Many-body quantum chaos in mixtures of multiple species

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Vijay Kumar & DR, *arXiv:2310.06811 (2023)*

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Outline

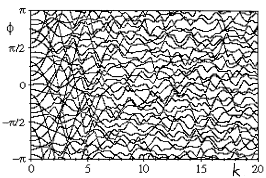
1. Many body quantum chaos (MBQC)
2. MBQC with single species
3. MBQC with mixtures of two species

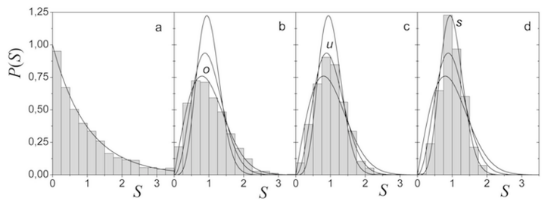
Quantum chaos : statistics of energy levels

Classical chaos : hypersensitivity of phase space trajectories to perturbations in initial conditions and long trajectories uniformly filling the available space.

Kicked top[1] : $\hat{H}(t) = \hat{H}_0 + \sum_n \hat{H}_1 \delta(t - n\tau), \ \hat{H}_0 = \hbar\omega \hat{J}_y, \ \hat{H}_1 = (\hbar k/2j)\hat{J}_z^2$

$\hat{U} = e^{-i(k/2j)\hat{J}_z^2} e^{-i\omega\tau \hat{J}_y}, \ \hat{U}|m\rangle = e^{-i\phi_m}|m\rangle$, spacing $S_m = \phi_{m+1} - \phi_m$





Quantum systems with integrable (nonintegrable) classical counterpart have quantum levels showing clustering or level crossing (level repulsion) when a parameter in the Hamiltonian is varied.

Level spacing distribution $P(S)$ for a kicked top under conditions of classically regular motion (e^{-S}) and chaos $(S^{\beta} e^{-cS^2}$ for $\beta = 1, 2, 4)$.

[1] Classical and quantum chaos for a kicked top, Haake, Kus & Scharf (1987)

Beyond one-particle : Many-body quantum chaos

Bohigas-Giannoni-Schmit (BGS) conjecture (1984) asserts that the spectral statistics of quantum systems whose classical counterparts exhibit chaotic behaviour are described by random matrix theory. Berry & Tabor (1976), (1977)

Research over twenty years could explain BGS conjecture for single-particle systems whose corresponding classical dynamics are fully chaotic.

A series of recent works could further establish such relationship for nonintegrable, extended, many-body systems where local degrees of freedom, e.g., qubits, fermions have no classical limit. [1]

These studies have analytically computed the spectral form factor (SFF) characterizing spectral fluctuations, and the derived SFF shows a good agreement with the RMT form, e.g., $K(t) = 2t - t\log(1 + 2t/t_H)$ for circular orthogonal ensemble (COE)

How (e.g., mechanism, nonuniversal behavior) & when (timescales) many-body quantum systems acquire a universal RMT form?

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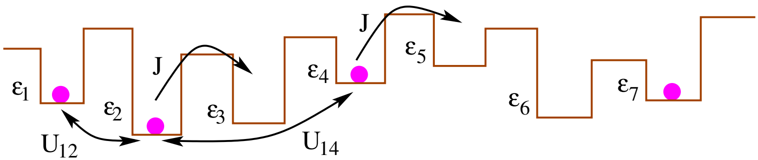
[1] P. Kos, M. Ljubotina & T. Prosen, Phys. Rev. X **8**, 021062 (2018)
A. Chan, A. De Luca & J. T. Chalker, Phys. Rev. Lett. **121**, 060601 (2018)
B. Bertini, P. Kos & T. Prosen, Phys. Rev. Lett. **121**, 264101 (2018)
A. J. Friedman, A. Chan, A. De Luca & J. T. Chalker, **123**, 210603 (2019)
D. Roy & T. Prosen, Phys. Rev. E **102**, 060202(R) (2020)

Periodically driven interacting single species

A 1D lattice of interacting spinless fermions or bosons with a time-periodic kicking in the nearest-neighbor coupling (hopping and pairing):



$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1 \sum_{m\in\mathbb{Z}} \delta(t-m),$$

$$\hat{H}_0 = \sum_{i=1}^{L} \epsilon_i \hat{n}_i + \sum_{i<j} U_{ij}\hat{n}_i\hat{n}_j,$$

$$\hat{H}_1 = \sum_{i=1}^{L}(-J\hat{a}_i^\dagger\hat{a}_{i+1} + \Delta\hat{a}_i^\dagger\hat{a}_{i+1}^\dagger + \text{H.c.}),$$

Number operator $\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$; creation operator of a fermion/boson $\hat{a}_i^\dagger$

$\Delta = 0$ or $\neq 0$ corresponds respectively to conservation or violation of a total fermion/boson number $\hat{N} = \sum_{i=1}^{L}\hat{n}_i$.

Spectral form factor (SFF) $K(t)$

Statistics of energy or quasienergy levels can be characterized by mean and fluctuations in the spectral density of energy or quasienergy.

Quasienergies of interest are the eigenphases φ_m of a unitary Floquet propagator $\hat{U}$ of evolution after one cycle: $\hat{U} = \mathcal{T}\exp(-i\int_0^1 dt\hat{H}(t))$

$\hat{U}|m\rangle = e^{-i\varphi_m}|m\rangle$ for $m = 1, 2, \ldots, \mathcal{N}$ (dimension of the Hilbert space)

Spectral density $\rho(\varphi) = \frac{2\pi}{\mathcal{N}}\sum_m \delta(\varphi - \varphi_m)$, $\langle\rho(\varphi)\rangle_\varphi \equiv \int_0^{2\pi}\frac{d\varphi}{2\pi}\rho(\varphi) = 1$

Pair correlation function $R(\vartheta) = \langle\rho(\varphi + \vartheta/2)\rho(\varphi - \vartheta/2)\rangle_\varphi - \langle\rho(\varphi)\rangle_\varphi^2$ provides a measure of spectral fluctuations.

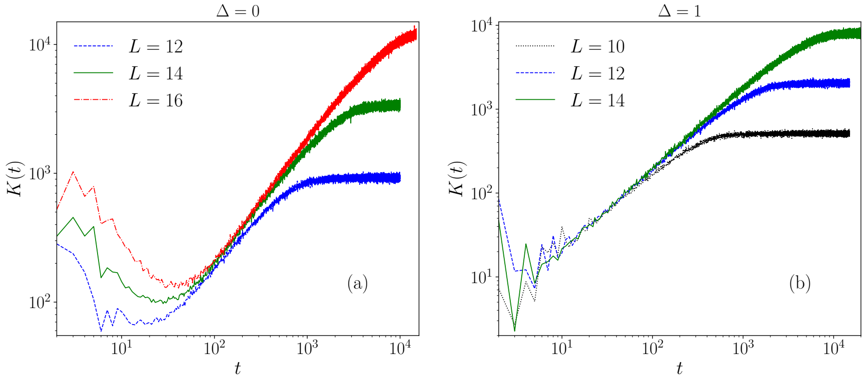
$$K(t) = \frac{\mathcal{N}^2}{2\pi}\int_0^{2\pi} d\vartheta\langle R(\vartheta)e^{-i\vartheta t}\rangle = \langle(\mathrm{tr}\hat{U}^t)(\mathrm{tr}\hat{U}^{-t})\rangle - \mathcal{N}^2\delta_{t,0}$$

where $\mathrm{tr}\hat{U}^t = \sum_m e^{-i\varphi_m t}$, and $\langle\ldots\rangle$ denotes an average over disorder.

$\hat{U}$ can be written as a two-step unitary Floquet propagator:

$$\hat{U} = \hat{V}\hat{W}, \quad \hat{W} = e^{-i\hat{H}_0} \text{ and } \hat{V} = e^{-i\hat{H}_1}$$

Exact numerically computed $K(t)$: fermions



Spectral form factor $K(t)$ for different system sizes L of the kicked spinless fermion chain with $(\Delta = 0)$ (a) and without $(\Delta = 1)$ (b) particle-number conservation. Here, $J = 1, U_0 = 15, \alpha = 1.5, \Delta\epsilon = 0.3$ and $N/L = 1/2$ for $\Delta = 0$. An averaging over 10^3 realizations of disorder is performed.

Mechanism to reach universal RMT form of $K(t)$

Let's consider a set of eigenbasis $|\underline{n}\rangle \equiv |n_1, n_2, \ldots, n_L\rangle$ of $\hat{H}_0$ and $\hat{W}$

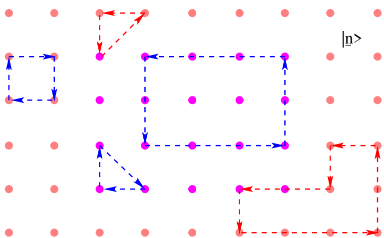
Using a random phase assumption (RPA) which essentially requires a long-range nature of the interaction in $\hat{H}_0$, the SFF can be written as

$$\begin{aligned} K(t) &= 2t\,\mathrm{tr}\mathcal{M}^t - t^2 f(\mathcal{M}, \mathcal{V}) + \mathcal{O}(t^3) \\ &= 2t(1 + \sum_{j=1}^{\mathcal{N}-1} \lambda_j^t) - \frac{2t^2}{\mathcal{N}} + \mathcal{O}(t^3) \end{aligned}$$

$\mathcal{M}_{\underline{n},\underline{n}'} = |\mathcal{V}_{\underline{n},\underline{n}'}|^2 = |\langle \underline{n}|\hat{V}|\underline{n}'\rangle|^2 = |\langle \underline{n}|e^{-iH_1}|\underline{n}'\rangle|^2$ is a $\mathcal{N} \times \mathcal{N}$ Markov matrix

$P_t(n) \equiv \langle \underline{n}|\mathcal{M}^t|\underline{n}\rangle$ is return probability to $|\underline{n}\rangle$ after t time steps

$P_t(n) \sim 1$ when $t \ll t^*$ and $P_t(n) \sim 1/\mathcal{N}$ when $t \geq t^*$



Mapping $\mathcal{M}$ to effective Hamiltonian

λ_j are found by (a) numerically diagonalizing $\mathcal{M}$, and (b) mapping $\mathcal{M}$ to an effective Hamiltonian in the Trotter regime, i.e., at small J, Δ.

Expand $\hat{V}$ in the Trotter regime of the Hamiltonian $\hat{H}_1$ [1]:

$$\begin{aligned}\mathcal{M} &= e^{-i\hat{H}_1} \bullet e^{i\hat{H}_1} \\ &= (\mathbb{1} - i\hat{H}_1 - \frac{1}{2}\hat{H}_1^2 + \dots) \bullet (\mathbb{1} + i\hat{H}_1 - \frac{1}{2}\hat{H}_1^2 + \dots) \\ &= \mathbb{1} + \hat{H}_1 \bullet \hat{H}_1 - \hat{H}_1^2 \bullet \mathbb{1} + \mathcal{O}(\hat{H}_1^4),\end{aligned}$$

where $\hat{H}_1 \bullet \hat{H}_1$ is an element-wise square of $\hat{H}_1$ in the Fock space basis.

For $J, \Delta \to 0$, $\mathcal{M}$ can be generated by anisotropic Heisenberg model.

$$\mathcal{M} = (1 - c_x L)\mathbb{1}_{\mathcal{N}} + \sum_{j=1}^{L} \sum_{\nu=x,y,z} c_\nu \sigma_j^\nu \sigma_{j+1}^\nu + \mathcal{O}(J^4, \Delta^4),$$

$c_x = (J^2 + \Delta^2)/2$, $c_y = c_z = (J^2 - \Delta^2)/2$. σ_j^ν: Pauli matrix at site j.

[1] D. Roy, D. Mishra & T. Prosen, Phys. Rev. E **106**, 024208 (2022)

Thouless time t^* to reach RMT form of $K(t)$

Eigenvalues λ_j may or may not depend on the dimension $\mathcal{N}$ of $\mathcal{M}$ which itself depends on L. Consider λ_j falls rapidly with increasing j and λ_1 scales with system size L as $1 - 1/t^*(L)$ where $t^*(L) \simeq L^\beta / D$[1]:

$$K(t) \simeq 2t(1 + \lambda_1^t) \simeq 2t(1 + (1 - 1/t^*(L))^t) \simeq 2t(1 + e^{-t/t^*(L)}).$$

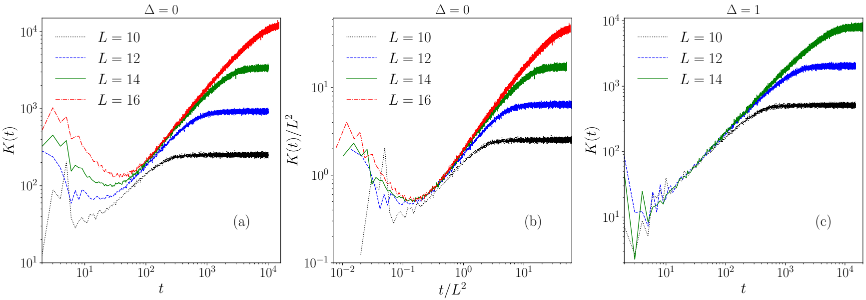
For $\Delta = 0$, isotropic Heisenberg model ($SU(2)$ symmetry) whose eigenenergy spectrum is gapless for any magnetization (any N). Eigenvalue of first "excited state" $\lambda_1 = 1 - c_1/L^2$ (one x-polarized magnon excitation with momentum $k = 2\pi/L$). $\beta = 2$ and Thouless time, $t^* \simeq L^2/c_1$. JHEP 7, 124 (2018), PRL **123**, 210603 (2019)

For $\Delta \neq J \neq 0$, anisotropic Heisenberg model which has a finite and system-size independent gap in the energy spectrum between the ground and first excited state. $\beta = 0$ and L-independent Thouless time.

For $\Delta = J \neq 0$, Ising model which has a finite and system-size independent gap in the energy spectrum between the ground and highly degenerate first excited state. $K(t) \simeq 2t(1 + \sum_{j=1}^{L} \lambda_j^t)$ and Thouless time $t^* \simeq \log L$: PRX **8**, 021062 (2018), PRL **121**, 060601 (2018)

[1] D. Roy & T. Prosen, Phys. Rev. E **102**, 060202(R) (2020)

Exact numerically computed $K(t)$: fermions

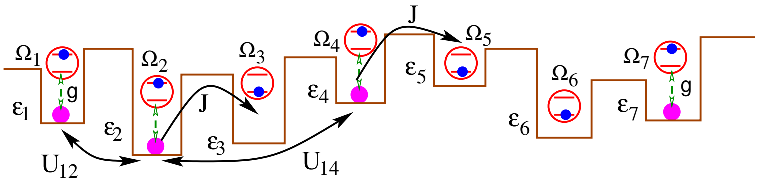


Spectral form factor $K(t)$ for different system sizes L of the kicked spinless fermion chain with $(\Delta = 0)$ (a,b) and without $(\Delta = 1)$ (c) particle-number conservation. Here, $J = 1, U_0 = 15, \alpha = 1.5, \Delta\epsilon = 0.3$ and $N/L = 1/2$ for $\Delta = 0$. An averaging over 10^3 realizations of disorder is performed. In (b) we show data collapse in scaled time t/L^2.

Temporal growth of $K(t)$ for $\Delta = 1$ at $t \ll t_H$ is independent of L which confirms our analytical prediction based on the RPA.

For $\Delta = 0$, we find a nice data collapse for various L and $t < t_H$ which confirms our above predicted L-dependence of $K(t)$ using the RPA.

Periodically driven mixtures of two species



A 1D lattice of fermions/bosons and qubits: mixing between two species and nearest-neighbor hopping of fermions/bosons are periodically modulated.
$\hat{H}(t) = \hat{H}_0 + \hat{H}_{\mathsf{JC}/R} \sum_{m\in\mathbb{Z}} \delta(t-m)$

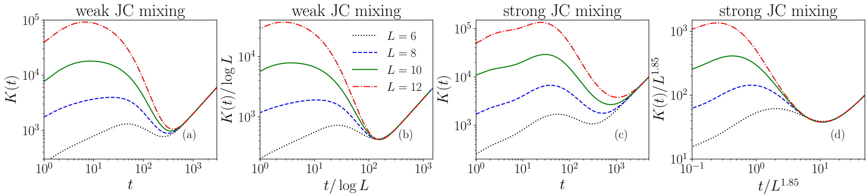
$$\hat{H}_0 = \sum_{i=1}^{L}(\epsilon_i \hat{n}_i + \Omega_i \hat{\sigma}_i^\dagger \hat{\sigma}_i) + \sum_{i<j} U_{ij}\hat{n}_i\hat{n}_j,$$

$$\hat{H}_{\mathsf{JC}} = \sum_{i=1}^{L} g(\hat{a}_i^\dagger \hat{\sigma}_i + \hat{\sigma}_i^\dagger \hat{a}_i) + \sum_{i=1}^{L}(-J\hat{a}_i^\dagger \hat{a}_{i+1} + \mathrm{H.c.}),$$

$$\hat{H}_{\mathsf{R}} = \sum_{i=1}^{L} g(\hat{a}_i^\dagger + \hat{a}_i)(\hat{\sigma}_i + \hat{\sigma}_i^\dagger) + \sum_{i=1}^{L}(-J\hat{a}_i^\dagger \hat{a}_{i+1} + \mathrm{H.c.}),$$

Total excitation number $\hat{N} = \sum_{i=1}^{L}(\hat{n}_i + \hat{\sigma}_i^\dagger \hat{\sigma}_i)$ is conserved for Jaynes-Cummings (JC) mixing but not for Rabi (R) mixing.

JC mixing : crossover in system-size scaling of t^*



$K(t)$ for different system sizes L with JC mixing between fermions and qubits for $g = 0.1, J = 0.4$ in (a,b), and $g = 0.4, J = 0.1$ in (c,d). We take half-filling $N/L = 1/2$. (b) and (d) show data collapse in scaled time $t/\log L$ and $t/L^{1.85}$.

$\mathcal{M}^{\mathsf{F}}_{\mathsf{J}C}$ has $SU(2)$ symmetry, and its eigenvalues (excluding largest) for $N = 1$:

$$\lambda_i = 1 - g^2 - J^2\big(1 - \cos\frac{2i\pi}{L}\big) + \sqrt{J^4\big(1 - \cos\frac{2i\pi}{L}\big)^2 + g^4},\ i = 1, 2 \ldots, L-1$$

For $(1-\cos\frac{2\pi}{L}) \ll (\frac{g}{J})^2 \Rightarrow L > l_c = \frac{\pi}{\sin^{-1}(\frac{g}{\sqrt{2}J})}$, $\lambda_1 \approx 1 - \frac{2\pi^2 J^2}{L^2} \Rightarrow t^* \propto \mathcal{O}(L^2)$

For $(1-\cos\frac{2i\pi}{L}) \gg (\frac{g}{J})^2$ at finite L $(< l_c)$, $\lambda_i \approx 1 - g^2 + (g^2/2J)^2 \csc^2(i\pi/L)$ for $i = 1, 2 \ldots, L-1$. Second largest eigenvalues for small g/J are approximately $L-1$ fold degenerate $\Rightarrow t^* \approx \mathcal{O}(\log L)$

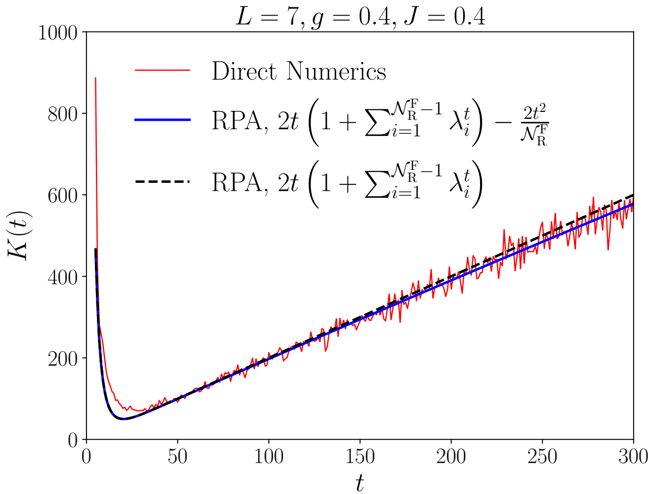
L-scaling of t^* & emergent symmetry of $\mathcal{M}$

For periodically driven (Floquet) models with homogeneous kicking in the Trotter regime:[1]

$\hat{H}(t)$	U(1) symmetric/JC-mixing		U(1) broken/R-mixing	
species	t^*	$\mathcal{M}$ symmetry	t^*	$\mathcal{M}$ symmetry
Fermion	L^2	$SU(2)$	$L^0, \log L$	$U(1)$, Ising
Qubit	L^2	$SU(2)$	$L^0, \log L$	$U(1)$, Ising
Boson	L^2	$SU(1,1)$	$\sim L^{0.7}$	$U(1)$
Fermion & qubit	$\log L$ to L^2	$SU(2)$	$\log L$	$U(1) \otimes u^{\otimes L}(1)$
Boson & qubit	$\log L$ to L^2	$\sim SU(1,1)$	$\sim \log L$	$u^{\otimes L}(1)$

[1] V. Kumar & D. Roy, *arXiv:2310.06811 (2023)*

Spectral form factor: exact vs. RPA



Comparison of the exact numerically computed SFF, $K(t)$ vs. t with that obtained using the RPA for Rabi mixing between fermions and qubits.

Summary

Study of spectral form factor infers how (e.g., mechanism, nonuniversal behavior) $\&$ when (timescales) many-body quantum systems acquire a universal RMT form.

L-dependence of Thouless time t^* crosses over from $\log L$ to L^2 with an increasing Jaynes-Cummings mixing between qubits and fermions or bosons in a finite-sized chain, and it finally settles to $t^* \propto \mathcal{O}(L^2)$ in the thermodynamic limit for any mixing strength.

Rabi mixing between qubits and fermions leads to $t^* \propto \mathcal{O}(\log L)$

V. Kumar $\&$ D. Roy, *arXiv:2310.06811 (2023)*

Mapping $\mathcal{M}$ to effective Hamiltonian : Bosons

Generating Hamiltonian in the Trotter regime of small J when $\Delta = 0$:

$$\mathcal{M} = \mathbb{1} + \sum_{i=1}^{L} \Big(J^2(\hat{K}_i^- \hat{K}_{i+1}^+ + \hat{K}_{i+1}^- \hat{K}_i^+) - 2J^2(\hat{K}_i^0 \hat{K}_{i+1}^0 - \frac{1}{4}) \Big) + \mathcal{O}(J^4)$$

in terms of $\hat{K}_i^0 = -(\hat{n}_i + 1/2),\ \hat{K}_i^+ = \hat{a}_i\sqrt{\hat{n}_i},\ \hat{K}_i^- = \sqrt{\hat{n}_i}\hat{a}_i^\dagger$, which satisfy the commutation relations of $SU(1,1)$ algebra

$$[\hat{K}_i^+, \hat{K}_j^-] = -2\hat{K}_i^0 \delta_{ij},\ [\hat{K}_i^0, \hat{K}_j^\pm] = \pm \hat{K}_i^\pm \delta_{ij}.$$

We have $[\hat{K}^\alpha, \mathcal{M}] = 0$, where $\hat{K}^\alpha = \sum_{i=1}^{L} \hat{K}_i^\alpha$, $\alpha \in \{+, -, 0\}$ satisfy $SU(1,1)$ algebra.

Generating Hamiltonian of the Markov matrix $\mathcal{M}$ has $SU(1,1)$ symmetry in the particle-number conserving case.

Numerics shows $\mathcal{M}$ has $SU(1,1)$ symmetry for arbitrary values of J

Due to $SU(1,1)$ symmetry of the generating Hamiltonian, its lowest excited states can be obtained as degenerate descendants of the single-particle $(N = 1)$ states, i.e., by applying the operator $\hat{K}^-$.

L-dependence of Thouless time ($\Delta = 0$): Bosons

Therefore, the L-dependence of λ_1 is independent of N when $\Delta = 0$.

$$\mathcal{M}|_{\Delta=0}^{N=1} = (\mathbb{1} - 2J^2) + \sum_{i=1}^{L} J^2(\hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_{i+1}^\dagger \hat{a}_i) + \mathcal{O}(J^4).$$

"Ground state" of $\mathcal{M}|_{\Delta=0}^{N=1}$ is a state with eigenvalue 1 and with zero momentum. Eigenenergy spectrum is gapless, and first "excited state" (with momentum $k = 2\pi/L$) goes as $\lambda_1 = 1 - c_2/L^2$.

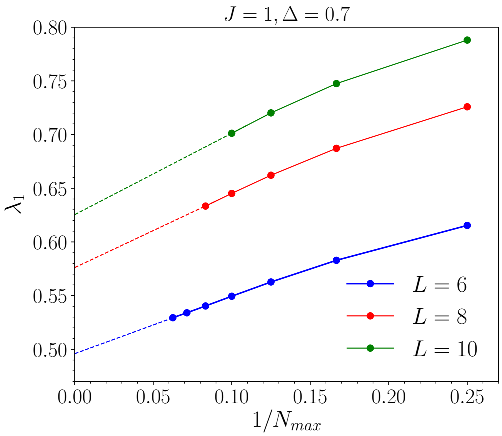
Thouless time, $t^* \simeq L^2/c_2$, for single boson and, due to $SU(1,1)$ symmetry, for any number of bosons in the particle number conserving model.

$J=1, \Delta=0, N/L=1/2$				$J=1, \Delta=0, N/L=1/4$			
L	λ_1	λ_2	λ_3	L	λ_1	λ_2	λ_3
8	0.8526	0.7486	0.6680	8	0.8526	0.7486	0.4847
10	0.9042	0.8283	0.7658	12	0.9329	0.8764	0.8278
12	0.9329	0.8764	0.8278	16	0.9619	0.9278	0.8970
14	0.9504	0.9071	0.8688	20	0.9755	0.9529	0.9320

$J = 1, \Delta = 0$: $\lambda_1 \sim 1 - 8.29/L^{1.94}$ (or $\lambda_1 \sim e^{-11.4/L^{2.05}}$) for $N/L = 1/2$, and $\lambda_1 \sim 1 - 9.0/L^{1.97}$ (or $\lambda_1 \sim e^{-10.5/L^{2.02}}$) for $N/L = 1/4$

L-dependence of Thouless time ($\Delta \neq 0$): Bosons

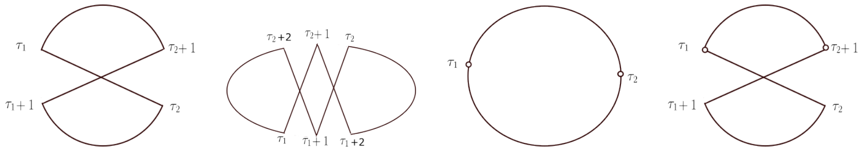
Generating Hamiltonian of $\mathcal{M}$ lacks $SU(1,1)$ symmetry when $\Delta \neq 0$. Consequently, λ_1 changes with N or $N_{\max}$ for a fixed L.



Dashed lines indicate a linear extrapolation of the last few large $N_{\max}$ points. These linear extrapolations give $\lambda_1 \sim 1 - 1.43/L^{0.58}$ or $e^{-2.89/L^{0.79}}$ at $1/N_{\max} \to 0$, which predicts a finite system-size dependence of the Thouless time (e.g., $t^* = \mathcal{O}(L^\gamma)$, $\gamma = 0.7 \pm 0.1$ when $J = 1, \Delta = 0.7$)

Second-order contributions

Single and double crossing diagrams for non-repeated basis states. Zero and single crossing diagrams for repeated basis states ($|\underline{n}_{\tau_1}\rangle = |\underline{n}_{\tau_2}\rangle$)



$$
\begin{aligned}
K_c(t) &= t^2(Z_X + Z_{XX} - Z_{OR} - Z_{XR}) \\
&= t^2\Big(\frac{t-3}{\mathcal{N}} + \frac{2}{\mathcal{N}}\sum_{i\neq 0}\frac{\lambda_i}{1-\lambda_i} + \frac{t-5}{\mathcal{N}} + \frac{2}{\mathcal{N}}\sum_{i\neq 0}\frac{\lambda_i^3}{1-\lambda_i} \\
&\quad -\frac{t-1}{\mathcal{N}} + \frac{2}{\mathcal{N}}\sum_{i\neq 0}\frac{\lambda_i}{1-\lambda_i} - \frac{t-5}{\mathcal{N}} - \frac{2}{\mathcal{N}}\sum_{i\neq 0}\frac{\lambda_i^3}{1-\lambda_i}\Big) \\
&= -\frac{2t^2}{\mathcal{N}}
\end{aligned}
$$