Pedagogical Lectures on MBL

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The Plan

Lecture 1: *Analytical tools for a proof of Anderson localization*
Lie-Schwinger rotations provide a graphical framework for stepwise diagonalization of the Hamiltonian. Nonperturbative regions are controlled probabilistically with moment estimates and the Markov inequality.

Lecture 2: *Existence of an MBL phase*
I will describe competing effects on the density of nonperturbative regions. In the RG, isolated nonperturbative regions can be eliminated, while nearby ones have to be merged. Percolation estimates ensure that these regions are compact and rare, maintaining a minimum exponential decay rate and forestalling the avalanche mechanism.

Lecture 3: *The MBL transition*
In order to understand the nature of the transition between the MBL and ETH phases, I will use a series of approximations to develop RG flow equations based on elimination and merging of nonperturbative regions. These equations resemble the Kosterlitz-Thouless (KT) flow equations, but there are important differences that place the MBL transition in a new universality class.

Outline of Lecture 3[1]

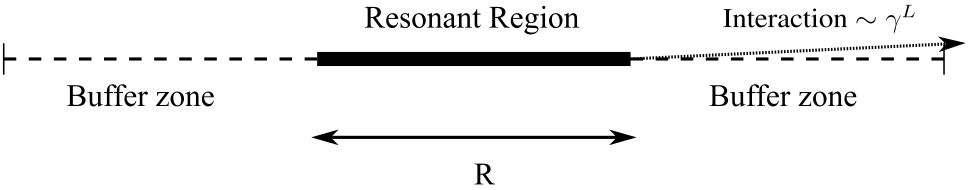
1. Insights from the MBL proof: Buffer zones; combining nearby resonant regions; effect of resonant regions on the decay rate
2. Simplified picture: Thermalized/Localized intervals
3. The transition out of the MBL phase
   3.1 Review of resonant regions, buffer zones, and the avalanche mechanism
   3.2 Approximate recursion relation leads to flow equations
   3.3 Correlation length exponent; comparison with KT

[1] Based on Morningstar, Huse, Imbrie: "Many-body localization near the critical point", PRE2020

Resonant regions (= Griffiths regions) need buffer zones

These are regions where we have failure of the bounds needed to control the rotations.

Buffer zones are needed so that the smallness $\sim \gamma^L$ of a graph crossing the buffer is much smaller than the typical $\Delta E = 2^{-R}$ in the resonant region.



The buffer zone is expected to be thermalized by the resonant region.

In 1-d the buffer zone has volume comparable to that of the resonant block, so we can diagonalize H in the combined region, eliminating internal interactions while keeping the level-spacing larger than the interactions with spins outside.

Renormalization group picture

In RG terms, the rotations removing terms in the Hamiltonian up to order γ^L is analogous to "integrating out" short distance degrees of freedom in traditional RG.

At the same time, resonant regions up to some size R are "eliminated" once L is large enough so that the remaining interaction terms are smaller than the level spacing in the region (with its buffer zone, total size $R + 2L$). At that point, the region hosts a "metaspin" which takes 2^{R+2L} values, but the interactions are so small that there is little hybridization with spins elsewhere.

Note two effects are in play:
(1) Elimination of smaller resonant regions reduces the density.
(2) Fattening of the buffer zones on the remaining regions can cause neighboring resonant blocks to merge. These effects increase the density.

My MBL proof shows that (1) dominates (2) deep in the weak coupling/strong disorder region, and the density goes to zero as $L \to \infty$.

Moving toward the transition: the avalanche effect

For weaker disorder/stronger interactions, the decay rate can be reduced to the point where no buffer size can insulate the resonant region from the rest of the chain: the avalanche instability[2].

Flip rates for off-diagonal matrix elements connecting the resonant region with spins outside the buffer zone should behave as $2^{-2L/\zeta}$ for some decay length ζ. For this to be small compared with the level spacing $\sim 2^{-(R+2L)}$, we need $\zeta^{-1} > 1$. Let $x = \zeta^{-1} - 1$ be the excess decay rate. Equating $2^{-2L/\zeta} = 2^{-2L(1+x)}$ with $2^{-(R+2L)}$, we find that the buffer size must satisfy

$$L \geq \frac{R}{2x}.$$

As γ increases, the excess decay rate $x \to 0$ and then the buffer size L will diverge.

At some point, then, increasing γ causes (2) to dominate (1); *i.e.* the fattening effect dominates the eliminations, and the density of resonant regions grows with L.

[2] *Many-body delocalization as a quantum avalanche*. Thiery, Huveneers, Müller, De Roeck, PRL '18

Simplified strong-disorder RG picture, I

At a given cutoff Λ, the line consists of alternating localized intervals (L-blocks) and thermalized intervals (T-blocks). Assumes bimodality is strongly attractive near transition. Assume the decay rate deficit x is constant in space.[3]

- L-blocks represent intervals where quasilocal basis changes have been defined.
- T-blocks have minimum length Λ; they represent intervals where the basis change cannot be defined due to too-strong interactions with the environment.
- As $\Lambda \to \Lambda + d\Lambda$, T-blocks of length $\in [\Lambda, \Lambda + d\Lambda]$ are *erased* (absorbed into neighboring L-blocks) if they are *isolated*, that is, separated by more than the buffer size Λ/x from other T-blocks.

[3]This approximation can be justified near the transition using Chayes-Harris arguments, once we have solved for the length divergence.

Simplified strong-disorder RG picture, II

- If a T-block is *not isolated*, then it pairs with a neighboring T-block that lies within the distance Λ/x to form a larger T-block (eliminating the intervening L-block). Such blocks do not have enough room to localize separately.

Simplified strong-disorder RG picture, III

- The avalanche parameter x flows downward with the RG because erased T-blocks interrupt the decay of interactions.

Functional RG

Due to the quenched (iid) randomness, we can assume that T-blocks appear "at random" with an exponential distribution in space for each subsequent T-block (outside of the minimum distance Λ/x as determined by the RG rules). Letting R_Λ denote the rate for this exponential distribution, we have that $R_\Lambda \exp(-R_\Lambda w)dw$ is the probability that length of an L-block lies in $[\Lambda/x + w, \Lambda/x + w + dw]$.

This rate can be broken down according to the length ℓ of the T-block that appears after the L-block: $R_\Lambda = \int_\Lambda^\infty r_\Lambda(\ell)d\ell$.

The full functional RG describes the flow with Λ of the function $r_\Lambda(\ell)$ and x:

$$\frac{dx}{d\Lambda} = -\frac{\Lambda r_\Lambda(\Lambda)(1+x)}{1+\Lambda R_\Lambda/x}$$

$$\frac{dr_\Lambda(L)}{d\Lambda} = \frac{1}{x}\left(\frac{dx}{d\Lambda} - R_\Lambda\right) r_\Lambda(L) + \frac{1}{x}\Theta(L-[2+x^{-1}]\Lambda)\int_\Lambda^{L-(1+x^{-1})\Lambda} d\ell r_\Lambda(\ell) r_\Lambda(L-\ell-\Lambda/x).$$

Reduction to two parameters

The rate $r_\Lambda(\Lambda)$ has dimensions $1/(\text{length})^2$, so let us define a dimensionless rate

$$y = y_\Lambda = \Lambda^2 r_\Lambda(\Lambda).$$

We anticipate that $y = 0$, $x \geq 0$ will be the MBL fixed line, due to the vanishing density of T-blocks. The phase transition will be governed by the point $x = y = 0$, where the fixed line becomes unstable because the interaction decay rate reaches the critical value for avalanches.

Recursion, I

The dominant mode of production of T-blocks of size Λ/x should be the combination of component T-blocks of size close to Λ. (The most efficient – least unlikely – way to create a T-block of a given length is to combine the smallest possible subblocks.)

Recursion, II

Fixing the distance between the outer edges of the pair of blocks at Λ/x, the inner edges remain free. The rate for a block with one edge free is R_Λ, so we obtain a recursion relation[4]

$$r_\Lambda(\Lambda/x) = R_\Lambda^2. \tag{1}$$

For the same reason, $r_\Lambda(\ell)$ should depend weakly on Λ between $x\ell$ and ℓ.

For dimensional reasons, $r_\Lambda(\ell) \approx y_\Lambda/\ell^2$ for $\Lambda \leq \ell \leq \Lambda/x$ and $R_\Lambda \approx \Lambda r_\Lambda(\Lambda)$.

Combining these facts with the recursion (1), we obtain a recursion for y:

$$y_{\Lambda/x} = (\Lambda/x)^2 r_{\Lambda/x}(\Lambda/x) \approx (\Lambda/x)^2 r_\Lambda(\Lambda/x) = (\Lambda/x)^2 R_\Lambda^2 = (\Lambda^2 r_\Lambda(\Lambda))^2/x^2 = \left(\frac{y_\Lambda}{x_\Lambda}\right)^2.$$

[4] This relation is consistent with dimensional analysis as r_Λ has dimension $1/\text{length}^2$ and R_Λ has dimension $1/\text{length}$.

Recursion, III

The above heuristic derivation of the recursion

$$y_{\Lambda/x} = \left(\frac{y_\Lambda}{x_\Lambda}\right)^2$$

can be confirmed quantitatively by analyzing the functional RG equations

$$\frac{dx}{d\Lambda} = -\frac{\Lambda r_\Lambda(\Lambda)(1+x)}{1+\Lambda R_\Lambda/x}$$

$$\frac{dr_\Lambda(L)}{d\Lambda} = \frac{1}{x}\left(\frac{dx}{d\Lambda} - R_\Lambda\right) r_\Lambda(L) + \frac{1}{x}\Theta(L - [2+x^{-1}]\Lambda)\int_\Lambda^{L-(1+x^{-1})\Lambda} d\ell\, r_\Lambda(\ell) r_\Lambda(L-\ell-\Lambda/x).$$

assuming that y and y/x are small. Within these approximations, the first equation gives the flow equation for x:

$$\Lambda\frac{dx}{d\Lambda} = -y.$$

Behavior of the recursion/flow

As is customary, we use $t = \log \Lambda$ to parametrize the RG.
The recursion/flow can then be written as:

$$\frac{dx}{dt} = -y, \qquad y_{\Lambda/x} = \left(\frac{y_\Lambda}{x_\Lambda}\right)^2, \tag{2}$$

with the equation for x representing the decrease in decay rate due to the erasure of T-blocks at the cutoff Λ ("rule of halted decay").

If we start on the curve $y = x^{2+\delta}$, then the image under the recursion is close to the curve $y = x^{2+2\delta}$. Hence the separatrix is asymptotic to the curve $y = x^2$.

The flow along the separatrix is then determined, with $x \sim t^{-1}$, $y \sim t^{-2}$.

Below the separatrix, x freezes and determines the scale jumps in forming thermal blocks, hence their fractal dimension $(\log 2)/(\log x^{-1})$.

Near and below the separatrix we do indeed have y/x small as long as y is small.

Diverging length

A diverging length may be defined as the point where an orbit departs the vicinity of the separatrix, from an initial small displacement δ_0. We find that this length is

$$\Lambda = e^t = \delta_0^{-\log_2 \log_2 \delta_0^{-1}}. \tag{3}$$

This evidently diverges faster than any power of δ_0, so we have in effect $\nu = \infty$.

The above form for the divergence of length at the critical point may be distinguished from the KT form: $\Lambda = \exp(\text{const} \cdot \delta_0^{-1/2})$.

An alternative approach to defining ν is to equate $\Lambda = \delta_0^{-\nu}$ and solve for ν as a function of Λ (this is with δ_0 considered as a function of Λ through eqn. 3). We find

$$\nu(\Lambda) \approx \log_2 \log \Lambda,$$

which represents the effective ν that is seen in a box of size Λ. For example, $\nu(10^6) \sim 3$, $\nu(10^{60}) \sim 7$.

Aside: $W(L) \sim L$ essentially ruled out by the theorem discussed yesterday.

Qualitative differences with the KT flow

Like the KT flow, there is logarithmic slowdown along the separatrix and $\nu = \infty$. However in that case progress is slow both along the separatrix and orthogonal to it.

Here we have exponential divergence from the separatrix along with slow progress along it, which leads to an unusual degree of sensitivity to the initial condition, if one wants to see the system remain critical at large length scales.

$\nu = \infty$ justifies neglect of spatial fluctuations in x

The Chayes *et al* inequality $\nu \geq 2$ is satisfied, which tells us that fluctuations of size $\Lambda^{-1/2}$ in x are small in comparison to the initial displacement $\delta_0^{-1/\nu}$ that is needed to depart the vicinity of the separatrix at scale Λ. In this way, we can justify the neglect of fluctuations in x in the derivation of these equations.

Equivalent flow equations

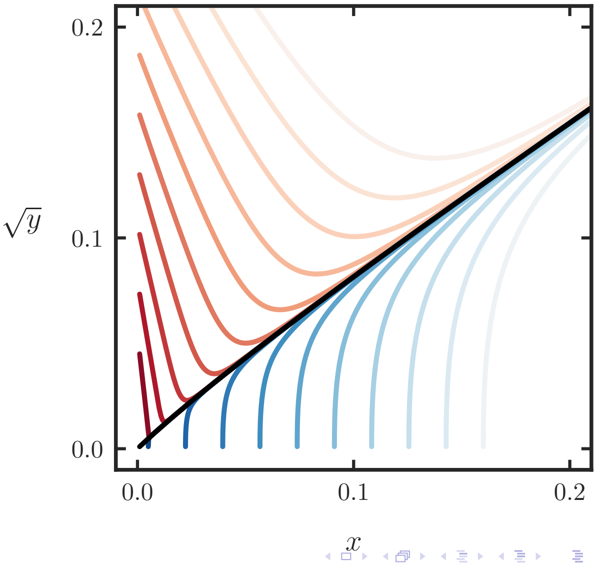
The following flow equation for y leads to the same critical behavior as the recursion:

$$\frac{dy}{dt} = -(\log 2)y\delta$$
$$= -(\log 2)y\left(\frac{\log y}{\log x} - 2\right).$$

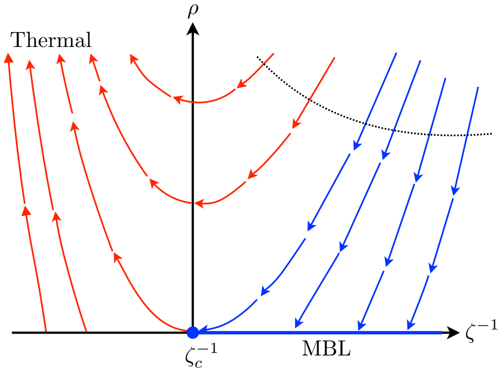
Note the nonanalyticity: Logs are needed in order to put the squaring operation on an infinitesimal level.

Unchanged flow equation for x:

$$\frac{dx}{dt} = -y$$



Parallels with the KT transition



Like the vortices, T-blocks represent nonperturbative effects, and the tendency of these effects to grow or shrink with the flow determines the phase reached from any starting point in the diagram. Vortex binding is analogous to T-block erasure as discussed above.

When bound, vortices renormalize the stiffness (screening). Likewise, when eliminated, T-blocks renormalize the decay rate (anti-screening).

Note: Earlier works (Dumitrescu et al, Goremykina et al., 2019) suggested a KT picture for the transition but assumed analytic flow equations. Our flow involves a factor $(\log y)/(\log x) - 2$, which puts this problem in a different universality class from KT.

Generalizations

A similar analysis[5] may be accomplished with correlated disorder wherein sums of N disorder variables scales as N^γ with $\gamma \neq \frac{1}{2}$ (*i.e.* with an exponent describing non-central-limit-theorem behavior). The critical theory was found to universal both for $\gamma > \frac{1}{2}$ (positively correlated disorder) and for $\gamma < \frac{1}{2}$ (hyperuniform disorder).

The nature of the transition for quasiperiodic "disorder" remains an open quesiton.

[5] Shi, Khemani, Vasseur, Gopalakrishnan: "Many body localization transition with correlated disorder" PRB2022