisors

- C: $y^2 = (z-\mu_1)\ldots(z-\mu_6)$

  $$\rho(\Phi_1, \Phi_2, \Phi_3) \in H^0(C, K^3) = \{(cy + c_0 + c_1 z + \ldots + c_3 z^3)\frac{dz^3}{y^3}\}$$

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- hyperelliptic involution $\tau(y) = -y$ acts trivially on $\mathcal{N}$

  (Narasimhan & Ramanan 1969)

  $$\Rightarrow \ \rho(\Phi_1,\Phi_2,\Phi_3) = c\frac{dz^3}{y^2}$$

  $\sim$ distinguished quartic surface in P^3

- semi-stable locus = Kummer surface Jac$(C)/\mathbf{Z}_2$

  = S-equivalent to $E = L \oplus L^*, \deg L = 0$

- non-trivial extension $L \to E \to L^*$ $(L^2 \neq \mathcal{O})$

  $\to H^0(C, \mathrm{End}_0\, E \otimes K) \to H^0(C, L^{-2}K) \to H^1(C, E \otimes LK)$

- $$\begin{array}{ccccc} H^0(C, L^{-2}K) & \to & H^1(C, K) & \subset & H^1(C, E \otimes LK) \\ \wr\| & \cong & \wr\| & & \\ \mathbf{C} & & \mathbf{C} & & \end{array}$$

- Higgs fields all preserve $L \Rightarrow$ lie in a 2-dimensional subalgebra

  $\Rightarrow \rho = 0$

- C genus 2, E rank 2, $\Lambda^2 E$ fixed, odd degree

  moduli space $\mathcal{N} \cong Q \cap Q_\mu$ intersection of quadrics

$$Q : \sum_{i=1}^{6} x_i^2 = 0, \qquad Q_\mu : \sum_{i=1}^{6} \mu_i x_i^2 = 0$$

- $\rho(\Phi_1, \Phi_2, \Phi_3) = (c_0 + c_1 z + \ldots + c_3 z^3)\dfrac{dz^3}{y^3}$

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- $\Phi \in H^0(C, \mathrm{End}_0\, E \otimes K)$ independent of $E \mapsto E \otimes L$, L^2 trivial

  $\Lambda^3 T \cong \mathcal{O}(2)$, quadrics invariant by $x_i \mapsto \pm x_i$ (even number)

$$\sum_{i=1}^{6} a_i x_i^2 = 0$$

  $(a_1, \ldots, a_6)$ modulo $(1, \ldots, 1), (\mu_1, \ldots, \mu_6)$: 4-dimensional

- C genus 3, E rank 2 $\Lambda^2 E$ trivial

  $\mathcal{N} \cong$ Coble quartic hypersurface $Q \subset \mathsf{P}^7$

  M.S.Narasimhan & S.Ramanan, 2Θ *systems on abelian varieties*, Vector bundles and algebraic varieties, Bombay 1984, OUP (1987) 415 – 427.

- on Q

  normal bundle

  $$0 \to T_Q \to T_{\mathsf{P}^7} \to \mathcal{O}(4) \to 0$$

  $$0 \to \Lambda^4 T_Q(-4) \to \Lambda^4 T_{\mathsf{P}^7}(-4) \to \Lambda^3 T_Q \to 0$$

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- on Q

  $$0 \to T_Q \to T_{\mathsf{P}^7} \to \mathcal{O}(4) \to 0$$

  $$0 \to \Lambda^4 T_Q(-4) \to \Lambda^4 T_{\mathsf{P}^7}(-4) \to \Lambda^3 T_Q \to 0$$

- $0 \to \mathcal{O}_{\mathsf{P}^7}(-1) \to \mathbf{C}^8 \to T_{\mathsf{P}^7}(-1) \to 0$

  $$0 \to \Lambda^3 T_{\mathsf{P}^7}(-4) \to \Lambda^4 \mathbf{C}^8 \to \Lambda^4 T_{\mathsf{P}^7}(-4) \to 0$$

  $$\Rightarrow H^0(\mathsf{P}^7, \Lambda^4 T_{\mathsf{P}^7}(-4)) \cong \Lambda^4 \mathbf{C}^8$$

- Heisenberg group $\mathbf{Z}_2 \to \Gamma \to \mathbf{Z}_2^6 \cong H^1(C, \mathbf{Z}_2)$

  acts on $\mathbf{C}^8$ and $\dim(\Lambda^4 \mathbf{C}^8)^\Gamma = 7$

  $\Rightarrow$ 7-dimensional invariant subspace of $H^0(Q, \Lambda^3 T)$

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  $\Rightarrow$ 7-dimensional invariant subspace of $H^0(Q, \Lambda^3 T)$

- $0 \to H^0(Q, \Lambda^4 T_{\mathrm{P}^7}(-4)) \to H^0(Q, \Lambda^3 T) \to H^1(Q, \Lambda^2 T^*) \to 0$

- $\dim H^1(Q, \Lambda^2 T^*) = g = 3$

- invariant subspace of $H^0(\mathcal{N}, \Lambda^3 T)$ of dimension $7 + 3 = 10$

  $\dim H^0(C, K^3) = 5g - 5 = 10$

- on Q

$$0 \to T_Q \to T_{\mathsf{P}^7} \to \mathcal{O}(4) \to 0$$

$$0 \to \Lambda^4 T_Q(-4) \to \Lambda^4 T_{\mathsf{P}^7}(-4) \to \Lambda^3 T_Q \to 0$$

- $0 \to \mathcal{O}_{\mathsf{P}^7}(-1) \to \mathbf{C}^8 \to T_{\mathsf{P}^7}(-1) \to 0$

$$0 \to \Lambda^3 T_{\mathsf{P}^7}(-4) \to \underline{\Lambda^4 \mathbf{C}^8 \to \Lambda^4 T_{\mathsf{P}^7}(-4)} \to 0$$

$$\Rightarrow H^0(\mathsf{P}^7, \Lambda^4 T_{\mathsf{P}^7}(-4)) \cong \Lambda^4 \mathbf{C}^8$$

- $\sigma_{ijk\ell} = \partial_i \wedge \partial_j \wedge \partial_k \wedge \partial_\ell \in \Lambda^4 T_{\mathsf{P}^7}(-4)$

  quartic $q = 0$, $dq \in H^0(Q, T^*_{\mathsf{P}^7}(4))$, $i_{dq}\sigma \in H^0(Q, \Lambda^3 T)$

- $[i_{dq}\sigma_{1234}, i_{dq}\sigma_{5678}] \neq 0$

- C genus 2, E rank 3 $\Lambda^2 E$ trivial

  $\mathcal{N} \cong$ double covering of P^8,

  branched over a sextic hypersurface

A. Ortega, *On the moduli space of rank 3 vector bundles on a genus 2 curve and the Coble cubic*, J. Algebraic Geom. **14** (2005) 327-356

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- $p : \mathcal{N} \to \mathsf{P}^8$,

  $p_* \mathcal{O}_{\mathcal{N}} \cong \mathcal{O}_{\mathsf{P}^8} \oplus \mathcal{O}_{\mathsf{P}^8}(-3)$

- $H^0(\mathcal{N}, \Lambda^3 T) \to H^0(\mathsf{P}^8, \Lambda^3 T \oplus \Lambda^3 T(-3))$

- $H^0(\mathsf{P}^8, \Lambda^3 T(-3)) \cong \Lambda^3 \mathbf{C}^9$

- invariants by Heisenberg group $\mathbf{Z}_3 \to \Gamma \to H^1(C, \mathbf{Z}_3)$

  $\cong \mathbf{C}^4 \cong$ anti-invariant part of $H^1(C, K^{-2})$

- $SL(3)$: degree 5 invariant form...

INTEGRABILITY:

intersection of quadrics

- intersection of quadrics X, dimension $2g-1$

  $=$ moduli space of semi-stable $Spin^c(2g)$ bundles

  on hyperelliptic C, genus g, invariant by involution τ

  (S.Ramanan 1981)

- Higgs fields adjoint representation, consider $SO(2g)$-bundles

  invariance $\Rightarrow$ degenerate orthogonal bundle on $\mathbf{P}^1$

  (U.Bhosle 1984)

- $H^0(C, K^3) = (p(z)y + q(z))\dfrac{dz^3}{y^3}$

$$\deg p = 2g-3, \deg q = 3g-3$$

$$\rho(\Phi_1, \Phi_2, \Phi_3) = q(z)\frac{dz^3}{y^3}$$

- Higgs field Φ rank 2

(V.Benedetti et al 2025)

moving frame $\Rightarrow$

$$\Phi(e_1) = -\sum_{i=1}^{N} \frac{x_i y_i}{z - \mu_i} e_1 - \sum_{i=1}^{N} \frac{y_i^2}{z - \mu_i} e_2$$

$$\Phi(e_2) = \sum_{i=1}^{N} \frac{x_i^2}{z - \mu_i} e_1 + \sum_{i=1}^{N} \frac{x_i y_i}{z - \mu_i} e_2$$

quasi-parabolic Higgs field on P^1

- Higgs field Φ rank 2

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quasi-parabolic Higgs field on P^1

- fix parabolic structure $y_i = c_i x_i$

cotangent vectors spanned by

$$\frac{x_i^2}{z-\mu_i} \begin{pmatrix} -c_i & -c_i^2 \\ 1 & c_i \end{pmatrix}$$

- $\mathrm{tr}([\Phi_1, \Phi_2], \Phi_3) = 2x_1^2 x_2^2 x_3^2 \dfrac{(c_1 - c_2)(c_2 - c_3)(c_3 - c_1)}{(z - \mu_1)(z - \mu_2)(z - \mu_3)}.$

$$= 2\frac{(x_2 y_1 - x_1 y_2)(x_3 y_2 - x_2 y_3)(x_1 y_3 - x_3 y_1)}{(z - \mu_1)(z - \mu_2)(z - \mu_3)}.$$

- $\rho = 2 \displaystyle\sum_{i<j<k} \frac{X_{ij} \wedge X_{jk} \wedge X_{ki}}{(z - \mu_i)(z - \mu_j)(z - \mu_k)} \qquad X_{ij} \in \mathfrak{so}(2g + 2)$

- polyvector fields $\quad \rho_i = \displaystyle\sum_{j<k, \neq i} \frac{X_{ij} \wedge X_{jk} \wedge X_{ki}}{(\mu_i - \mu_j)(\mu_i - \mu_k)}.$

- $\mathrm{tr}([\Phi_1,\Phi_2],\Phi_3) = 2x_1^2x_2^2x_3^2\frac{(c_1-c_2)(c_2-c_3)(c_3-c_1)}{(z-\mu_1)(z-\mu_2)(z-\mu_3)}.$

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- polyvector fields $\rho_i = \sum_{j<k,\neq i}\frac{X_{ij}\wedge X_{jk}\wedge X_{ki}}{(\mu_i-\mu_j)(\mu_i-\mu_k)}.$

Why do these commute?

Compare: Knizhnik-Zamolodchikov

- $A_i = \sum_{j \neq i} \frac{\Omega_{ij}}{\mu_i - \mu_j} \qquad \Omega_{ij} = \Omega_{ji}$

  $[A_i, A_j] = 0$ for all μ if for distinct i, j, k, ℓ

  $[\Omega_{ij}, \Omega_{k\ell}] = 0, \qquad [\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0$

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- $\rho_i = \sum_{j<k, \neq i} \frac{\Omega_{ijk}}{(\mu_i - \mu_j)(\mu_i - \mu_k)} \qquad \Omega_{ijk}$ alternating

- $[\rho_i, \rho_j] = 0$ if $[\Omega_{ijk}, \Omega_{\ell mn}] = 0$ when indices are distinct

  or have two in common

  or $[\Omega_{123}, \Omega_{145} + \Omega_{245} + \Omega_{345}] = 0$ etc.

Our example: $\Omega_{ijk} = X_{ij} \wedge X_{jk} \wedge X_{ki}$

- distinct indices $\sim$

  $\mathfrak{so}(3)$ for orthogonal 3-dimensional subspaces

  $\Rightarrow \Omega_{123}, \Omega_{456}$ commute

- $[X_{12} \wedge X_{23} \wedge X_{31}, X_{12} \wedge X_{24} \wedge X_{41}] = 0$

  compute in $\mathfrak{so}(4)$

- $[\Omega_{123}, \Omega_{145} + \Omega_{245} + \Omega_{345}] = 0$

  $\mathfrak{so}(5) \cong \mathfrak{so}(5)^*$ bi-invariant 3-form commutes with everything

Note:

- $g > 4$, rank 2 $H^0(\mathcal{N}, \Lambda^2 T) = 0$ but

$$\dim H^0(Q \cap Q_\mu, \Lambda^2 T) = \binom{n+3}{4}$$

- $$\sigma = (\mu_1 + \mu_2 - \mu_3 - \mu_4) X_{12} \wedge X_{34} +$$

$$+ (\mu_1 + \mu_3 - \mu_4 - \mu_2) X_{13} \wedge X_{42} + (\mu_1 + \mu_4 - \mu_2 - \mu_3) X_{14} \wedge X_{23}$$

- σ and $[\sigma, \sigma]$ not Γ-invariant

Poisson structures:

F.Loray, J. V.Pereira & F.Touzet, *Foliations with trivial canonical bundle on Fano 3-folds*, Math. Nachr. **286** (2013) 921—940.

P.Belmans : $\dim H^0(Q \cap Q_\mu, \Lambda^2 T) = \binom{n+3}{4}$

- $\mathfrak{so}(4) = \Lambda^2 \mathbf{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-$

- trace-free symmetric tensors: $\Lambda^2_+ \otimes \Lambda^2_-$

- $\mathrm{diag}(\mu_1, \mu_2, \mu_3, \mu_4) - \frac{1}{4}\sum_{i=1}^{4} \mu_i I$

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- $\dim H^0(X, \Lambda^3 T) = [0, 19, 98, 322, 840, 1890, 3828, 7161...]$

HOMOLOGY AND COHOMOLOGY

- $H^0(\mathcal{N}, \Lambda^3 T) \subset H^*(\mathcal{N}, \Lambda^* T)$ tangential cohomology

- exterior multiplication + Schouten bracket

- $H^0(\mathcal{N}, \Lambda^3 T) \subset H^*(\mathcal{N}, \Lambda^* T)$ tangential cohomology

- exterior multiplication + Schouten bracket

- contraction with classes in $H^*(\mathcal{N}, \Lambda^* T^*)$

  e.g.

$$H^1(\mathcal{N}, T) \otimes H^1(\mathcal{N}, \Lambda^2 T^*) \to H^2(\mathcal{N}, T^*)$$

$$H^1(C, K^{-1}) \otimes H^0(C, K) \to H^1(C, \mathcal{O})$$

$$H^0(\mathcal{N}, \Lambda^3 T) \otimes H^1(\mathcal{N}, \Lambda^2 T^*) \to H^1(\mathcal{N}, T)$$

$$H^1(C, K^{-2}) \otimes H^0(C, K) \to H^1(C, K^{-1})$$

- odd integrable system $\Rightarrow$

  Schouten-commuting algebra acting on $H^*(\mathcal{N}, \Lambda^* T)$

- e.g. action of $\rho \in H^0(\mathcal{N}, \Lambda^3 T)$ on $H^1(\mathcal{N}, T)$

  $\Rightarrow$ class in $H^1(\mathcal{N}, \Lambda^3 T)$

  $=$ obstruction to extending in deformation of complex structure

  $= 0$

- $c_1 > 0 \Rightarrow H^q(\mathcal{N}, \Lambda^p T) = 0$ for $q > p$ (Kodaira-Nakano)

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- rank 2, $H^0(\mathcal{N}, \Lambda^2 T) = 0$ $g > 4$

  ($\Rightarrow$ no noncommutative versions)

- commuting polyvector fields

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- $H^1(\mathcal{N}, T)$ deformations

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- $H^1(\mathcal{N}, T)$ deformations

- $H^0(\mathcal{N}, \Lambda^{top} T)$ Verlinde formula

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- $H^0(\mathcal{N}, \Lambda^{top} T)$ Verlinde formula

- what else? exterior products?

- HKR isomorphism:

  Hochschild homology $\cong H^*(M, \Lambda^* T^*)$

  Hochschild cohomology $\cong H^*(M, \Lambda^* T)$

- additively: multiplicatively compose with $\sqrt{\text{td}}$

Hochschild cohomology $\bigoplus_{p+q=n} H^q(\mathcal{N}, \Lambda^p T) \cong HH^n(\mathcal{N})$

- $HH^n(M)$ is an invariant of the derived category $\mathcal{D}^b(M)$

- For rank 2 $\mathcal{D}^b(\mathcal{N})$ semi-orthogonal decomposition

  with components $\mathcal{D}^b(C^{(k)})$ $0 \leq k \leq g-1$

  ($C^{(k)}$ = symmetric product)

  J.Tevelev & S.Torres, *The BGMN conjecture via stable pairs*, Duke Math.J. **173** (2024) 3495–3557.

  J.Tevelev, *Braid and phantom*, arXiv:2304.01825v3

- $\cdots \to HH^m(\mathcal{N}) \to HH^m(C^{(k)}) \to \cdots$

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- $\cdots \to HH^m(\mathcal{N}) \to HH^m(C^{(k)}) \to \cdots$

- $HH^2(\mathcal{N}) \cong H^2(\mathcal{N}, \mathcal{O}) \oplus H^1(\mathcal{N}, T) \oplus H^0(\mathcal{N}, \Lambda^2 T)$

$HH^2(C^{(k)}) \cong H^2(C^{(k)}, \mathcal{O}) \oplus H^1(C^{(k)}, T) \oplus H^0(C^{(k)}, \Lambda^2 T)$

- For rank 2 $\mathcal{D}^b(\mathcal{N})$ semi-orthogonal decomposition

  with components $\mathcal{D}^b(C^{(k)})$ $0 \leq k \leq g-1$

  ($C^{(k)}$ = symmetric product)

  J.Tevelev & S.Torres, *The BGMN conjecture via stable pairs*, Duke Math.J. **173** (2024) 3495–3557.

  J.Tevelev, *Braid and phantom*, arXiv:2304.01825v3

- $\cdots \to HH^m(\mathcal{N}) \to HH^m(C^{(k)}) \to \cdots$

- $HH^2(\mathcal{N}) \cong H^2(\mathcal{N}, \mathcal{O}) \oplus H^1(\mathcal{N}, T) \oplus H^0(\mathcal{N}, \Lambda^2 T)$

$$\downarrow$$

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- $H^1(\mathcal{N},T) \cong H^1(C,K^{-1}) \cong H^1(C^{(k)},T)$

- $\sqrt{\text{td}} = (p,p)$ classes

  $H^p(\mathcal{N},\Lambda^p T^*) \otimes H^q(\mathcal{N},\Lambda^q T) \to H^{p+q}(\mathcal{N},\Lambda^{q-p} T) = 0$

  $\Rightarrow$ usual exterior product in $H^*(\mathcal{N},\Lambda^* T)$

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- products of classes in $H^1(\mathcal{N}, T) \subset HH^2(\mathcal{N})$

  map to products of classes in $H^1(C^{(k)}, T)$

  $\Rightarrow$ non-trivial k-fold products for $k \leq g - 1$

- **Conjecture:** These vanish for $k \geq g$

  (Note: if $g = 2$, $H^2(\mathcal{N}, \Lambda^2 T) = 0$)

Note: INTERSECTION X^{2g-1} OF TWO QUADRICS

- $\mathcal{D}^b(X) = \{\mathcal{D}^b(C), \mathcal{O}_X(1), \ldots, \mathcal{O}_X(2g)\}$,

  C hyperelliptic genus g

  A.Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. in Math. **218** (2008) 1340–1369.

- $H^p(X, \Lambda^q T)$=0 for $p \geq 2$ (exact sequences from Q, Q_μ)

P.Belmans + 8, *On decompositions for Fano schemes of intersections of two quadrics,* arXiv:2403.12517v2

$X = F_k(Q_1 \cap Q_2)$

$= k$-dimensional linear subspaces ($\dim(Q_1 \cap Q_2)$ odd)

Conj: For $k = 0, \ldots, g-2$, $\mathcal{D}^b(X)$ has a semi-orthogonal decomposition with copies of $\mathcal{D}^b(C^{(i)})$ for $0 \leq i \leq k+1$

- $F_{g-2}(Q_1 \cap Q_2) =$

  moduli space $\mathcal{N}$ of quasiparabolic rank two bundles on P^1

- do g-fold products of $H^1(\mathcal{N}, T)$ vanish?