A universal moduli space

Nigel Hitchin
Mathematical Institute
University of Oxford

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THE HYPERKÄHLER METRIC

- curve C $g \geq 2$

- Higgs bundles (E, Φ), $\Phi \in H^0(C, \mathrm{End}\, E \otimes K)$

- Hermitian metric: equations $F_A + [\Phi, \Phi^*] = 0$

  equivalently $\nabla_A + e^{i\theta}\Phi + e^{-i\theta}\Phi^*$ is a flat connection for all θ

- moduli space $\mathcal{M}$: infinite-dimensional hyperkähler quotient

- complex structures I, J, K

  I: moduli space of Higgs bundles

  J: moduli space of flat connections

  Hom$(\pi_1(C), GL(n, \mathbf{C}))/GL(n, \mathbf{C})$

- $(\mathcal{M}, I)$, complex symplectic form $\omega_2 + i\omega_3$, depends on complex structure of C

  $(\mathcal{M}, J)$ complex symplectic form $\omega_3 + i\omega_1$, independent of complex structure of C

- isometric circle action $\Phi \mapsto e^{i\theta}\Phi$

- holomorphic on $(\mathcal{M}, I)$, preserves Kähler form ω_1

  vector field X, moment map $f = -\|\Phi\|^2/2$

- on $\mathcal{M}_B = (\mathcal{M}, J)$ $\omega_2 =$ Kähler form $\omega_2 = -dJdf$

$f : \mathcal{M}_B \to \mathbf{R}$ determines the hyperkähler metric

- $\mathcal{T}$ = Teichmüller space

  complex manifold, parametrizes complex structures on C

- action of mapping class group $\pi_0(\mathrm{Diff}(C))$

- $f : \mathcal{M}_B \times \mathcal{T} \to \mathbf{R}$

  determines all hyperkähler metrics

  – the geometry of the universal family

L.Álvarez-Cónsul, M.Garcia-Fernandez, O.García-Prada & S.Trautwein, *Universal Hitchin moduli spaces*, **arXiv**:2512.07553

B.Collier, J.Toulisse & R.Wentworth, *Higgs bundles, isomonodromic leaves and minimal surfaces*, **arXiv**:2512.2512.07152

- $(\mathcal{M}, I) = \mathcal{M}_{Dol} =$ moduli space of Higgs bundles (E, Φ)

  $\Rightarrow$ holomorphic family over $\mathcal{T}$

- isomonodromic leaf = Higgs bundles defining the same

  point in $(\mathcal{M}, J) = \mathcal{M}_B =$ character variety

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- **Aim:** Use f to construct the holomorphic family on $\mathcal{M}_B \times \mathcal{T}$

CONNECTIONS

- fibration $\pi : E \to B$, fibres $\cong M$

  Ehresmann connection = horizontal subbundle H of TE

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- adjoint action = covariant derivative on $C^\infty(T_F)$

  T_F = tangent bundle along fibres

- $\nabla_X Y = [\tilde{X}, Y]$, $\tilde{X}$ = horizontal lift of X

- $E = \mathcal{M}_B \times \mathcal{T} \to \mathcal{T} = B$

  flat connection ∇_B, preserving J and ω_3, ω_1

- acts on Hamiltonian vector fields wrt ω_1

  $h : \mathcal{M}_B \times B \to \mathbf{R}$ = section of the adjoint bundle

  $\nabla_B h$ = derivative in the direction of the base B

- $$\dot{f} + i\dot{g} = -\frac{1}{2}\int_C \mu \, \mathrm{tr}\, \Phi^2 \qquad \mu \in \Omega^{0,1}(C, K^*)$$

  $\nabla_B f = \beta$, a section of $\pi^* T_B^*$

  $\dot{f} = i_Y \beta$, $Y \in T_B \cong H^1(C, K^*)$

- circle action along fibres, preserves ω_1

  does not preserve ∇_B

- transform $\nabla_B \Rightarrow S^1$-family of flat symplectic connections

- connections form an affine space – average over the group

  invariant connection $\nabla_A = \nabla_B + c$, c section of $\pi^* T^*_B$

  e.g. function h, $\nabla_A h = d_B h + \{c, h\}$ (Poisson bracket)

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- ω_1 constant ($\nabla_B \omega_1 = 0$)

$$\nabla_B\{f, h\} = \{\nabla_B f, h\} + \{f, \nabla_B h\}$$

$$\Rightarrow (\mathcal{L}_X \nabla_B)h = \{\nabla_B f, h\} = \{\beta, h\}$$

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- $$\varphi = \beta + i\gamma = -\frac{1}{2} \int_C \mu \, \mathrm{tr}\, \Phi^2$$

  $\Phi \mapsto e^{i\theta}\Phi$ acts as $e^{2i\theta}$ on φ

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- $\mathcal{L}_X \nabla_A = 0$ (invariance) $\Rightarrow \mathcal{L}_X(\nabla_B + c) = 0$

  $\Rightarrow \mathcal{L}_X c = -\beta = \mathcal{L}_X \gamma/2$

$$\boxed{c = \frac{1}{2}\gamma}$$

- $\nabla_B = \nabla_A + \frac{1}{2}\gamma = \nabla_A - \frac{i}{4}(\varphi - \bar{\varphi})$

- S^1 family of flat connections

$$\nabla_\theta = \nabla_A - \frac{i}{4}(e^{2i\theta}\varphi - e^{-2i\theta}\bar{\varphi}).$$

- Flatness:

$$d_A\varphi = 0, \qquad \{\varphi, \varphi\} = 0, \qquad F_A + \frac{1}{8}\{\varphi, \bar{\varphi}\} = 0$$

(ω_1 Poisson brackets)

THE HIGGS BUNDLE ANALOGY

- ∇_A – “unitary connection”

  Lie group – Hamiltonian diffeomorphisms of $(\mathcal{M}, \omega_1)$

  Lie algebra – functions $h : \mathcal{M} \to \mathbf{R}$

- S^1-invariance $\Rightarrow$

  ∇_A acting on functions commutes with $h \mapsto \{f, h\}$

  $\Rightarrow \nabla_A f = 0$

  $\Rightarrow$ connection reduces to stabilizer of $S^1 \subset \mathrm{Ham}(\mathcal{M})$

- $\bar{\partial}_A$ – holomorphic structure on complexification of adjoint bundle

- φ – "Higgs field"

  $\{\varphi, \varphi\} = 0$, ω_1 Poisson bracket

- $\varphi = \sum \varphi_i dt_i$

  I-holomorphic, ω_1 type $(1,1)$

- g, h holomorphic

  Hamiltonian vector field of g: $X_g = \sum \omega_1^{i\bar{j}} \frac{\partial g}{\partial z_i} \frac{\partial}{\partial \bar{z}_j}$

  $\{g, h\} = X_g(h) = 0$

- S^1-action $\varphi \mapsto e^{2i\theta}\varphi$

  "variation of Hodge structure"

- invariant by $\varphi \mapsto -\varphi$

- Lie algebra $C^\infty(\mathcal{M}) \otimes \mathbf{C}$

  $= (f) \oplus$ odd weights $\oplus$ even weights

But...

- S^1-family of "unitary" connections

- involution on Lie algebra $h \mapsto \bar{h}$

$$F_A + \frac{1}{8}\{\varphi, \bar{\varphi}\} = 0 \quad \sim F_A - [\Phi, \Phi^*] = 0$$

- $\sim$ (pluri)harmonic map to compact G or symmetric space

THE CONNECTION ∇_A

- $\nabla_A f = 0$ and $\nabla_A X = 0$

- parallel translation: path $g : [0,1] \to \mathcal{T}$

  integrate a time-dependent vector field to give a Hamiltonian isotopy from the fibre at $t = 0$ to $t = 1$

- $\mathcal{M}$ noncompact but note: $|f| \leq N$ compact since f is proper

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- **Corollary**: fixed points of S^1, or of a subgroup, are diffeomorphic on varying the complex structure

  e.g. cyclic Higgs bundles

- curvature F_A: section of $\pi^*\Lambda^2 T^*_B \subset \Lambda^2 T^*_{\mathcal{M}\times B}$

$$F_A + \frac{1}{8}\{\varphi, \bar{\varphi}\} = 0 \qquad \varphi = -\frac{1}{2}\int_C \mu \, \mathrm{tr}(\Phi^2)$$

- $[\mu] \in H^1(C, K^*) = T^{1,0}_B \Rightarrow \varphi$ type $(1,0)$

$\Rightarrow F_A$ type $(1,1)$

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- $0 = \underline{\{\varphi, \varphi\}} = \{\beta + i\gamma, \beta + i\gamma\}$

  $\Rightarrow \{\beta, \gamma\} = 0, \{\beta, \beta\} = \{\gamma, \gamma\}$

- $\underline{\{\varphi, \bar{\varphi}\}} = \{\beta + i\gamma, \beta - i\gamma\} = 2\{\gamma, \gamma\}$

  $\Rightarrow \underline{F_A = -\{\gamma, \gamma\}/4}$

$d_A \varphi = 0$

- $d_B \varphi = \{\gamma, \varphi\}/2$

$$\Rightarrow d_B \gamma = \{\gamma, \gamma\}/2 = -2F_A$$

$$d_A\varphi = 0$$

- $d_B\varphi = \{\gamma, \varphi\}/2$

$$\Rightarrow d_B\gamma = \{\gamma, \gamma\}/2 = -2F_A$$

- I_B = complex structure on B = Teichmüller space

$$I_B(\beta + i\gamma) = i(\beta + i\gamma) \Rightarrow d_B I_B d_B f = 2F_A$$

- $2F_A = d_B I_B d_B f = -2i \sum_{a,b} \frac{\partial^2 f}{\partial t_a \partial \bar{t}_b} dt_a \wedge d\bar{t}_b$

Levi form/ complex Hessian

$$F_A + \frac{1}{8}\{\varphi, \bar{\varphi}\} = 0$$

- $$\{\varphi, \bar{\varphi}\} = \sum_{a,b,k,\ell} \omega_1^{k\bar{\ell}} \frac{\partial \varphi_a}{\partial z_k} \frac{\partial \bar{\varphi}_b}{\partial \bar{z}_\ell} dt_a \wedge d\bar{t}_b$$

- ω_1 positive definite Hermitian form

  complex matrix $J_{ak} = \dfrac{\partial \varphi_a}{\partial z_k}$

- $-2f = \|\Phi\|^2 \sim$ energy of equivariant harmonic map

  $\Rightarrow$ Levi form is non-negative

D.Toledo, *Hermitian curvature and plurisubharmonicity of energy on Teichmüller space,* GAFA **22** (2012), 1015–1032.

D.Toledo, *Hermitian curvature and plurisubharmonicity of energy on Teichmüller space,* GAFA **22** (2012), 1015–1032.

O.Tošic, *Non-strict plurisubharmonicity of energy on Teichmüller space,* IMRN (2024)(9), 7820–7845.

- kernel of Levi form = null space of $J_{ak} = \dfrac{\partial \varphi_a}{\partial z_k}$

$$= \{Y \in T_B : i_Y X_\varphi = 0\}$$

(X_φ = Hamiltonian vector field of φ wrt ω_1)

= critical point of $i_Y \varphi$

(= quadratic function of the integrable system)

- **Thm:** (Tošic) Define $R : \mathcal{M} \times B \to \mathcal{M}$ by

$$R : (E, \Phi) = (E, i\Phi).$$

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- **Prf:** The kernel of DR is the intersection of the horizontal space of $\nabla_B = \nabla_A + \gamma/2 = \nabla_A - \frac{i}{4}(\varphi - \bar{\varphi})$ and its transform by $e^{i\theta} = i$.

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$$\nabla_A - \frac{i}{4}(e^{2i\theta}\varphi - e^{-2i\theta}\bar{\varphi}).$$

Transform $= \nabla_A - \gamma/2$

$\Rightarrow$ intersection is defined by $i_Y X_\gamma = 0$.

$\mathcal{M} \times \mathcal{T}$ AS A COMPLEX MANIFOLD

- complex structure I on fibres

  complex structure I_B on base $B = \mathcal{T}$

- horizontal subbundle $H \cong \pi^* T_B$, $T_{\mathcal{M}\times B} \cong T_F \oplus H$

  almost complex structure $(I, \pi^* I_B)$

Thm: This almost complex structure is integrable.

- $\pi : \mathcal{M} \times B \to$ is holomorphic
- $S^1 \subset \mathbf{C}^*$-action is holomorphic
- φ is a holomorphic 1-form

Prf:

- $\bar\partial_{\mathcal{M}\times B} = \bar\partial_F + \bar\partial_A$: we need $\bar\partial^2_{\mathcal{M}\times B} = 0$

  $\bar\partial^2_F = 0$ (fibres holomorphic)

  $\bar\partial^2_B = 0$ (B holomorphic and F_A type $(1,1)$)

- $\omega^c = \omega_2 + i\omega_3$ defines complex structure on fibres

  (θ type $(1,0)$ iff $(\omega^c)^n \wedge \theta = 0$)

- $\bar\partial_F\bar\partial_A + \bar\partial_A\bar\partial_F = 0$ if $\underline{\bar\partial_A\omega^c} = 0$

- tangent direction Y on B, $i_Y\beta = b$

  variation of $\omega_2 + i\omega_3 = \dot{\omega}_2$ (ω_3 fixed)

$$\omega_2 = -dJdf \Rightarrow \dot{\omega}_2 = -dJdb$$

- $\nabla_A = \nabla_B + \frac{1}{2}\mathcal{L}_{X_c}$ where $i_Y\gamma = c$

- $\nabla_Y \omega^c = -dJdb + \frac{1}{2}dJd(b + ic) = \frac{1}{2}dJd(-b + ic)$

  $(0, 1)$ component vanishes

EXAMPLE

- E rank 2, $\Lambda^2 E$ fixed, odd degree

  C genus 2: $y^2 = (z-\mu_1)\cdots(z-\mu_6)$

- moduli space of stable bundles $\mathcal{N} = Q \cap Q_\mu$

  $$\sum_{i=1}^{6} x_i^2 = 0 = \sum_{i=1}^{6} \mu_i x_i^2$$

  $T^*\mathcal{N} \subset \mathcal{M}$ moduli space of Higgs bundles

- moduli of complex structures on $C \sim$

  $\{\mu_1, \ldots, \mu_6\}$ modulo $PSL(2, \mathbf{C}) \sim \{\mu_1, \mu_2, \mu_3, 0, 1, \infty\}$

- $$\varphi = 4\sum_{i=1}^{6}\sum_{j\neq i}\frac{(x_iy_j - x_jy_i)^2}{\mu_j - \mu_i}d\mu_i$$

$x \wedge y \in \Lambda^2\mathbf{C}^6 \in T^*Q$ coadjoint orbit of $SO(6, \mathbf{C})$

A.Beauville, A.Höring, J.Liu & C.Voisin, *Symmetric tensors on the intersection of two quadrics and Lagrangian fibration*, Moduli (2024)

REAL FORMS

- G real form of G^c

- character variety: $\mathrm{Hom}(\pi_1(C), G)/G$

  Higgs bundles: $H \subset G$, maximal compact $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$

  principal H^c-bundle, $\Phi \in H^0(C, \mathfrak{m}^c \otimes K)$

- circle action $\Phi \mapsto e^{i\theta}\Phi$

- $\mathcal{M}^r \times B$, average connection etc.

EXAMPLE: $G = SL(2, \mathbf{R})$

- Hom$(\pi_1(C), SL(2, \mathbf{R}))$ uniformizing representation
- Higgs bundle:

  $K^{1/2} \oplus K^{-1/2}$, $\Phi(u, v) = (qv, u)$, $q \in H^0(C, K^2)$
- holomorphic fibration: $H^0(C, K^2) \to \mathcal{M}^r \times B \to B \cong \mathcal{T}$

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- holomorphic fibration: $H^0(C, K^2) \to \mathcal{M}^r \times B \to B \cong \mathcal{T}$
- $\varphi \in H^0(\mathcal{M}^r \times B, \pi^* T_B^*)$ defines $\mathcal{M}^r \times B \cong T^* B$

  $=$ cotangent bundle of Teichmüller space

  (φ becomes the canonical 1-form)

A HERMITIAN METRIC

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- $\Lambda^2 T_F^* \cong \Lambda^2 H^* \qquad \omega_1 \mapsto \omega_1 - \frac{1}{2} d_F \gamma$

- Kähler forms ω_1 on T_F, $\pi^*\omega$ on H

  ω Weil-Petersson form

$$\tilde{\omega} = \pi^*\omega + \omega_1 - \frac{1}{2} d_F \gamma$$

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  define a Hermitian metric so that these are orthogonal

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- $d\tilde{\omega} \neq 0, \qquad \partial\bar{\partial}\tilde{\omega} \neq 0...$

THE QUANTUM LINE BUNDLE

- $\omega_1 - \frac{1}{2} d\gamma$ is closed and

$$= \omega_1 - \frac{1}{2} d_F \gamma - \frac{1}{2} d_B \gamma$$

- $\frac{1}{2} d_B \gamma = -F_A$ type $(1,1)$

- $\Rightarrow \ \omega_1 - \frac{1}{2} d\gamma =$ curvature of a unitary connection on a holomorphic line bundle L

- ω_1 = curvature of prequantum line bundle on $\mathcal{M}$

  connection ∇ independent of complex structure

- $$d_B + \nabla - \frac{1}{2}\gamma \;\; = \text{connection on } L$$

  restricts to ∇ on each fibre $\mathcal{M} \subset \mathcal{M} \times B$

- L = holomorphic extension of prequantum line bundle

- ∇ on prequantum line bundle

  invariant under action of Hamiltonian functions

$$h \cdot s = \nabla_{X_h} s + ihs$$

- ∇ on prequantum line bundle

  invariant under action of Hamiltonian functions

$$h \cdot s = \nabla_{X_h} s + ihs$$

- symplectic connection $\nabla_A = \nabla_B - \frac{1}{2}\gamma$

  Lie algebra of Hamiltonian functions on $\mathcal{M}$

- preserves ∇ on L

- $\nabla_L^{0,1} = \bar{\partial}_A + \nabla^{0,1}$

  holomorphic sections $\nabla^{0,1} s = 0, \bar{\partial}_A s = 0$