Supersymmetric ground states of 3d $\mathcal{N} = 4$ theories on a Riemann surface

Heeyeon Kim (Rutgers)

Based on 1812.05567, 1912.09591, 2105.08783 with Mathew Bullimore and Andrea Ferrari

Introduction

Setup:

$$\text{3d } \mathcal{N} = 4 \text{ gauge theories on } \Sigma_g \times \mathbb{R}$$

$$\Big\downarrow \text{ topological A- or B-twist}$$

$$\text{1d } \mathcal{N} = 4 \text{ quantum mechanics on } \mathbb{R}.$$

We obtain two types of quantum mechanics on $\mathbb{R}$, which are sigma models

$$\mathbb{R} \to \mathcal{M} \ ,$$

where $\mathcal{M}$ is the moduli space of vortices on Σ_g. We will study the Hilbert space of these quantum mechanics.

Moduli space of vortices

More precisely, for a 3d $\mathcal{N} = 4$ theory (G, R), $\mathcal{M}$ is the moduli space of solutions (A, ϕ) to

$$*F_A + e^2 \left(\mu_{\mathbb{R}}(\phi) - \tau\right) = 0 \ , \quad \bar{\partial}_A \phi = 0$$

modulo G.

- $\mathcal{M}$ is the moduli space of generalized vortices on Σ_g.
- Algebraically, $\mathcal{M}$ can be identified with moduli space of stable quasi-maps $\phi : \Sigma_g \to M_H$, where M_H is the Higgs branch of the theory.
- These moduli spaces play important roles in quantum K-theory, geometric Langlands correspondence, etc,.

Twisted Index

Twisted index is the Witten index of the effective quantum mechanics

$$I_{\text{twisted}} = \text{tr}_{\mathcal{H}}(-1)^F \prod_i y_i^{J_i}$$

This is well-defined as long as the theory is gapped and can be computed systematically via supersymmetric localization [Nekrasov,Shatashvilli 14][Benini,Zaffaroni 15,16][Closset,H.K 16]

$$\begin{aligned} I_{\text{twisted}} &= \sum_{P(x)=0} \mathcal{H}(x,y)^{g-1} \\ &= \oint_{JK} \frac{dx}{x} Z_{\text{1-loop}}(x,y) H(x,y)^g \end{aligned}$$

One of the goals is to provide geometric meaning of the twisted indices in terms of enumerative invariants of $\mathcal{M}$.

Contents

1. Moduli space of 3d $\mathcal{N} = 4$ gauge theories on a Riemann surface
2. Twisted indices and enumerative geometry of quasi-maps
   - Mirror symmetry
   - Level structure
   - Wall-crossing
3. Space of SUSY ground states

3d $\mathcal{N} = 4$ gauge theories on Σ

3d $\mathcal{N} = 4$ quiver gauge theories

Consider quiver gauge theories $\mathcal{T} = (G, R)$ with

$$G = \prod_I U(V_I)\ , \quad R = M \oplus M^*$$

- Higgs branch M_H and Coulomb branch M_C
- Global symmetry $G_H \times G_C \supset T_H \times T_C$. Generic $(m, \zeta) \in \mathfrak{t}_H \times \mathfrak{t}_C$
- Higgs branch $M_{H,\zeta} = M \oplus M^* /\!/\!/_\zeta G$
- The global symmetry $U(1)_t = U(1)_H - U(1)_C$

Assumption

The fixed loci of T_H action on $M_{H,\zeta}$ are isolated points for generic ζ, m.

e.g., $G = U(k)$, $M = \mathbb{C}^N$ with $k < N$

3d $\mathcal{N} = 4$ quiver gauge theories on Σ_g

Two topological twists on Σ_g

- A-twist $\to$ A-type quantum mechanics on $\mathbb{R}$ ($\mathcal{N} = (2,2)$ multiplets)
- B-twist $\to$ B-type quantum mechanics on $\mathbb{R}$ ($\mathcal{N} = (0,4)$ multiplets)

They are exchanged by 3d mirror symmetry.

The moduli space of solutions is decomposed into

$$\mathcal{M} = \bigcup_d \mathcal{M}_d \ , \qquad d = \int_\Sigma \mathrm{tr}(F) \in \Lambda_C \ ,$$

where $\mathcal{M}_d$ (with respect to two supercharges) parametrizes solutions to

$$\boxed{\mathcal{M}_d = \left\{(A,X,Y) \middle| *F_A + e^2(\mu_\mathbb{R} - \tau) = 0 \ , \bar{\partial}_A X = \partial_A \bar{Y} = 0 \ , \mu_\mathbb{C} = 0\right\}}$$

where $X \in \Gamma(P \times_G M \otimes K^{r/2})$, $Y \in \Gamma(P \times_G M^* \otimes K^{r/2})$ with $r = 1, 0$ for the A- and B-twist respectively.

Moduli space of quasi-maps

Algebraically, $\mathcal{M}_d$ is the moduli space of (twisted) stable quasi-maps

Quasi-maps

$\phi : \Sigma_g \to M_H$ with a collection of following data

- holomorphic vector bundle E
- holomorphic sections $\phi = (X, Y) \in H^0(P \times_G (M \oplus M^*))$
- Subject to $\mu_{\mathbb{C}}$

modulo $G_{\mathbb{C}}$ transformation.

A quasi-map ϕ is stable if $\phi(p) \in \mu_{\mathbb{C}}^{-1}(0)^s = M_H$ for all $p \in \Sigma_g$ but finitely many points on Σ_g.

Moduli space of quasi-maps

From the BPS equations, we find

- B-twist

$$\mathcal{M}_d = (\text{moduli space of degree } d \text{ stable quasi-maps to } M_d)$$

- A-twist

$$\mathcal{M}_d = (\text{moduli space of degree } d \text{ twisted stable quasi-maps to } M_d)$$

[B.Kim][Ciocan-Fontanine, B.Kim, Maulik][Okounkov]...

Twisted indices and enumerative geometry of quasi-maps

Virtual tangent space

What does the path integral compute?

Fock spaces of massless degrees of freedom of effective QM fit into following pair of complexes

$$H^0(E_V) \to t^{1/2} H^0(E_X \oplus E_Y) \to t H^0(E\varphi)$$
$$H^1(E_V) \to t^{1/2} H^1(E_X \oplus E_Y) \to t H^1(E\varphi)$$

where $E_V = P \times_G \mathfrak{g}$, $E_X = P \times_G M \otimes K_\Sigma^{r/2}$ and $E_\varphi = P \times_G \mathfrak{g} \otimes K_\Sigma^{1-r}$.

(K-theory class of the complexes) $=$ T_{vir} (virtual tangent bundle)

For the A-twist, due to Serre duality

$$H^0(E) \cong H^1(K_\Sigma \otimes E^*)^* \ ,$$

The moduli space has (-1)-shifted symplectic structure

$$(T^{\text{vir}}_{\mathcal{M}})^\vee \cong t^{-1}(T^{\text{vir}}_{\mathcal{M}})[1] \ .$$

Virtual Euler characteristic

The path integral of the guage theories can be identified with the virtual Euler characteristics

$$I_{\text{twisted}} = \chi_{\text{vir}} := \sum_{d \in \Lambda_C} q^d \int_{\mathcal{M}_d} \hat{A}(T_{\text{vir}}) \ ,$$

where $q = e^{2\pi i \zeta}$.

Localization

Global symmetry $T_H \times U(1)_t$ acts on the moduli space $\mathcal{M}_d$.

- Fixed loci of $U(1)_t$ action:

$$\mathcal{M}_d|_{\text{fixed},U(1)_t} := \mathcal{M}_d^T = (\text{QMaps } X : \Sigma_g \to L_H = M/\!\!/_\zeta G \subset M_H)$$

For the A-twist, this provide an interpretation of the twisted index as the virtual χ_{-t}-genus of $\mathcal{M}^T$:

$$\begin{aligned} I_A(t) &= \chi_{-t}^{\text{vir}}(\mathcal{M}^T) \\ &= \sum_d q^d \int_{\mathcal{M}_d^T} \hat{A}\left(T^{\text{vir}}|_{\mathcal{M}_d^T}\right) \wedge \text{ch}\left(\wedge^\bullet (T^{\text{vir}}|_{\mathcal{M}_d^T})^\vee\right) \end{aligned}$$

Localization

- Fixed loci of $T_H = U(1)^{\text{rk}(G)} \subset G_H$:

  The vector bundle is decomposed into

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_{\text{rk}(G)} \ , \quad \deg(L_i) = m_i \ , \ \text{tr}(m) = d$$

  and there exist rk(G)-many non-vanishing components of X

$$X_{i=1,\cdots,r} \neq 0 \ , \quad Y = 0$$

  The fixed loci $\{(L_1, X_1) \ , \cdots \ , (L_{\text{rk}(G)}, X_{\text{rk}(G)})\}$ parametrize solutions to abelian vortex equations:

$$\mathcal{M}_d|_{\text{fixed}, T_H} := \mathcal{M}_{I,m} = \prod_{i=1}^{\text{rk}(G)} \text{Sym}^{m_i + r(g-1)} \Sigma_g$$

  and

$$T_{\text{vir}}|_{\text{fixed}, T_H} = T\mathcal{M}_{I,m} + \mathcal{N}_{I,m}^{\text{vir}} \ .$$

Twisted indices and virtual Euler characteristics

$$\begin{aligned} I_{A,B} &= \sum_d q^d \int_{\mathcal{M}} \hat{A}(T^{\text{vir}}) \\ &= q^{\text{tr}(m)} \sum_I \int_{\mathcal{M}_{I,m}} \hat{A}(TM_{I,m}) \wedge \text{ch}(\hat{S}^\bullet N^\vee_{I,m}) \end{aligned}$$

Identity due to Don Zagier [Thaddeus 94]

$$\int_{\text{Sym}^n\Sigma_g} A(\eta) e^{\sigma B(\eta)} = \underset{u=0}{\text{res}}\ du \frac{A(u)(1+B(u))^g}{u^{n+1}} \ ,$$

where $\eta \in H^2(\text{Sym}^n\Sigma_g)$, $\sigma = \sum_{i=1}^g \xi_i \xi_{i+g}$, $\xi_i \in H^1(\text{Sym}^n\Sigma_g)$.

Using this, we can show

$$I_{A,B} = \sum_{m \in \Lambda_G^\vee} q^{\text{tr}(m)} \oint_{JK} du\ Z^m_{\text{1-loop}}(u,y) H(u,y)^g \ .$$

[Bullimore, H.K., Ferrari 18]

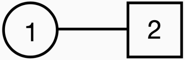
Mirror symmetry

$\boxed{\mathcal{T}1(G_1, R_1)}$		$\boxed{\mathcal{T}2(G_2, R_2)}$
A-twist		B-twist
B-twist	$\longleftrightarrow$	A-twist
M_H		M_C
M_C		M_H
G_H (m)		G_C (ζ)
t		t^{-1}

The duality implies highly non-trivial identities between the indices:

$$I_A(m, \zeta, t)[\mathcal{T}1] = I_B(\zeta, m, t^{-1})[\mathcal{T}2]$$
$$I_A(m, \zeta, t)[\mathcal{T}2] = I_B(\zeta, m, t^{-1})[\mathcal{T}1]$$

Example



$$
\begin{aligned}
I_A|_{g=2} &= -\frac{(1+t)[t(a+a^{-1}-2)(q+q^{-1}-2)+4(1-t)^2]}{t^{1/2}(t-a)(t-a^{-1})} \\
I_B|_{g=2} &= -\frac{(1+t)[t(a+a^{-1}-2)(q+q^{-1}-2)+4(1-t)^2]}{t^{1/2}(t-q)(t-q^{-1})}
\end{aligned}
$$

We have

$$
I_A[a,q,t] = I_B[q,a,t^{-1}]\ .
$$

Level structure

One can also construct a 3d gauge theory that computes

$$\chi(\mathcal{M}^T, \mathcal{L}^k) \ ,$$

where $\mathcal{L}$ is the virtual canonical bundle on $\mathcal{M}^T$. This can be done by considering $\mathcal{N} = 2$ theories with effective chern-simons level k_{eff}.

There exists a systematic way to write down an ultraviolet gauge theory that computes this quantity:

3d $\mathcal{N} = 4$ theories with M_H $\qquad \xrightarrow[t \to 0, \infty]{}$ 3d $\mathcal{N} = 2$ theory with CS levels

For example, for $M_H = T^* Gr(N, M)$, this process gives $N = 2$ SQCD with the following UV CS coupling:

$$k_{U(1)} = -Nk \ , k_{SU(N)} = (-N + 2M)k$$

Topological vacua

There exists an subtlety in identifying the twisted index of these gauge theories with the holomorphic Euler characteristic $\chi(\mathcal{M}^T, \mathcal{L}^k)$. Consider an $\mathcal{N} = 2$ gauge theory with $k_{\text{eff}} \neq 0$. Moduli space of the BPS equations are in general in the form of

$$\mathcal{M}_{\text{vortex}} \cup \mathcal{M}_{\text{top}} \ .$$

Points on $\mathcal{M}_{\text{top}}$ are solutions with unbroken gauge symmetries. The low energy theory is described by effective Chern-Simons theory.

$\longrightarrow \mathcal{M}_{\text{top}}$ should be described as a quotient stack.

Two branches of solutions

For $G = U(1)$, $k_{\text{eff}} \neq 0$, the BPS equations have two branches of solutions.

- $\phi \neq 0$

$$*F_A + e^2(\mu_{\mathbb{R}} - \tau) = 0 \ , \quad \bar{\partial}_A \phi = 0$$

  This branch can be identified with $\mathcal{M}_d^{\phi \neq 0} = \text{Sym}^d \Sigma_g$. $G = U(1)$ completely broken.

- $\phi = 0$

$$*F_A = \frac{2\pi}{\text{vol}(\Sigma)} d$$

  This branch parametrizes holomorphic line bundles of degree d. $\mathcal{M}_d^{\phi=0} = \mathfrak{P}ic^d \Sigma_g$. $G = U(1)$ unbroken.

Effective QM at $\phi = 0$ is the sigma model into $\mathfrak{P}ic^d \Sigma_g$. The fluctuation of ϕ provides a $\mathbb{C}^*$-equivariant complexes $\mathcal{E}$ over the moduli space.

The twisted index and topological vacua

The twisted index of the gauge theory computes

$$I_d = I_d^{\text{vortex}} + I_d^{\text{top}} \ ,$$

where

$$I_d^{\text{vortex}} = \sum_I \int_{\mathcal{M}_d^I} \hat{A}(\mathcal{M}_d^I) \wedge \frac{\text{ch}(\mathcal{L}^k)}{\text{ch}(\widehat{\wedge}^\bullet \mathcal{E})}$$

and

$$\begin{aligned} I_d^{\text{vor}} &= \int_{\mathcal{M}_d^{\text{top}}} \hat{A}(\mathcal{M}_d^{\text{top}}) \wedge \frac{\text{ch}(\mathcal{L}^k)}{\text{ch}(\widehat{\wedge}^\bullet \mathcal{E})} \\ &= \int_{x=0,\infty} \frac{dx}{x} \int_{T^{2g}} \hat{A}(T^{2g}) \wedge \frac{\text{ch}(\mathcal{L}^k)}{\text{ch}(\widehat{\wedge}^\bullet \mathcal{E})} \end{aligned}$$

One can check that the sum of two contributions reproduces the residue integral representation of the twisted index

$$I_{\text{twisted}} = \sum_{d\in\mathbb{Z}} q^d \oint_{JK} du \ Z_{CS}(u) Z^d_{1\text{-loop}}(u,y) H(u,y)^g \ .$$

[Bullimore, H.K., Ferrari, Xu 20]

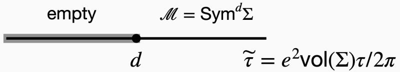
Wall-crossing

In general, such moduli spaces depend on a parameter $\tau \in \mathfrak{t}_C$

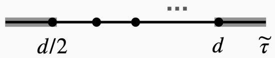
$$\mathcal{M}_d^\tau = \left\{(A,\phi) | *F_A + e^2(\mu_\mathbb{R} - \tau) = 0 \ , \bar{\partial}_A \phi = 0 \ , \mu_\mathbb{C} = 0\right\}$$

Description of $\mathcal{M}_d^\tau$ changes discontinuously as τ crosses the walls.

Example 1 $G = U(1)$, $\phi \in H^0(L)$: moduli space of abelian vortices



Example 2 $G = U(2)$, $\phi \in H^0(E)$: moduli space of rank 2 stable pair



[Bradlow 90][Thaddeus 94][Bertram]...

Wall-crossing from path integral

- How do we understand the wall-crossing in the 3d gauge theory point of view?
- Universal formula for

$$I(\tau_* + \epsilon) - I(\tau_* - \epsilon) = ?$$

ζ vs. τ

$$\zeta \int d^3x \sqrt{g}\ D \qquad\qquad \tau \int d^3x \sqrt{g}\ (D - F_{z\bar{z}})$$

- Singularities where tensionless domain walls exist [see Bullimore's lecture]
- Description of Hilbert space may jump
- Supersymmetric observables are holomorphic in complexified ζ [Seiberg,Intriligator 2013]
- No wall-crossing in the twisted indices

- Varying τ is a Q-exact deformation
- Walls are co-dimension one
- Supersymmetric observables are piecewise constant in the space of τ

Remarks

- Some of the supersymmetric ground states appear/disappear as τ crosses the walls.
- I_{twisted} can jump discontinuously at walls, for theories with $k_{\text{eff}} = 0$. In this case, the theory contains gauge-singlet monopole operators, which generate non-compact Coulomb branch at walls. (e.g., $\mathcal{N} = 4$ theories)
- For theories with $k_{\text{eff}} \neq 0$, recall

$$I^{d}_{\text{twisted}} = I^{d}_{\text{vortex}} + I^{d}_{\text{top}}$$

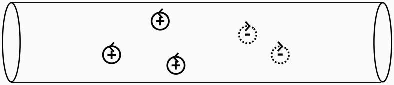
The individual term jumps at walls, but the sum is invariant.

Example: wall-crossing for $G = U(1)$, $k_{\text{eff}} = 0$

The wall-crossing formula can be derived in the point of view of the effective quantum mechanics following [Hori, HK, Yi 14]

The index can be written as a sum of theh residues of certain choice of poles, which is prescribed by the Jeffrey-Kirwan residue.

$$I^{d}_{\text{twisted}} = q^{d} \oint_{JK(\eta)} du\ g(u)$$



One can show that

$$\eta = -\frac{2\pi d}{e^2} + \text{vol}(\Sigma)\tau\ .$$

$$\boxed{I(\tau^* + \epsilon) - I(\tau^* - \epsilon) = q^{d^*} \underset{u=\pm\infty}{\text{res}}\ du\ Z^{d}_{\text{1-loop}}(u, y) H(u, y)^{g}}$$

[Bullimore, H.K., Ferrari 19]

Example: wall-crossing for $G = U(2)$

One can study the index associated with moduli space of rank 2 stable pairs $\mathcal{M}_d^\tau$. The 3d $\mathcal{N} = 4$ theory whose index computes $\chi_{-t}(\mathcal{M}_d^\tau)$ is

$$G = U(2)\ , R = F \oplus \bar{F} \ \ \text{(1 hypermultiplet)}$$

we find

$$I[\mathcal{M}_d^\tau] = \sum_{m=0}^{d-[\widetilde{\tau}]-1} \underset{u_2=\pm\infty}{\text{res}} \underset{u_1=0}{\text{res}}\ du_2 du_1\ Z_{\text{1-loop}^m}(u) H(u)^g\ .$$

One can check that this formula agrees with the result of [Munoz, Ortega, Vazquez-Gallo 07]

Space of SUSY ground states

Half-BPS states

From the structure of supersymmetry multiplets, we can identify the space of half-BPS states $\mathcal{H}_{A,B}$, graded by two R-symmetries.

- A-twist

$$\mathcal{H}_A = \bigoplus_{d \in \pi_1(G)} q^d \mathbb{H}^\bullet(\mathcal{M}_d^T, P_d)$$

  When $\mathcal{M}_d$ is smooth, it reduces to $H^\bullet_{\mathrm{dR}}(\mathcal{M}_d^T)$.

- B-twist

$$\mathcal{H}_B = H^{0,\bullet}_{\bar{\partial}}(M_H, (\wedge^\bullet T_{M_H})^g)$$

  as in Rozansky-Witten theory.

They can be computed after deforming the supercharges $Q \to e^{-m\mu_H} Q e^{m\mu_H}$, where μ_H is the moment map for the T_H action.

Perturbative ground states

Taking the limit $m \to \infty$ in a given chamber, the wavefunctions are localized around the critical loci. Under our assumption, the critical loci are compact and smooth. For the A-twist,

$$\mathcal{M}_d^I = \prod_{i=1}^{\mathrm{rk}(G)} \mathrm{Sym}^{m_i + r(g-1)} \Sigma_g \ .$$

Therefore the perturbative ground states are

$$\mathcal{H}_A = \bigoplus_{I,d} q^d H^{\bullet}_{\mathrm{dR}}(\mathcal{M}_d^I) \ .$$

For the B-twist, the fixed locus are isolated points $\mathcal{M}^I = v^I$. The perturbative ground states are

$$\mathcal{H}_B = \bigoplus_{I} \widehat{\mathrm{Sym}}^{\bullet} V_I \ .$$

Supersymmetric ground states

In general, these states are subject to instanton corrections, due to tunneling between fixed loci.

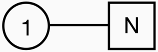
However, by studying the Morse index of each fixed point, one can show that there are no instanton corrections for both twists, for the half-BPS states.

Therefore the perturbative states are exact ground states.

[Bullimore, H.K., Ferrari 21]

Example

Consider SQED



We find

$$\mathcal{H}_{A,B} = \bigoplus_{I=1}^{N} \widehat{\mathrm{Sym}}^{\bullet} V_I^{A,B} \ ,$$

where

$$V_I^A = qt^{-i}(\mathbb{C} \oplus \mathbb{C}^g[-1] \oplus t\mathbb{C}^g[-1] + t\mathbb{C}[-2])[2i]$$

and

$$V_I^B = N_I^+ \oplus (N_I^-)^\vee \oplus \left(N_I^+[-1] \oplus N_I^-[-1]^\vee\right) \otimes \mathbb{C}^g$$

$$N_I^+ = \bigoplus_{j>i} \frac{x_i}{x_j}\mathbb{C} \oplus \bigoplus_{j<i} t^{-1}\frac{x_j}{x_i}\mathbb{C}[2]$$

$$N_I^- = \bigoplus_{j<i} \frac{x_i}{x_j}\mathbb{C} \oplus \bigoplus_{j>i} t^{-1}\frac{x_j}{x_i}\mathbb{C}[2]$$

Remarks

- The dimensions of the graded Hilbert space reproduces the twisted indices. For the A-twist,

$$\mathcal{I}_A = \sum_d q^d \sum_{p,q} (-1)^{p+q} t^p h^{p,q}(\mathbb{H}^\bullet(\mathcal{M}_d^T, P_d)) \ .$$

- Mirror symmetry can be checked at the level of the graded Hilbert space.